ALGORITHM CONSTRUCTION AND NUMERICAL SOLUTION BASED ON THE GRADIENT METHOD OF ONE INVERSE PROBLEM FOR THE ACOUSTICS EQUATION

The paper considers the problem of continuation of solutions of hyperbolic equations from a part of the domain boundary. These problems include the Cauchy problem for a hyperbolic equation with data on a timelike surface. In the inverse problems, the inhomogeneities are located at some depth under the medium layer, the parameters of which are known. In this case, an important tool for practitioners are the problems of continuation of geophysical fields from the Earth's surface towards the lay of inhomogeneities. In equations of mathematical physics, solution of the continuation problem from part of the boundary is in many cases strongly ill-posed problems in classes of functions of finite smoothness. The ill-posedness of this problem is considered, that is, the example of Hadamard, a Cauchy problem for a hyperbolic equation, is given. The physical formulation of the continuation problem is considered and reduced to the inverse problem. The definition of the generalized solution is formulated and the correctness of the direct problem is presented in the form of a theorem. The inverse problem is reduced to the problem of minimizing the objective functional. The objective functional is minimized by the Landweber method. By the increment of the functional, we consider the perturbed problem for the direct problem. We multiply the equation of the perturbed problem by some function and integrate by parts, we obtain the formulation of the conjugate problem. After that, we get the gradient of the functional. The algorithm for solving the inverse problem is listed. A finite-difference algorithm for the numerical solution of the problem is presented. The numerical solution of the direct problem is performed by the method of inversion of difference schemes. The results of numerical calculations are presented

Keywords: inverse problems, continuation problem, acoustic equation, numerical experiment, Landweber method

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1. Introduction

The study of wave processes is important for the successful development of many areas of science and technology. The most interesting and closest to the real physical processes are two-dimensional problems of mathematical modeling of wave processes. In various applications, fundamental research and technology, the fact that the transmission of sound is influenced by the medium through which it passes, as well as intermediate bodies and inhomogeneities, is used, and, therefore, information about it is provided. Many physical processes in acoustics are simulated as boundary problems, in which some characteristics of the acoustic field are determined on boundary surfaces or in the whole spatial domain at the initial moment of time.

Significant interests, especially for many branches of geophysics and ocean acoustics, are the issues of formation of acoustic fields with specified properties and the study of the features of sound wave propagation in inhomogeneous media.

In many inverse problems, the target inhomogeneities are located at some depth under a layer of the medium whose parameters are known. In this case, an important tool for practitioners are the problems of continuation of geophysical fields from the Earth's surface in the direction of the occurrence of heterogeneities. Continuation problems in mathematical physics equations with a part of the boundary are in many cases strongly ill-posed problems in classes of functions of finite smoothness. These problems include the Cauchy problem for a hyperbolic equation with Cauchy data on a time-like surface [1].

The development of seismic methods of investigation of the Earth's interior in recent years to search for minerals and experiments to determine a more detailed internal structure of the Earth has led to a revision of views on the theoretical, methodological, and technical capabilities used by seismology. By now, geophysics has suggested various methods of using gravity data to calculate static corrections, which are reduced to solving a hyperbolic type of equation. Therefore, numerical solutions of inverse problems for the hyperbolic type of equation are essential.

2. Literature review and problem statement

Finding causes by knowing their consequences, such as mine detection, medical imaging, weather forecasts or climate change forecasts, remote sensing and geophysical research constitute the idea of solving inverse wave equation problems. Pierce's contributions to acoustic theory include fundamental research on wave mechanics, vibration of the structure. Particular areas of investigation ranged from atmospheric acoustics, waves on shells immersed in liquid, interaction of sound with structures, sonic shocks, sound diffraction and scattering, marine sediment mechanics, and noise control [2]. The paper shows the applications of the general acoustic equation and illustrates the problem statement and solutions.

The invention of radar and sonar forced scientists to question whether it was possible to determine more about a scattering object than just its location, that is the question of reconstructing its exact images. However, small changes in effects can lead to large differences in causes, or the same effect can be obtained for more than one cause. Such problems are related to the inverse problem of scattering, and it gradually became clear that these problems are of obvious physical interest but are ill-posed. New methods of solving geophysical investigations have been developed at this time. Modern geophysical equipment is being introduced in [3], which allows rapid geophysical information processing and interpretation. Currently, geophysical survey methods mainly use sonars for qualitative surveying, which requires calibration of results from in situ drilling and sampling [4]. At the same time, with the development of such geophysical methods considered in [4], the mathematical theory of geophysical problems developed. The development of the field of geophysical research and other related sciences has led to the need to create a science of methods for solving the inverse problem of geophysics. Depending on the purpose of the study, the complexity of the geological structure and the completeness of the initial information, mathematical models and partial derivative equations are chosen. In particular, the type of mathematical model is studied and algorithms for reconstructing the shape of location and boundary parameters for inaccessible targets in acoustics are investigated in [5]. Physical interpretations of the considered problems are given and geophysical applications for some of them are described [5, 6]. The importance of inverse problems in geophysical research is shown in connection with this topical issue. The actuality of the problem of numerical solution of the inverse problem was also noted.

The more accurately the direct problem model reflects real processes, the more reasonable the method of interpretation and the more accurate the result. The lack of data for the process under study leads to the fact that not all the parameters included in the direct problem model are known. Consequently, to construct the inverse problem, additional information is needed. Moreover, it is not always possible to find an accurate solution to the inverse problem because of the mathematical difficulties that appear, and they lead the difficulties to a numerical solution. However, there is still no unified universal mathematical theory for solving inverse problems.

The problem considered in the paper relates to the problem of the continuation of the acoustic equation. Here it is important to note that the existence of a generalized solution of the direct problem is guaranteed by the conditional stability theorem of the continuation problem for the Helmholtz equation, proved in [7]. The Helmholtz equation is obtained by decomposing the acoustics equation into a Fourier series in time, where the state is homogeneous. The algorithm for solving the continuation problem for the Maxwell equation was discussed in [1]. This paper considers the continuation problem for the electromagnetic wave equation and shows only the algorithm for the numerical solution in general. The regularization approach to the continuation problem using the iterative method for the Laplace equation was proposed in [8]. In [9], the hyperbolic integro-differential equation of acoustics is considered. The direct problem consists in determining from the initial boundary value problem for this equation the acoustic pressure created by a concentrated excitation source located on the boundary of the spatial domain. For this direct problem, the inverse problem is investigated, which consists in determining the one-dimensional kernel of the integral term by the known solution of the direct problem at the point $x=0$ at $t>0$. The above papers suggest ways to solve the problem. The results of the numerical solution for the continuation problem are not shown in detail.

The general factor is that both electromagnetic and acoustic waves are described by wave equations, and both cause reflection, scattering and diffraction effects when the wave interacts with heterogeneity in the medium. The work [10] considers the advancement of radio wave tomography methods. The physical and mathematical models of systems designed to restore images of hidden objects, using tomographic processing of remote measurements of scattered acoustic and ultrasonic wave radiation were described. In various applications of engineering and science, fuzzy nonlinear equations play a vital role. In [11], an optimization algorithm based on the Euler method for solving fuzzy nonlinear equations is presented. Compared to some other algorithms, the algorithm discussed in this paper is clearly better suitable for obtaining a solution with higher accuracy. In this regard, it can be seen that the study of the continuation problem for the acoustic equation, the numerical solution, and obtaining the result is an actual issue in general.

3. The aim and objectives of the study

The aim of the study is to solve numerically the continuation problem for the acoustic equation, with data on a time-like surface.

To achieve the aim, the following objectives are accomplished:

– to show the ill-posedness of the initial problem and reduce it to an equivalent inverse problem with respect to some direct (correct) problem;

– by introducing a perturbed problem, to obtain a formula for calculating the gradient and a formulation of the conjugate problem, and construct an algorithm for the inverse problem using the iterative Landweber method;

– using the Fourier series on the second spatial variables, the direct and conjugate problem is reduced to the vector form. And also write a finite-difference approximation of these problems with a triangular domain;

– to present the numerical solution of the direct and inverse problem, compare the obtained results of the inverse problem with the exact solution.

4. Materials and methods

The considered problem relates to the continuation problem of the acoustic equation, with data on a time-like surface. The performance of standard numerical methods for such equations depends on the wavelength of the sound field. In particular, at high frequencies, the requirement of a certain number of discretization points per wavelength often leads to an impractical computational problem. There is an obvious need for an accurate method for modeling timedomain wave fields in complex geometric shapes. Recent applications for modeling such mechanical wave fields in the time domain have been in the fields of exploration seismology, electromagnetism, and medical ultrasound.

We associate the direct problem of mathematical physics with the classical boundary value problems of mathematical physics characterized by the need to find a solution that satisfies a given partial differential equation and some initial and boundary conditions. In inverse problems, the equation and initial boundary conditions are not completely specified, but some additional information is provided. In such a classification of inverse problems of mathematical physics, we can talk about coefficient, boundary (unknown boundary conditions), and evolutionary (associated with the fact that the initial condition is not given) inverse problems of mathematical physics. Inverse problems are often ill-posed in the classical sense. Typical is the violation of the requirement of continuous dependence of the solution on the initial data.

The main idea of the projection method is that the multidimensional inverse problem is projected onto a finite-dimensional subspace generated by some orthogonal system of functions. The resulting finite system of one-dimensional inverse problems can be solved numerically, using the method of finite-difference schemes. The main problem in this way is to justify the existence of a finite system of one-dimensional inverse problems and to estimate the speed of convergence of a finite system of one-dimensional inverse problems to the exact solution of the original multidimensional inverse problem when the parameter N of the Fourier series segment length in the basis function decomposition tends to infinity [12].

To evaluate the effectiveness of the methods of finding an extremum of the investigated function, computational experiments and comparative analysis of the methods were performed. The minimization was carried out by the Landweber iteration method with a constant step. The step of descent was chosen from the condition $\alpha \in (0.1/\|A\|^2)$ [12]. The difficulty in solving the direct and conjugate problems is that the formulation of these problems is given in a matrix-vector form and the solution area is triangular. For this purpose, a special difference scheme was chosen, with the time step and the spatial variables being the same, that is, the difference scheme pattern is a "rhombus". And to increase the accuracy, the initial condition is approximated with the second order. For the convenience of software implementation, the "rhombus" pattern is transformed into a "square". The programming was realized in Python, and the results of the numerical experiment are given.

5. Research results of the numerical solutions and algorithms of the direct and inverse problems

5. 1. Direct and inverse problems
The complete system of linear general equations of acoustics for the pressure and velocity of particles has the form (1):

$$\begin{cases} \rho \dfrac{\partial v}{\partial t} + \nabla p = 0, \\ \beta \dfrac{\partial p}{\partial t} + \nabla v = 0, \end{cases} \tag{1}$$

where ρ is the density of the medium, v is the particle velocity, p is the wave pressure, β is the compressibility of the medium. We reduce the system (1) to a single equation (2), with respect to the value p.

$$c^{-2}(x,y)\frac{\partial^2 p}{\partial t^2} = \Delta p - \nabla \ln(\rho(x,y))\nabla p. \tag{2}$$

By making the transformation $u = v(x,y) \cdot e^{\ln(x,y)/2}$, we can reduce it to the following equation:

$$v_{tt} = v_{xx} + v_{yy} - a(x,y) \cdot v, \tag{3}$$

where $a(x,y) = -\dfrac{1}{2}\Delta \ln \rho(x,y) + \dfrac{1}{4}\left|\nabla \ln \rho(x,y)\right|^2$.

Consider the continuation problem for the acoustics equation in the domain: $\Omega = \Delta(T) \times (0,L)$, where $\Delta(T) = \{(x,t): x \in R^2, t \in (0,T)\}$, $T, L \in R$:

$$u_{tt} = u_{xx} + u_{yy} - a(x,y)u, \tag{4}$$

$$u_x(0,y,t) = g(y,t), \tag{5}$$

$$u(0,y,t) = f(y,t). \tag{6}$$

Physical problem statement (4)−(6). Let the acoustic wave source (5) be switched on at the boundary of the medium $x=0$ of the studied region Ω at the time $t=0$ [13].

The response of the medium (6) is measured on the surface $x=0$ during the time $t \in (0, 2T)$.

We assume that the source function $g(y, t)$ is finite and its carrier lies inside $(0, L)$ and L is large enough to:

$$u(x,0,t) = u(x, L_y, t) = 0, \quad (x,t) \in \Delta(L_x). \tag{7}$$

Hadamard's example of the ill-posedness of the continuation problem.

The continuation problem (4)−(6) is ill-posed according to Hadamard. Let $\rho(x, y) \equiv$ const, then the solution of the problem (8), (9)

$$u_{tt} = u_{xx} + u_{yy}, \tag{8}$$

$$u(0,y,t) = \frac{1}{k}\cos(k\sqrt{2}y)\cos(kt), \quad u_x(0,y,t) = 0. \tag{9}$$

It is easy to see that for $k \to \infty$ the data $u_k(0, y, t) = (1/k)\cos(k\sqrt{2}y)\cos(kt)$ tend to zero, while the solution (10)

$$u_k(x,y,t) = \frac{1}{k}\frac{e^{kx}+e^{-kx}}{2}\cos(k\sqrt{2}y)\cos(kt), \tag{10}$$

increases indefinitely in an arbitrary small neighborhood of $x=0$.

Direct and inverse problems.

For the considered ill-posed problem (4)−(7), the corresponding well-posed problem is constructed:

$$u_{tt} = u_{xx} + u_{yy} - a(x,y)u, \quad (x,y,t) \in \Omega, \tag{11}$$

$$u_x(0,y,t)=g(y,t), \quad y\in\left[0,L_y\right], \quad t\in[0,2T], \tag{12}$$

$$u(x,y,x)=q(x,y), \quad x\in(0,T), \quad y\in(0,L), \tag{13}$$

$$u(x,0,t)=u\left(x,L_y,t\right)=0, \quad (x,t)\in\Delta(T). \tag{14}$$

where it is necessary to define $u\in H^1(\Omega)$, by given $\rho\in C^1((0,T)\times(0,L))$, $q\in H^1((0,T)\times(0,L))$, and $g\in H^1((0,L)\times(0,2T))$. Problem (5)–(8) is called direct.

Thus, instead of the original problem (1)–(4), we will study the inverse to (11)–(14) problem, in which it is required to define the function $q\in H^1((0,T)\times(0,L))$, from relations (11)–(14) for given functions $\rho\in C^1((0,T)\times(0,L))$, $g\in H^1((0,L)\times(0,2T))$, and additional information on solving a direct problem:

$$u(0,y,t)=f(y,t). \quad y\in[0,L],\, t\in[0,2T]. \tag{15}$$

Definition. Let $\rho\in C^1((0,T)\times(0,L))$, $q\in H^1((0,T)\times(0,L))$, $g\in H^1((0,L)\times(0,2T))$. And so let the $u\in H^1(\Omega)$ function call a generalized solution of the direct problem (11)–(14), if for any $\omega\in H^1(\Omega)$, such that

$$\omega(x,y,2T-x)=0, \quad x\in[0,T], \quad y\in[0,L], \tag{16}$$

$$\omega(x,0,t)=\omega(x,L,t)=0, \quad (x,t)\in\overline{\Delta}(T). \tag{17}$$

There is an equality

$$\iiint_{\Omega}\left(\omega_t u_t-\omega_x u_x-\omega_y u_y+a(x,y)\cdot u\cdot\omega\right)\mathrm{d}x\mathrm{d}y\mathrm{d}t=$$
$$=\int_0^L\int_0^T-\omega(x,y,x)q_x(x,y)\mathrm{d}x\mathrm{d}y+$$
$$+\int_0^L\int_0^{2T}\omega(0,y,t)g(y,t)\mathrm{d}t\mathrm{d}y.$$

Theorem 5.1. (well-posedness of the direct problem). Let $\rho\in C^1((0,T)\times(0,L))$, $q\in H^1((0,T)\times(0,L))$, $g\in H^1((0,L)\times(0,2T))$. Then the direct problem (11)–(14) has a unique generalized solution $u\in H^1(\Omega)$, and the following estimate is true

$$\|u\|^2(t)\le 6\left(3+2\sqrt{2}\right)\cdot e^{4Mt}\cdot T^2\cdot\left(\|q\|^2(T)+\|g\|^2(2T)\right), \tag{18}$$

here $t\in(0,T)$.

The theorem of the well-posedness of the direct problem was proved in [13].

5. 2. Solution of the inverse problem for the acoustic equation by the Landweber method

We introduce the operator A as follows

$$A:q(x,y)\mapsto f(y,t), \tag{19}$$

$$A:H^1(0,T)\mapsto H^1(0,2T). \tag{20}$$

Then we will write the inverse problem (5)–(9) in operator form:

$$Aq=f. \tag{21}$$

Let's introduce the objective functional

$$J(q_n)=\|Aq_n-f\|_{L_2}^2=\int_0^L\int_0^{2T}\left[u(0,y,t;q_n)-f(y,t)\right]^2\mathrm{d}y\mathrm{d}t. \tag{22}$$

We minimize the objective functional (22) by the Landweber method.

$$q_{n+1}=q_n-\alpha_n J'q_n, \tag{23}$$

here $\alpha\in\left(0,\dfrac{1}{\|A\|^2}\right)$, descent parameter [14, 15].

Landweber's method has the benefit of being remarkably easy to apply.

We will set the increment $q_n+\delta q_n$ then we will enter the following notation

$$\delta u=\tilde{u}-u=u(x,y,t;q_n+\delta q_n)-u(x,y,t;q_n). \tag{24}$$

Using the notation (24), we calculate the increment of the objective functional $J(q)$. By the increment of the functional we consider the perturbed problem for the direct problem.

$$J(q_n+\delta q_n)-J(q_n)=$$
$$=\int_0^{L_y}\int_0^{2L_x}\left[u(0,y,t;q_n+\delta q_n)-f((y,t)\right]^2\mathrm{d}y\mathrm{d}t-$$
$$-\int_0^{L_y}\int_0^{2L_x}\left[u(0,y,t;q_n)-f(y,t)\right]^2\mathrm{d}y\mathrm{d}t=$$
$$=\int_0^{L_y}\int_0^{2L_x}\left[u(0,y,t;q_n+\delta q_n)-u(0,y,t;q_n)\right]\times$$
$$\times\left[\begin{array}{l}u(0,y,t;q_n+\delta q_n)-f(y,t)+\\+u(0,y,t;q_n)-f(y,t)\end{array}\right]\mathrm{d}y\mathrm{d}t=$$
$$=\int_0^{L_y}\int_0^{2L_x}\delta u(0,y,t;q_n)2\left[\begin{array}{l}u(0,y,t;q_n)-\\-f(y,t)\end{array}\right]\mathrm{d}y\mathrm{d}t+o\left(\|\delta u\|\right). \tag{25}$$

Let us consider the statement of the perturbed problem to the problem (11)–(14)

$$\tilde{u}_{tt}=\tilde{u}_{xx}+\tilde{u}_{yy}-a(x)\tilde{u}, \tag{26}$$

$$\tilde{u}_x(0,y,t)=g(y,t), \tag{27}$$

$$\tilde{u}(x,y,x)=q_n+\delta q_n. \tag{28}$$

$$\tilde{u}(x,0,t)=u\left(x,L_y,t\right)=0. \tag{29}$$

To obtain the problem with respect to $\delta u(0, y, t; q_n)$, we subtract the problem (11)–(14) from the problem (26)–(29), and then we get the following relations:

$$\delta u_{tt}=\delta u_{xx}+\delta u_{yy}-a(x,y)\delta u, \tag{30}$$

$$\delta u_x(0,y,t)=0, \tag{31}$$

$$\delta u(x,y,x)=\delta q_n. \tag{32}$$

$$\delta u(x,0,t)=u\left(x,L_y,t\right)=0. \tag{33}$$

Consider the expression (30) and multiply it by an arbitrary function $\psi(x,y,t)$. The resulting expression is integrated over the domain Ω.

$$0 = \iiint_{\Omega}\left(\delta u_{tt} - \delta u_{xx} - \delta u_{yy} + a(x,y)\delta u\right)\psi \mathrm{d}x\mathrm{d}y\mathrm{d}t =$$

$$= \int_0^L\int_0^T\int_x^{2T-x} \psi\delta u_{tt}\mathrm{d}t\mathrm{d}x\mathrm{d}y -$$

$$- \int_0^L\int_0^T\int_0^t \psi\delta u_{xx}\mathrm{d}x\mathrm{d}t\mathrm{d}y - \int_0^L\int_T^{2T}\int_0^{2T-t} \psi\delta u_{xx}\mathrm{d}x\mathrm{d}t\mathrm{d}y -$$

$$- \int_0^T\int_x^{2T-x}\int_0^L \psi\delta u_{yy}\mathrm{d}y\mathrm{d}t\mathrm{d}x + \iint_{\Delta(L_x)} a(x,y)\delta u\cdot\psi\,\mathrm{d}x\mathrm{d}t\mathrm{d}y.$$

Let's integrate this expression in parts:

$$0 = \int_0^L\int_0^T\begin{bmatrix}\left(\psi\delta u_t\right)(x,y,2T-x)-\\ -\left(\psi\delta u_t\right)(x,y,x)-\\ -\left(\psi_t\,\delta u\right)(x,y,2T-x)+\\ +\underline{\underline{\left(\psi_t\,\delta u\right)(x,y,x)}}+\int_x^{2T-x}\psi_{tt}\delta u\mathrm{d}t\end{bmatrix}\mathrm{d}x\mathrm{d}y -$$

$$- \int_0^L\int_0^T\begin{bmatrix}\left(\psi\delta u_x\right)(t,y,t)-\left(\psi\,\delta u_x\right)(0,y,t)-\\ -\underline{\underline{\left(\psi_x\delta u\right)(t,y,t)}}+\\ +\left(\psi_x\delta u\right)(0,y,t)+\int_0^t\psi_{xx}\delta u\mathrm{d}x\end{bmatrix}\mathrm{d}t\mathrm{d}y -$$

$$- \int_0^L\int_T^{2T}\begin{bmatrix}\left(\psi\delta u_x\right)(2T-t,y,t)-\left(\psi\,\delta u_x\right)(0,y,t)-\\ -\left(\psi_x\delta u\right)(2T-t,y,t)+\\ +\left(\psi_x\delta u\right)(0,y,t)+\int_0^{2T-t}\psi_{xx}\delta u\mathrm{d}x\end{bmatrix}\mathrm{d}t\mathrm{d}y -$$

$$- \int_0^T\int_x^{2T-x}\begin{bmatrix}\left(\psi\delta u_y\right)(x,L,t)-\left(\psi\delta u_y\right)(x,0,t)-\\ -\left(\psi_y\delta u\right)(x,L,t)+\\ +\left(\psi_y\,\delta u\right)(x,0,t)+\int_0^L\psi_{yy}\delta u\mathrm{d}y\end{bmatrix}\mathrm{d}t\mathrm{d}x +$$

$$+ \int_0^L\int_0^T\int_x^{2T-x} a(x,y)\psi\delta u\mathrm{d}x\mathrm{d}t\mathrm{d}y.$$

Considering (23) and (25), and due to the fact that

$$\psi_x(x,y,2T-x) - \psi_t(x,y,2T-x) =$$
$$= \frac{d\psi}{dx}\Big|_{\frac{dt}{dx}=-1} = \psi_t(x,y,2T-x)$$

is the derivative in the direction $t = 2T - x$;

$$\psi_x(x,y,x) + \psi_t(x,y,x) = \frac{d\psi}{dx}\Big|_{\frac{dt}{dx}=1} = \psi_t(x,y,x),$$

the derivative in the direction $x=t$;

$$\delta v_x(x,y,x) + \delta v_t(x,y,x) = \frac{d\delta u}{dx}\Big|_{\frac{dt}{dx}=1} = (\delta q)_t(x,y),$$

the derivative in the direction $t=x$. Integrate it in parts and get

$$0 = \iiint\left(\psi_{tt} - \psi_{xx} - \psi_{yy} + a(x,y)\cdot\psi\right)\delta u\mathrm{d}x\mathrm{d}y\mathrm{d}t +$$

$$+ \int_0^L\int_0^T\psi(x,y,2T-x)\left[\delta u_t - \delta u_x\right]\mathrm{d}x\mathrm{d}y +$$

$$: + \int_0^L\int_0^T\psi_t(x,y,x)\delta u(x,y,x)\mathrm{d}x\mathrm{d}y +$$

$$+ \int_0^L\int_0^T\delta u(x,y,2T-x)\psi_t(x,y,2T-x)\mathrm{d}x\mathrm{d}y +$$

$$+ \int_0^L\int_0^T\delta u(x,y,x)\psi_t(x,y,x)\mathrm{d}x\mathrm{d}y -$$

$$- \int_0^L\int_0^{2T}\delta u(0,y,t)\psi_x(0,y,t)\mathrm{d}t\mathrm{d}y -$$

$$- \int_0^T\int_x^{2T-x}\left[\left(\psi\delta u_y\right)(x,L_y,x) - \left(\psi\,\delta u_y\right)(x,0,t)\right]\mathrm{d}t\mathrm{d}x.$$

Given that this expression is identically equal to zero, we get the following expressions

$$\psi_{tt} - \psi_{xx} - \psi_{yy} + a(x,y))\psi = 0, \tag{34}$$

$$\psi(x,y,2T-x) = 0, \tag{35}$$

$$\psi(x,L,t) = \psi(x,0,t) = 0, \tag{36}$$

$$0 = \int_0^L\int_0^T\delta u(x,y,x)2\psi_t(x,y,x)\mathrm{d}x\mathrm{d}y -$$
$$- \int_0^L\int_0^{2T}\delta u(0,y,t)\psi_x(0,y,t)\mathrm{d}x\mathrm{d}y. \tag{37}$$

Thus, given the expression (25) from the expression (37), we get the following expression:

$$\psi_x(0,y,t) = 2\left(u(0,y,t) - f(y,t)\right), \tag{38}$$

$$\langle \delta q_n, J'q_n\rangle = \int_0^L\int_0^T 2\psi_t(x,y,x)\cdot\delta q\mathrm{d}x\mathrm{d}y. \tag{39}$$

From there follows the formulation of the conjugate problem.

$$\psi_{tt} = \psi_{xx} + \psi_{yy} - a(x,y)\psi, \tag{40}$$

$$\psi(x,y,2T-x) = 0, \tag{41}$$

$$\psi(0,y,t) = 2\left(u(0,y,t) - f(y,t)\right) = \mu(y,t), \tag{42}$$

$$\psi(x,L,t) = \psi(x,0,t) = 0. \tag{43}$$

Theorem 4. 1. The functional $J(q)$ at the derived point q has a Frechet derivative.

$$J'q_n = 2\psi_t(x,y,x). \tag{44}$$

Proof. By definition, the Frechet derivative of the functional:

$$\psi(q_n + \delta_n) - (_n) = \langle \prime_n, \delta_n\rangle + o(\|\delta\|), \tag{45}$$

from (25)

$$J(q_n + \delta q_n) - J(q_n) =$$

$$= \int_0^{L_y} \int_0^{2T} \delta u(0,y,t;q_n) 2 \big[u(0,y,t;q_n) - f(y,t) \big] dy dt +$$

$$+ o(\|\delta u\|).$$

In view of the estimate (18), we get

$$o(\|\delta u\|) \approx o(\|\delta q\|). \tag{46}$$

Thereby,

$$J'q_n = 2\psi_t(x,y,x), \tag{47}$$

where $\psi(x,y,x)$ is the solution of the conjugate problem (40)–(43). The theorem is proved.

Calculating the gradient of the functional, we can build an algorithm for solving the inverse problem for the acoustics equation.

An algorithm for solving the inverse problem:

1. Choosing the initial approximation q_0.

2. Suppose that q_0 is already known. Then we solve the direct problem (11)–(14).

3. We calculate the value of the functional

$$J(q_n) = \|Aq_n - f\|_{L_2}^2 = \int_0^L \int_0^{2T} \big[u(0,y,t;q_n) - f(y,t) \big] dy dt.$$

4. If the value of the objective functional is not small enough, then we solve the conjugate problem (40)–(43).

5. We calculate the gradient of the functional $J'(q_n) = 2\psi_t(x, y, x)$.

6. Calculate the following approximation $q_{n+1} = q_n - \alpha J'(q_n)$ and turn to step 2.

Generally, the element q_0 is chosen to take into account the a priori information about the desired solution. The uniqueness of the Landweber method is that the implementation of this method is very simple and the iterative process for a given boundary inverse problem converges.

5. 3. A finite difference scheme for solving a direct problem

We will look for the solution of the direct problem (11)–(14) in the form of a Fourier series with respect to the variable y:

$$u(x,y,t) = \sum u^k(x,t) e^{iky}, \quad a(x,y) = \sum a^k(x) e^{iky}.$$

Then we get

$$\sum u_{tt}^k e^{iky} = \sum u_{xx}^k e^{iky} - \sum k^2 u^k e^{iky} - \sum a^n(x) e^{iny} u^k e^{iky}.$$

Let's introduce the notation

$$n + k = k', \quad n = k' - k, \quad u_{tt}^k = u_{xx}^k - k^2 u^k - \sum a_{k'-k} u^k.$$

$$U = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \dots \\ u_N \end{pmatrix},$$

$$A(x) = \begin{pmatrix} a_0 & a_N & a_{N-1} & \dots & a_1 \\ a_1 & a_0 & a_N & \dots & a_2 \\ a_2 & a_1 & a_0 & \dots & a_3 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_N & a_{N-1} & a_{N-2} & \dots & a_0 \end{pmatrix},$$

$$E_N = \begin{pmatrix} 0^2 & 0 & 0 & \dots & 0 \\ 0 & 1^2 & 0 & \dots & 0 \\ 0 & 0 & 2^2 & \dots & 0 \\ \dots\dots\dots\dots\dots\dots\dots \\ 0 & 0 & 0 & \dots & N^2 \end{pmatrix}.$$

Then we get

$$U_{tt} = U_{xx} - (A(x) + E_N) U, \tag{48}$$

$$U_x(0,t) = G(t), \tag{49}$$

$$U(x,x) = Q(x). \tag{50}$$

We replace the derivatives in the equation (38) with finite difference analogs and get

$$\frac{U_i^{k+1} - 2U_i^k + U_i^{k-1}}{h^2} = \frac{U_{i+1}^k - 2U_i^k + U_{i-1}^k}{h^2} -$$

$$- (A_i + E_N) \cdot \frac{U_{i+1}^k + U_{i-1}^k}{2},$$

$$U_i^{k+1} = \left(I_N - \frac{h^2}{2} \cdot (A_i + E_N) \right) \cdot (U_{i+1}^k + U_{i-1}^k) - U_i^{k-1},$$

$$U_i^{k+1} = M_i \cdot (U_{i+1}^k + U_{i-1}^k) - U_i^{k-1},$$

where $M_i = I_N - \dfrac{h^2}{2} \cdot (A_i + E_N)$.

We also write down the boundary condition (49)

$$U_x(0,t) = G(t),$$

$$U(h,kh) = U_1^k = U_0^k + hU_x\big|_{x=0} + \frac{h^2}{2} U_{xx} =$$

$$= U_0^k + hG^k + \frac{h^2}{2}(U_{tt} + A(0) + E_N) \cdot U\big|_{x=0},$$

$$U_0^{k+1} = P_0^{-1} \cdot (U_k^1 - hG^k) - U_0^{k-1},$$

where $P_0^{-1} = I_N + \dfrac{h^2}{2} \cdot (A_0 + E_N)$.

Condition on the characteristic $U_i^i = Q_i$.

For a convenient numerical solution from the "rhombus" pattern, we turn to the "square" pattern.

$$U_j^n = M_j \cdot (U_j^{n-1} + U_{j-1}^n) - U_j^{n-1}, \quad j = \overline{2,N}, \quad n = \overline{1,j-1}. \tag{51}$$

$$U_j^j = P_0^{-1} \cdot (U_j^{j-1} - hG^{j-1}) - U_{j-1}^{n-1}, \quad j = \overline{1,N}. \tag{52}$$

$$U_j^0 = Q^{j-1}, \quad j = \overline{0,N}. \tag{53}$$

An algorithm for solving the direct problem.

1. We calculate by the formula (53) $U_0^0 = Q_0$ and $U_1^i = Q^1$.

2. We calculate by the formula (52)

$$U_1^1 = P_0^{-1}\left(U_0^0 - hG^0\right) - U_0^0.$$

3. And so on along the cycle $j=2,...,N$, we calculate by the formula (53).

4. We calculate by the formula (51) U_n^j.

5. Along the characteristic by formula (52) U_j^j, then go to step 3.

5. 4. Results of the numerical experiment

Let $T=1$, $L=1$, $N=30$. We choose the function

$$\rho(x,y) = e^{\frac{(x-0.5)^2 + y^2}{0.02}},$$

$$g(x,y) = 1 - (x-0.5)^2 + (y-0.5)^2,$$

$$q(x,y) = 0.2 + 0.1 \cdot \cos(4\pi x) \cdot y^2.$$

First, we give a numerical solution of the direct problem $U(x, t)$ for different components k (Fig. 1–3).
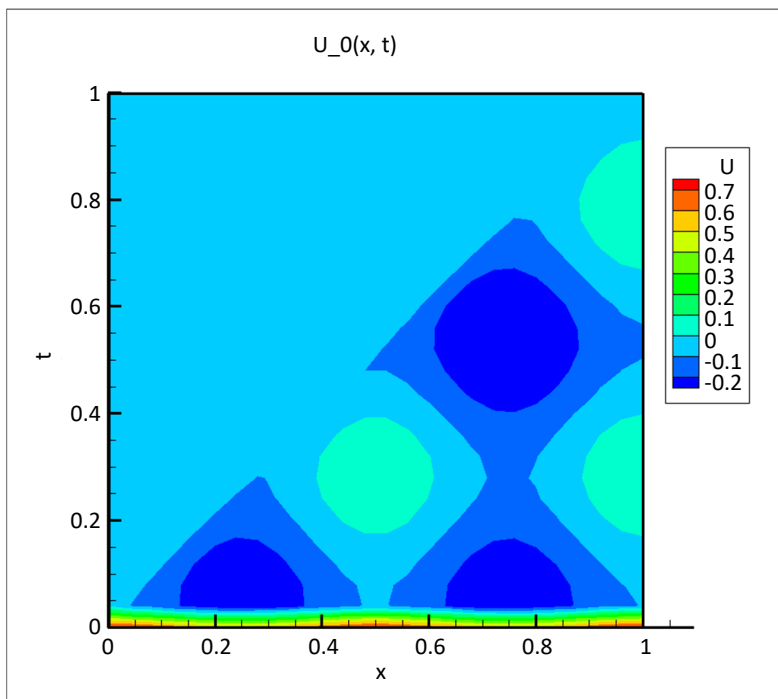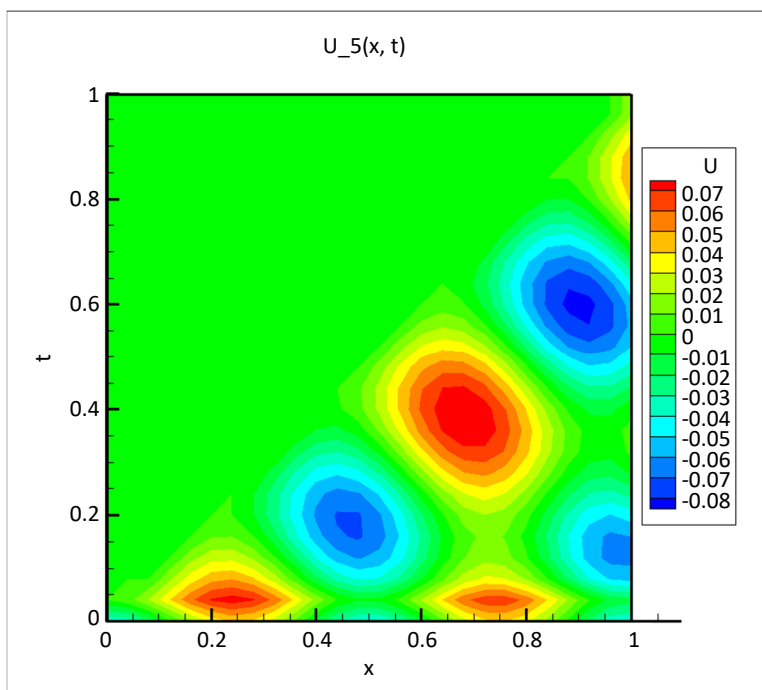

Fig. 1. Graph of the function $U(x,t)$, at $k=0$

Fig. 2. Graph of the function $U(x,t)$, at $k=5$

Fig. 4, 5 show the algorithm of the inverse problem, the Landweber iterative method converges (Fig. 4, 5).

Fig. 6 shows a graph of the difference function of the exact and approximate solution of the inverse problem.

The approaches described in the research can be applied to obtain a numerical solution of wide classes of optimal control problems for describing processes using inverse problems of hyperbolic type.
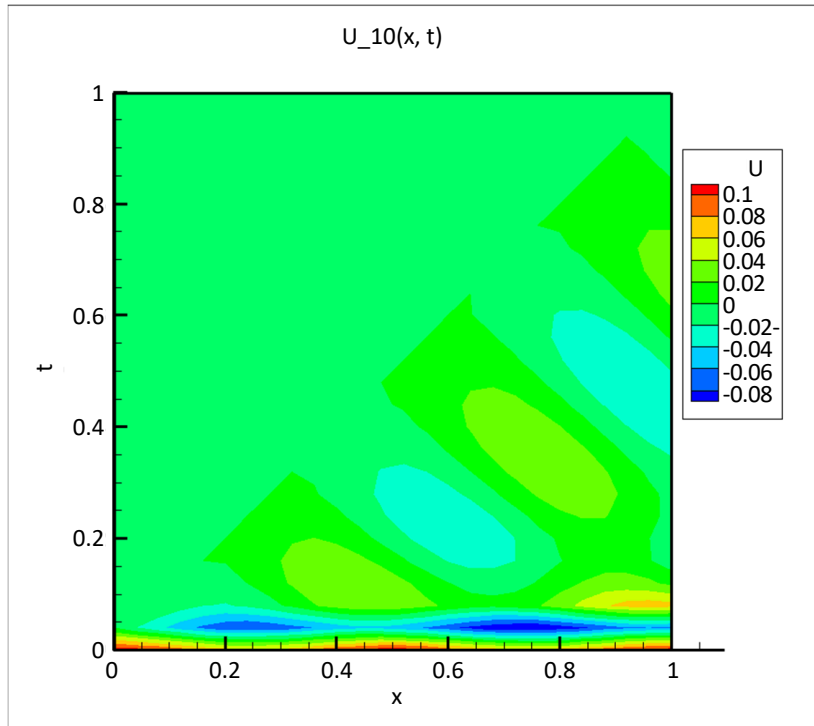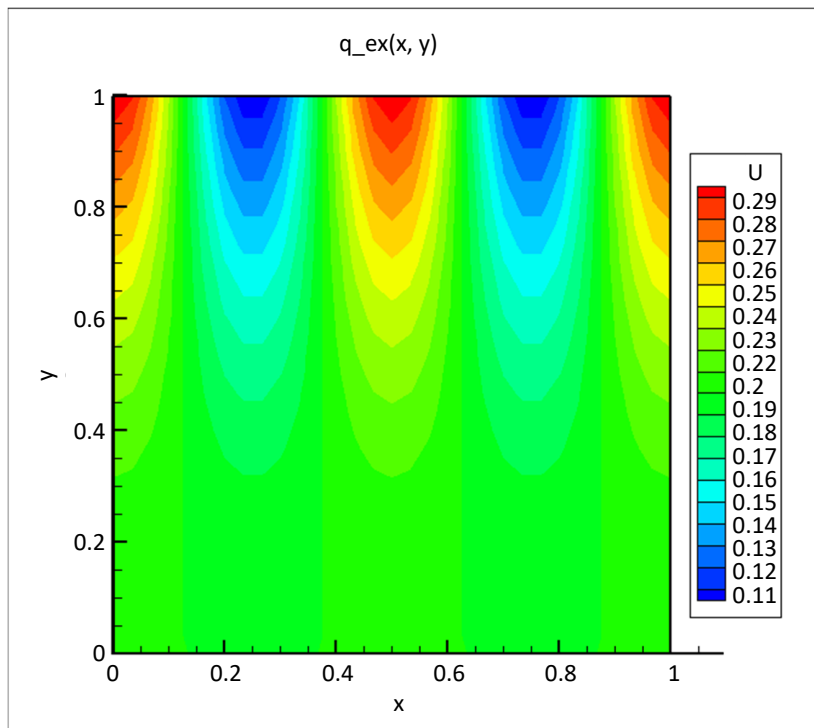

Fig. 3. Graph of the function $U(x,t)$, at $k=10$
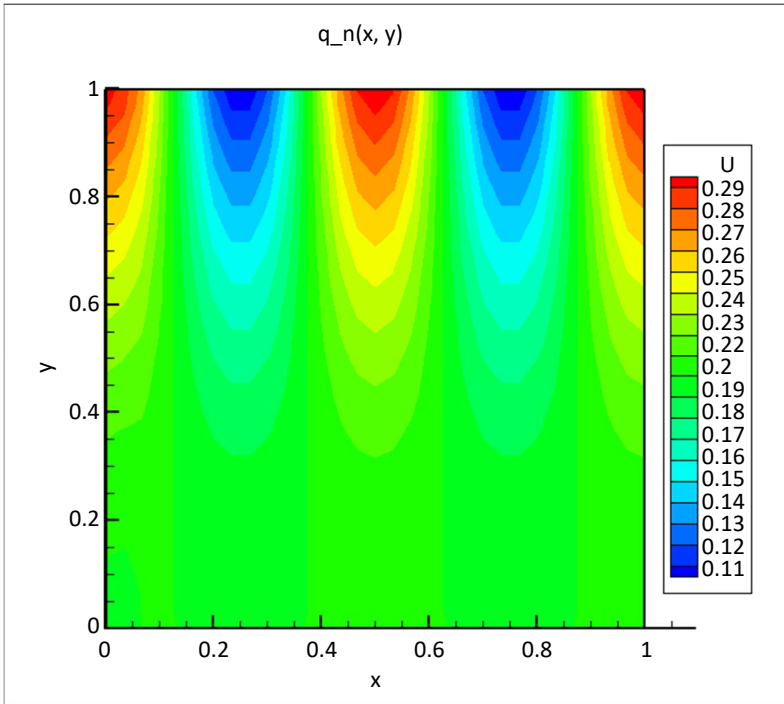

Fig. 4. Graph of the function $q_{ex}(x, y)$

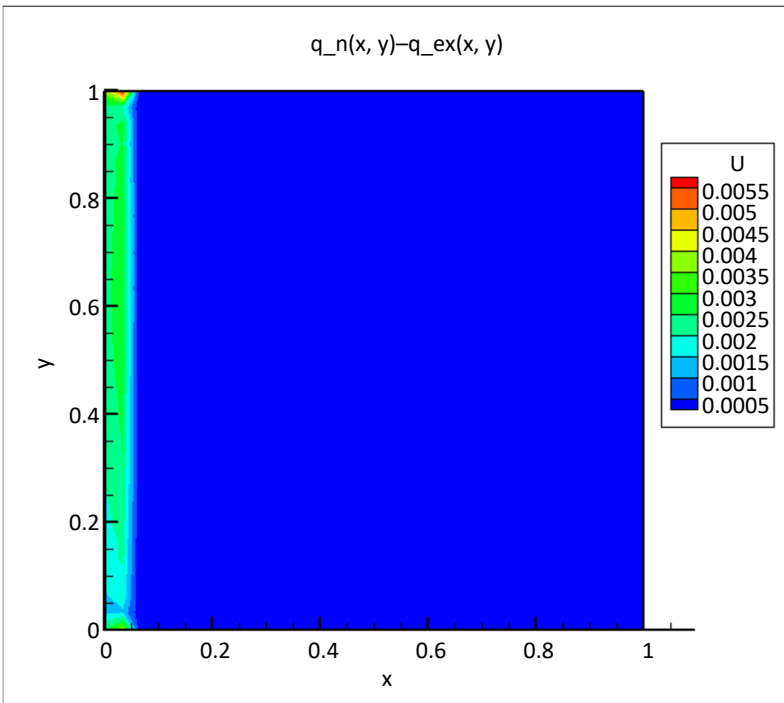Fig. 5. Graph of the function $q_n(x, y)$

Fig. 6. Graph of the function $q_n(x, y) - q_{ex}(x, y)$

6. Discussion of the results of the numerical methods for solving the direct and inverse problem

This paper constructs an algorithm for numerically solving the continuation problem for the acoustic equation with Cauchy data on a time-like surface. The gradient method chosen to solve the inverse problem converges completely, as shown by the numerical results. The following results are also in place.

Because of the instability of the Cauchy problem for the wave equation, the continuation problem is reduced to a boundary inverse problem for the acoustics equation in the time-triangular domain. Formulating the definition of the generalized solution of the direct problem, the stability of the generalized solution in the time-triangular domain is shown. An evaluation of the stability of the generalized solution of the direct problem, which shows the correctness of this problem, is presented.

Many researchers use some form of gradient methods to solve the inverse boundary value problem numerically. In this problem, the complexity lies in the fact that the solution area is in a time-triangular prism. Therefore, the derivative in direction was used to calculate the gradient. The conjugate problem (40)–(43) is formulated, which tends to zero when the functional decreases. And the gradient of the functional is calculated by the derivative in the direction $t=x$ of the solution of the conjugate problem (44). Other gradient methods could be applied to solve the inverse problem, such as the conjugate gradient method, the steepest descent method. The advantage of the Landweber method is that it is very easy to apply.

For the numerical solution of the direct problem, the projection method is used. The two-dimensional problem is reduced to a one-dimensional problem with a triangular area in a matrix-vector form. Then we write a finite-difference approximation of the problem with the same time step and spatial variables. An algorithm for the direct problem is built. It should be noted that such inverse problems also use the method of inverse difference schemes. This method is very convenient in comparison to the gradient method, for a one-dimensional problem.

According to the graph of the function U (Fig. 1–3), we can see that the maximum and minimum values of the function decrease. Therefore, for the numerical solution of the inverse problem after Fourier series decomposition, we take only 10 components. To test the algorithm, we set the exact solution and compute the corresponding additional information f. Then $q_0(x, y)=0.1$ will be the initial approximation; we try to reconstruct the original exact solution using the Landweber iteration with $\alpha=0.01$. The value of the functional decreases. The total number of iterations has a value of $n=8,317$. From this, we can see that the number of iterations of the Landweber method is large. In the future, other gradient methods can be used, such as the conjugate gradient method. Fig. 6 shows that the values of the exact and approximate solutions of the inverse problem are different at the boundary. This is explained as a Gibbs phenomenon. To solve the inverse problem, it was possible to apply other gradient methods, such as the conjugate gradient method, the steepest descent method. The advantage of the Landweber method is

that it is applied very simply. Since the continuation problem is ill-posed, it is reduced to an inverse problem relating to the correct direct problem. A special place is occupied by the solution of the inverse problem by the gradient method. Therefore, the convergence of the gradient method is a very important issue. When using the gradient method, the direct and conjugate problems are solved several times, so it is important to choose the correct, accurate, and efficient method for solving the direct problem numerically. The number of iterations may be greater than the Landweber method compared to other gradients.

7. Conclusions

1. The example of Adamar shows the ill-posedness of the continuation problem for the acoustic equation under consideration. So we reduced the initial problem to an equivalent inverse problem with respect to some direct problem. The theorem with an estimate of the stability of the generalized solution is written for the correctness of the direct problem. It is important to show the correctness of the direct problem, since the area in question is a time-dependent triangle.

2. The inverse problem is written in the form of the operator equation, and the problem of minimization of the objective functional is introduced. The iterative Landweber algorithm is proposed for minimization. The perturbed problem is obtained as a direct problem. Using all the conditions of the perturbed problem and the properties of the derivative in direction, we obtained the gradient of the objective functional and the formulation of the conjugate problem. The Landweber method proposed in this paper with a specially selected step provides a monotonic decrease of the functional values.

3. In the paper, for the numerical solution of the direct and conjugate problem, the projection method is applied, which projects a two-dimensional problem onto a one-dimensional problem. A finite system of one-dimensional inverse problems is obtained. A finite-difference approximation is written for this problem. Since the inverse problem is solved by the gradient method, the projection method is used as an effective method for numerically solving direct and conjugate problems. After that, a special finite-difference scheme was used, which solved the problem in the one-dimensional triangular domain.

4. The number of coefficients of the Fourier series is chosen by the solution of the direct problem. Calculations of the inverse problem show that the functional is minimized and the solution of the inverse problem approaches the solution of the initial problem.

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