In the paper, we apply the vanishing viscosity method for an approximate solution to the Riemann problem. This approach gives the effects of the accuracy of the solution and the speed of convergence by discrediting time and spatial variables.

The obtained method ensures the smoothness of the solution without taking into account the capillary pressure. The results confirm the negligible influence of cross-link conditions compared to the classical Darcy approach.

The proposed solutions of the new approach are intended to improve the methods and schemes of discretization both in space and in time. This is achieved by minimizing viscosity, and discretization in space and time. These factors are of paramount importance for studying phenomena with variable saturation in the transient mode and analyzing water/oil flows and migrations in real time, since discretization in space and time affects the accuracy and convergence of calculations. Our result in the form of obtaining viscous solutions of the filtration process is interesting from a theoretical point of view. From a practical point of view, numerical modeling allows early prediction of performance. Thus, the applied aspect of using the obtained scientific result is the possibility of improving the process by taking into account the influence of phases of fluid flows

Keywords: pressure-dependent viscosity, Buckley-Leverett model, Riemann problem

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CONSTRUCTION OF APPROXIMATE SOLUTIONS TO THE RIEMANN PROBLEM FOR TWO-PHASE FLOW OF IMMISCIBLE LIQUIDS BY MODIFYING THE VANISHING VISCOSITY METHOD

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1. Introduction

Modeling of multiphase flows in porous media is of great importance in many applications, such as groundwater storage, transportation in a protective shell, and, in particular, improving oil recovery in the oil industry. Modeling the dynamics of non-aqueous immiscible fluid, and/or water into an aquifer medium, immiscible fluid migration in the unsaturated soil/porous medium to the saturated zone is challenging.

Improvement of modeling and schemes of two-phase flow is achieved by minimizing viscosity, and discreteness in space and time. These factors are of paramount importance for studying phenomena with variable saturation in the transient mode and analyzing water/oil flows and migrations in real time, since discretization in space and time affects the accuracy and convergence of calculations. With the increase in computing speed and performance of modern multicore computers, the progress of GPUs has accelerated computing, and new powerful programming tools (mostly open source) have emerged that can improve big data modeling and management tools. This is an excellent and unique moment and an opportunity to simulate in detail the fate of pollutants for a flow with variable saturation and a flow jump, as in the Riemann problem.

Based on the above, the topic of studying the dynamics of immiscible liquids is very relevant from the point of view of the applied aspect.

2. Literature review and problem statement

The paper [1] describes the mechanism of displacement and the advantages of water over gas as a displacing agent and certain conclusions are drawn relative to the changing character of the displacement as depletion and on the effects of the properties of the fluids and of producing conditions on the ultimate oil recovery. The mathematical equations needed are derived by applying Darcy's law to the flowing phases, and by material balance considerations. In both the Buckley and Leverett method and the method discussed in [2], a linear sand section is assumed. Thus, the exploitation contemplates oil displacement as an immiscible phase and the method described in this paper can be applied equally well to the evaluation of oil recovery by linear water flooding or water drive, in [3] an alternative method via the characteristic technique to solve the Buckley and Leverett equation is used.

Solutions are derived from the horizontal, steady flow of two viscous incompressible fluids in the work [4]. One-dimensional unidirectional displacement of a nonwetting fluid is shown to occur increasingly like a shock front as pore size distribution widens; and the radial displacement of a nonwetting phase normally resident at low concentration is shown to be an inefficient process.

The classical mathematical model of multiphase flows was proposed in [5], where Darcy's law was generalized for single-phase flow.

To read scientific studies on the relationships between phases in multiphase flow modeling, refer to [6] for analysis and links to these papers. In particular, in the paper, the waterfront position is clearly shown to be significantly different from one model to the other. The authors proposed a method that allows taking into account the relationship between phases using any classical software capable of solving the classical Darcy system. Obviously, several other aspects need to be investigated such as the injection rate, the effect of gravity, etc.

Fluid flow through a porous medium is common in many areas of technology and science. At the same time, the problem of single-phase flow has been well studied both from an engineering and mathematical point of view [7]. The classical Darcy's law, widely used for practical purposes, can be obtained by modeling a sluggishly current incompressible flow. In practice, a porous medium is considered a periodic array of cells filled with Newtonian fluid. The problem is formulated at the cell scale (microscale) and then scaled by homogenization in the entire area, providing the classical Darcy's law.

According to Darcy's equation, a porous solid has a resistance to the liquid in the pores, which is directly proportional to the speed of the liquid relative to the solid, usually called the drag coefficient.

Oil production in most cases occurs when it is displaced in the pore space of the productive reservoir by water or gas. This process is used in natural operating modes and in artificial methods of maintaining reservoir pressure by flooding or gas injection. The theory of isothermal filtration serves as the basis for calculating such processes. In [8], the history of the development of the modern theory of shock waves is considered. Several attempts at an early theory quickly collapsed for a lack of foundations in mathematics and thermodynamics.

Simulation can be without taking into account nonlinear effects [9], assuming that the flow of two immiscible fluids is separated by a smooth boundary layer. The aim of the work [10] was to analyze the flow of a mixture of two immiscible liquids whose viscosity depends on pressure, generalizing the classical Buckley-Leverett model, a typical example being crude oils. As a result, they obtain a system of equations for saturation and pressure, which is reduced to the classical Buckley-Leverett equation when both viscosities are independent of pressure. Below in the paper, we modify the model with a transition to the Riemann problem. The work [11] presents a high-resolution numerical model that simulates three-phase immiscible fluid flow in both unsaturated and saturated zone in a porous aquifer. Some models consider the exchange of momentum between the phases of flows of two immiscible fluids in a porous medium. Creeping flow models are sometimes used that include an explicit relationship between both phases by adding cross-terms to the generalized Darcy's law. The results [12] indicate that the effects of momentum exchange on two-phase flow may increase with the permeability of the porous medium when the influence of the fluid-fluid interfaces becomes similar to that of the solid-fluid interfaces.

Several fundamental laws of physics take the form of a conservation equation.

The lack of regularity is a major source of complexity since most of the standard differential calculus tools are not applicable. Special methods are needed, in particular, the main building block is the so-called Riemann problem, in which the initial data are piecewise constant with one jump at the origin.

Definitions of the standard Riemann semigroup and viscosity solutions for a nonlinear hyperbolic system of conservation laws are given in [13]. These definitions were motivated by a natural hypothesis. Namely, viscosity solutions (characterized in terms of local integral estimates) must exactly coincide with the limits of approximations of vanishing viscosity. In the paper, we adopt a similar definition of viscosity solutions and prove that the above conjecture is indeed true. Our results apply to the more general case of (possibly nonconservative) quasilinear strictly hyperbolic systems. In particular, we obtain the uniqueness of the vanishing viscosity limit.

For a comprehensive account of the recent uniqueness and stability theory, we refer to [1, 8].

The mathematical model in [14] consists of the usual equations derived from the mass conservation of both fluids along with the Darcy-Muskat and the capillary pressure laws. The major difficulties related to this model are in the nonlinear degenerate structure of the equations, as well as in the coupling in the system.

Many numerical experiments assume convergence in the local limit. However, recent analytical results state that:

a) in the general case, convergence does not take place, since counterexamples can be given;

b) convergence can be restored by adding viscosity to both local and non-local equations.

Guided by the analytical results, the role of numerical viscosity in the numerical study of the local limit of nonlocal conservation laws is very important. In particular, [15] shows that the numerical viscosity of Lax-Friedrichs-type schemes jeopardizes the reliability of the numerical scheme and erroneously determines convergence in cases where convergence is excluded by analytical results.

The Buckley-Leverett model (1942) is the basis of many modern studies. By applying the Buckley-Leverett displacement mechanism, a mathematical model was developed to predict the efficiency of flooding in stratified reservoirs [16]; a model is developed for the prediction of waterflooding performance in stratified reservoirs using the Buckley-Leverett displacement theory [17] and a theoretical relationship is deduced to describe dynamic changes of the front of water injection, water saturation of producing well [18].

The model from [19] is a neural network that is cooperatively trained to match any available experimental data and obey the governing physical laws. This approach is being used as a new way to model and correlate with the history of flow and transport problems in porous media. The methodology is used to model the 2-phase immiscible transport problem (Buckley-Leverett). An additional benefit of the approach is that it is highly scalable and can leverage different computing architectures, including CPUs, GPUs, and distributed clusters. All of the above works emphasize the relevance of studying the modern theory of Buckley and Leverett.

A high-resolution numerical model simulating a three-phase immiscible fluid flow in both unsaturated and saturated zones in a porous aquifer was proposed in [11]. There, the main approach to the numerical solution of the problem is based on the (complete) explicit evolution of discretized (in space) variables.

Numerical modeling of the flow of immiscible fluids is of great importance in many areas for the proper management of underground resources, in particular water. Very relevant is the recently presented high-resolution numerical model that simulates a three-phase immiscible fluid flow in both unsaturated and saturated zones in a porous aquifer [20].

In the theory of immiscible two-phase flow presented in [21], the conservation of mass is provided by general equations, which require some additions for a porous medium. The basic equation can be derived from the relative permeability data. It turns out to have a surprisingly simple form when expressed in the correct variables [22]. The resulting system of equations can then be solved for a structured porous medium. However, the question remains what happens when the porous medium has a nontrivial structure along its entire length.

Recent work [23] is devoted to the study of the Cauchy problem for a system of differential equations describing the unsteady flow of a compressible fluid in a homogeneous and inhomogeneous porous medium with a general nonlinear filtration law in a three-dimensional space. In [23–25], using the methods of four-dimensional mathematics, a special four-dimensional space of space-time coordinates was developed, as well as a functional space of regular functions, and analytical conditions were obtained in the general form of the law of nonlinear filtration for which the Cauchy problem has a solution.

We now summarize the review and provide a brief analysis to deduce the purpose of our work. The study of the general theory of two-phase immiscible flow in heterogeneous porous media is a complex task with applications to petroleum engineering and hydrogeology. As already noted in the works cited above, many methods have been proposed for various aspects of this problem. Although the physical mechanisms of the development of large-scale regularities in such flows have been studied only partially. The circle of ideas has been applied in contexts, usually where solutions of high quality are of critical importance and where conventional difference methods, which excel for smooth solutions, perform poorly. When two immiscible fluids compete for the same pore space, we are dealing with the immiscible two-phase flow in porous media. The main problem in studying porous media is to find the correct description of an immiscible two-phase flow at the continuum level, that is, at scales where a porous medium can be considered as a continuum. It requires the solution of the Riemann problem for the hyperbolic system in order to advance the tracked discontinuities in the solution. In two space dimensions, the propagation of a tracked discontinuity is achieved by local splitting of the hyperbolic operator in normal and tangential directions. As the solution is smooth in the tangential direction on each side of a discontinuity, it suffices to look at the RP solution in one space dimension, corresponding to an analysis of the propagation of the discontinuity in the normal direction. Equations in one spatial dimension can be written as the familiar Buckley-Leverett equation. Research at the pore level of an immiscible two-phase flow is developing at a very high rate due to advances in experimental methods combined with an explosive increase in computing power. For the case of a nonlinear fractional flow function $f(s)\in C^2$ having at most a finite number of inflection points (as in immiscible flow), the Riemann problem solution is described in terms of a single family of waves consisting of both rarefaction waves (across which s varies smoothly) and shock waves (across which s varies discontinuously). By adding a small viscosity to the original system, the correct parabolic equation can be obtained. The problem is that it is necessary to prove the uniqueness of the limit of vanishing viscosity when the viscous term parabolic equation tends to zero, which we will deal with below.

3. The aim and objectives of the study

The aim of this study is to identify regularities of the flow of a mixture of two immiscible fluids whose viscosity depends on pressure, generalizing the classical Buckley-Leverett model. Such extension is mainly motivated by the possibility to obtain the convergence of the viscous approximation, and the uniqueness of the solution of the original Buckley-Leverett equation. Therefore, the scientific component of the aim is to develop an alternative approach to the uniqueness and stability of solutions with vanishing viscosity, where $v=v(x)$ is piecewise smooth with a finite number of jumps. In light of the results of experimental observations of the flow of mixtures of oil and/or water through sands, certain conclusions are drawn:

– relative to the changing character of the displacement as depletion proceeds and on the effects of the properties of the fluids;

– and of producing conditions on the ultimate oil recovery.

To achieve the aim, the following objectives are accomplished:

– to introduce an auxiliary boundary value problem called the Riemann problem, where the initial data are piecewise constant with a single jump at the origin;

– to apply the method of vanishing viscosity to solve the Riemann problem, obtain an approximate solution to the general Cauchy problem;

– to present a numerical solution of the problem under consideration to confirm or refute the convergence of the found solution to a function well approximated by the solution of the corresponding Riemann problem.

4. Materials and methods

Consider filtration of a two-phase liquid in a porous medium in water-pressure mode. The field is covered by a network of wells and their location schemes can be different. The oil-bearing formation is considered unlimited, of constant thickness, the porous medium is non-deformable, and the ratio of capillary pressure to the total hydrodynamic pressure drop is small, which allows considering the problem to obey the classical Buckley-Leverett model.

High-precision modeling of immiscible two-phase flows in porous media is very important. But even with such high-precision numerical modeling, the lack or fuzziness of information, for example, about: the relative permeability and functions of the capillary pressure in them [26]; single-phase turbulent flow [27]; two-phase pressure drops for homogeneous separated flow [28], does not allow a detailed comparison with experiments.

This is the mass conservation equation for two phases (oil and water):

$$\frac{\partial \rho_o S_o \phi}{\partial t} + \nabla \cdot (\rho_o \nu_o) = 0,$$

$$\frac{\partial \rho_\varpi S_\varpi \phi}{\partial t} + \nabla \cdot (\rho_\varpi \nu_\varpi) = 0, \tag{1}$$

with the natural physical constraint $S_0+S_w=1$, where ϕ – the effective porosity of the reservoir; ρ_0, S_0 and ρ_ϖ, S_ϖ – the density and saturations of oil and water, respectively; v_0, p_0 and v_ϖ, p_ϖ – the superficial velocity and the pressure of the oil and the water phases, respectively.

The Cauchy problem for a system of conservation laws in one space dimension takes the form:

$$u_t + f(u)_x = 0, \tag{2}$$

$$u(0,x) = \tilde{u}(x), \tag{3}$$

here $u=(u_0, u_\varpi)$ is the vector of conserved quantities (oil and water, respectively), while the components of $f=(f_0, f_\varpi)$ are the fluxes of oil and water, respectively. We assume that the flux function f: $\boldsymbol{R}^2 \to \boldsymbol{R}^2$ is smooth and that the system is strictly hyperbolic; i.e., at each point u, the Jacobian matrix $A=Df(u)$ has n real, distinct eigenvalues $\lambda_1(u)<...<\lambda_n(u)$. For t, u, f:

a) the time variable $t\in R^+$, the space variable is one dimensional and for the time being, we let x vary on the whole R, so $x\in R$;

b) the unknown is u: $R^+\times \boldsymbol{R}^N$;

c) the flux is f: $\boldsymbol{R}^N \to \boldsymbol{R}^N$ is smooth (as smooth as needed, as a matter of fact, $f\in C^2$ is usually enough).

One can then select bases of right t and left eigenvectors $r_i(u)$, $li(u)$ normalized so that:

$$|r_i| \equiv 1,\ l_i \cdot r_i = \begin{cases} 1 \text{ if } i=j, \\ 0 \text{ if } i\neq j. \end{cases}$$

The Riemann problem, in which the initial data is piecewise constant with one jump at the origin, is written in the following form:

$$u(0,x) = \begin{cases} u^- \text{ if } x<0, \\ u^+ \text{ if } x>0. \end{cases} \tag{4}$$

The viscosity solution of a Cauchy problem is unique and coincides with the limit of Glimm and front-tracking approximations for a strictly hyperbolic system of conservation laws satisfy the standard assumptions.

For each $i\in\{1, ..., n\}$, the i-th characteristic field is either linearly degenerate, so that:

$$D\lambda_i(u)\cdot r_i(u) = 0,$$

for all u, or else it is *genuinely nonlinear*; i.e.,

$$D\lambda_i(u)\cdot r_i(u) > 0,$$

for all u.

For the time being we focus on the Cauchy problem obtained by coupling 1 (2) with the datum (3), where $\tilde{u}(x)$ is a given function $\tilde{u}(x): R \to \mathbf{R}^N$.

Lemma 1 (Rankine-Hugoniot conditions). Fix $\lambda\in R$, u^-, u^+ in R^N. The function:

$$\nu(t,x) = \begin{cases} \nu^- \text{ if } x<\lambda t, \\ \nu^+ \text{ if } x>\lambda t, \end{cases} \tag{5}$$

is a weak solution of (2) if and only if the so-called Rankine-Hugoniot conditions hold, i.e. [3–6].

$$\mathbf{f}(\mathbf{u}^+ - \mathbf{u}^-) = \lambda(\mathbf{u}^+ - \mathbf{u}^-). \tag{6}$$

Definition. Assume that system (1) has at least one entropy-entropy flux pair (η, q) with η convex. A locally bounded function $u(x)$: $R^+\times R \to R^N$ is an entropy admissible solution of (2) if it is a weak solution of (1) and the inequality:

$$\eta(\mathbf{u})_t + q(\mathbf{u})_x \leq 0, \tag{7}$$

is satisfied in the sense of distributions for every entropy-entropy flux pair (η, q) with η convex. In other words,

$$\int_0^\infty \int_R \eta(\mathbf{u})_{\varphi(t)} + q(\mathbf{u})_{\varphi(x)} \geq 0, \tag{8}$$

for every $\varphi \in C_c^\infty((0,\infty)\times R)$, such that $\varphi(t, x)\geq 0$ every (t, x).

Lemma 2. Fix $\lambda\in R$, u^-, u^+ in $\boldsymbol{R}^N$ and let $\boldsymbol{v}$ be the function defined as in (5). Then the entropy inequality (8) is satisfied for every test function $\varphi\geq 0$ if and only if:

$$\lambda\left(\eta(\mathbf{u}^+ - \mathbf{u}^-)\right)_t \geq q(\mathbf{u}^+) - q(\mathbf{u}^-). \tag{9}$$

The so-called Riemann problem is a particular type of Cauchy problem, which is obtained by coupling the system of conservation laws (2) with an initial datum in the form (5), where u^- and u^+ are given states in $\boldsymbol{R}^N$. Note that the Cauchy problem (2), (3) has a solution in the form (5), provided $\lambda\in R$ satisfies the Rankine-Hugoniot conditions (5). Also, we recall *Lemma 2*, which says that $\boldsymbol{v}$ in (5) is entropy admissible if and only if (8) is satisfied for every entropy-entropy flux pair (η, q) with η convex.

A long-standing conjecture is that the entropic solutions of the hyperbolic system (2) coincide with the limits of solutions to the parabolic system:

$$u_t + f(u)_x = \varepsilon u_{xx}, \tag{10}$$

when the viscosity coefficient $\varepsilon \to 0$ because of the recent unique results, it looks indeed very plausible that the vanishing viscosity limit should single out the unique «good» solution of the Cauchy problem, satisfying the appropriate entropy conditions. In earlier literature, results in this direction were based on three main techniques: *Comparison principles for parabolic equations; Singular perturbations; Compensated compactness.*

In our point of view, to develop a satisfactory theory of vanishing viscosity limits, the heart of the matter is to establish *a priori* BV bounds on solutions $u(t, x)$ of (9) ε, uniformly valid for all $t\in[0, \infty]$ and $\varepsilon>0$. This is indeed what we will accomplish in the present paper. Our results apply, more generally, to strictly hyperbolic 2×2 systems with viscosity, not necessarily in conservation form:

$$u_t + A(u)u_x = \varepsilon u_{xx}.$$

The modeling of multiphase flows in porous media is of major importance in many fields of applications. Particularly in enhanced oil recovery applications of petroleum engineering. The classical mathematical models for multiphase flows are based on a straightforward generalization of Darcy's law for a single-phase flow [9]. A natural question arises about the importance of the influence of one phase on another phase. In some applications, it is shown that the coupling effects are small and therefore they can be neglected.

We show below the construction of the solution of the weak admissible solution of the Riemann problem (1)–(3).

Remark. In the case of one-dimensional flow of incompressible immiscible liquids under conditions where capillary pressure and the influence of gravity can be ignored, the displacement process allows a simple mathematical description.

It is known that if $\varepsilon>0$ is the coefficient of viscosity, then the viscous friction force acting on each particle of the porous medium $x(t)$ and related to the unit of mass can be assumed to be equal to $\varepsilon\cdot u_{xx}$. Then returning to the mathematical model of Buckley-Leverett (then instead of $u(t, x)$, we will write in $s(t, x)$ – water saturation):

$$s_t+s\cdot s_x=\varepsilon\cdot s_{xx}, \tag{11}$$

where $F'(s)=1/2s$ is the Leverett function.

The assumed method at $\varepsilon\to 0$ is called the «vanishing viscosity» method. Given that:

$$s_t=\left(\varepsilon\cdot s_x-\frac{s^2}{2}\right)_x,$$

we introduce the potential $u(x, t)$ defined by the equality (11):

$$du+\left(\varepsilon\cdot s_x-\frac{s^2}{2}\right)_x dt,$$

In this case,

$$u_x=s,$$

$$u_t=\varepsilon\cdot s_x-\frac{s^2}{2}=\varepsilon\cdot u_{xx}-\frac{u^2 x}{2},$$

that is, the function $u(x, t)$ satisfies the equation:

$$u_t+\frac{1}{2}u_x^2=\varepsilon\cdot u_{xx}. \tag{12}$$

Make a replacement in (12):

$$u=-2\varepsilon\cdot\ln z.$$

Then:

$$u_t=-2\varepsilon\cdot\frac{z_t}{z},$$

$$u_x=-2\varepsilon\cdot\frac{z_x}{z},$$

$$u_{xx}=-2\varepsilon\cdot\frac{z_{xx}}{z}+2\varepsilon\cdot\frac{z_x^2}{z^2}.$$

Equation (11) will take the form:

$$-2\varepsilon\cdot\frac{z_t}{z}+2\varepsilon^2\cdot\frac{z_x^2}{z^2}=-2\varepsilon^2\cdot\frac{z_{xx}}{z}+2\varepsilon^2\cdot\frac{z_x^2}{z^2},$$

in other words, the thermal conductivity equation is obtained regarding $z(x, t)$:

$$z_t=\varepsilon\cdot z_{xx}. \tag{13}$$

This method is often called the Florin-Hopf-Cole transformation. From the substitutions made, it follows that the solution of equation (10) has the form:

$$s=u_x=-2\varepsilon\cdot\frac{z_x}{z},$$

where $z(x, t)$ is the solution (13).

Suppose that a wave of the form propagates through an injection well:

$$s(x,t)=s_-+\frac{s_+-s_-}{2}\cdot\left(1+\operatorname{sign}(x-\omega t)\right)=\begin{cases}s_-, ats<\omega t\\ s_+, ats>\omega t\end{cases}, \tag{14}$$

where ω=const. Suppose that there is a generalized solution to the equation of the form (2) in the sense of fulfilling the integral identity. To do this, it is necessary and sufficient that the condition is met on the break line ω=const:

$$\omega=\frac{dx}{dt}=\frac{F(s_+)-F(s_-)}{s_+-s_-}. \tag{15}$$

The idea of the «vanishing viscosity» method, in this case, is that this solution (discontinuous) of the form (14) is acceptable. That is, for $x\neq\omega$ solutions of $s^\varepsilon(x, t)$ of the equation:

$$s_t^\varepsilon=+\left(F(s^\varepsilon)\right)_x=\varepsilon\cdot s_{xx}^\varepsilon, \tag{16}$$

for $\varepsilon\to 0$, it is obtained as a pointwise limit.

Given the structure of the solution $s(x, t)$, we will look for a solution to equation (16) in the form:

$$s^\varepsilon(x,t)=u(\xi),\xi=\frac{x-\omega t}{\varepsilon}. \tag{17}$$

Substituting a solution of this type in (16), we get that the function $u(\xi)$ is the solution of the equation:

$$-\omega\cdot\upsilon'+\left(F(\upsilon)\right)'=\upsilon''. \tag{18}$$

At $x\neq\omega t$, the function $s^\varepsilon=\upsilon\left(\frac{x-\omega t}{\varepsilon}\right)$ pointwise approximates for $\varepsilon\to 0$ function $s=(x, t)$ of the form (6) if and only if the function $\upsilon(\xi)$ satisfies the boundary conditions:

$$s(-n,t)=s_-,s(n,t)=s_+, \tag{19}$$

where n is a sufficiently large distance from the well.

It should be noted that $\upsilon(t)$ is not the only solution, i.e. there can be $\tilde{\upsilon}=\upsilon(\xi-\xi_0)$, for any $\xi_0\in R$.

Integrating (18), we get:

$$\upsilon'=-\omega\cdot\upsilon+\Phi(\upsilon)+C=\tilde{\Phi}(\upsilon)+C,\ C=\text{const}. \tag{20}$$

If these conditions are met, the solutions of equation (18) that interest us are given by the formula.

Following the method of Gelfand, for an autonomous equation (20) with a smooth right part $\tilde{\Phi}(\upsilon)+C$ to have a solution that tends to the constants s_- at $n\to-\infty$ and s_+ at $n\to+\infty$, it is necessary and sufficient to meet the following conditions:

a) s_- and s_+ – special points of the original equation, i.e., zero on the right side of the equation (20):

$$\Phi(\upsilon)+C=\tilde{\Phi}(\upsilon)+C=0,$$

that is, as a result, we have:

$$\tilde{\Phi}(s_-)=\tilde{\Phi}(s_+)=-C;$$

b) another option between s_- and s_+, there are no other special points and the right part (20) on the specified interval:

1) positive at $s_-<s_+$ the solution increases, i.e.

$$\tilde{\Phi}(\upsilon)-\tilde{\Phi}(s_-)>0,\ \forall\upsilon\in(s_-,s_+). \tag{21}$$

2) negative at $s_->s_+$, i.e. the solution decreases:

$$\tilde{\Phi}(\upsilon)-\tilde{\Phi}(s_+)<0,\ \forall\upsilon\in(s_+,s_-). \tag{22}$$

If these conditions are met, the solutions of equation (10) that interest us are given by the formula:

$$\int\limits_{\upsilon_0}^{\upsilon}\frac{d\upsilon}{\tilde{\Phi}(\upsilon)-\tilde{\Phi}(s_-)}=\xi-\xi_0,$$

where $\upsilon_0=\dfrac{s_++s_-}{2}$ – the location of wells.

The given conditions (21), (22) are an analytical record of the tolerance condition. By varying s_-, s_+ and $F(s)$, various converging sequences of valid generalized solutions can be constructed. At the same time, any point-to-point limits of acceptable solutions are also considered acceptable.

5. Results of the study of the accuracy of the coincidence of the viscous solution with the limits of approximations of vanishing viscosity

5. 1. Introduce an auxiliary boundary value problem called the Riemann problem, where the initial data are piecewise constant with a single jump at the origin

We introduced the auxiliary Riemann problem as follows: the Riemann problem (4) is the initial value problem when the initial data consists of two constant states u^- and u^+ separated by a jump discontinuity at $x=0$ (we use lower 10 cases u for the unknown because it is a scalar). That is, the initial value problem (2) and (3), where (3) is implied as (4):

$$u(0,x)=\begin{cases}u^- & \text{if } x<0,\\ u^+ & \text{if } x>0.\end{cases}$$

Weak solutions to the Cauchy problem (2) and (3) were constructed in the famous work of Glimm. This global existence result is true for small initial data and under the additional assumption from section 4 above.

5. 2. Applying a modification of the vanishing viscosity method to the solution of the Riemann problem constructed in the previous step above, we obtain an approximate solution to the general Cauchy problem

A basic step is thus the analysis of the vanishing viscosity solution to a general Riemann problem. The construction given here extends the previous results to general, nonconservative hyperbolic systems. As in the cases considered in [26] for a given left state s^-, there exists a Lipschitz continuous curve of right states s^+, which can be connected to u^- by i-waves. These right states are here obtained by looking at the fixed point of a suitable contractive transformation. Remarkably, our center manifold plays again a key role in defining this transformation. Our main results at this stage are presented as follows.

As a result, we get that the solution $s(x, t)$ can jump from s_- to s_+ (in the direction of increasing x). That is, this jump occurs during the transition from the water phase to the oil phase. In this case, the conditions for an acceptable gap are met (Fig. 1):

a) for $s_-<s_+$, the graph of the function $F(s)$ on the segment $[s_-, s_+]$ must be located below the chord with the ends $(s_-, F(s))$ and $(s_+, F(s_+))$;

b) in the case of $s_->s_+$, the graph of the function $F(s)$ on the segment $[s_+, s_-]$ must be located no higher than the chord with the ends $(s_-, F(s))$ and $(s_+, F(s_+))$.

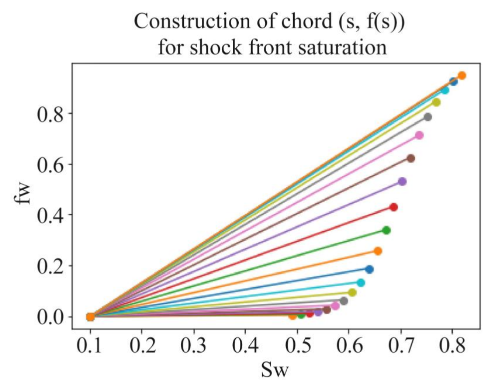


Fig. 1. Construction of chord (s, $F(s)$) for front saturation

The obtained conditions make it possible to regulate filtration processes in the bottom hole formation zone taking into account the initial information, in particular, some data from Table 1.

Table 1

Initial data used in modeling with the one-dimensional Buckley-Leverett problem

Parameter	Value
Porosity	0.28
Oil viscosity	1.e-4 kg/ms
Water viscosity	0.5e-4 kg/ms
Oil density	881 kg/m^3
Water density	1,000 kg/m^3
Water relative perm calculation for a given water saturation	11.174
Oil relative perm calculation for a given water saturation	3.326
Cross-sectional area	0.4 m^2

The position of the shock as it progresses at different time steps is shown in Fig. 2.

Fig. 3 shows a difference in the water saturation front S_{wf}. Therefore, one can see a difference in the water saturation front position. Observe that, in this case, the cross-terms are not equal.

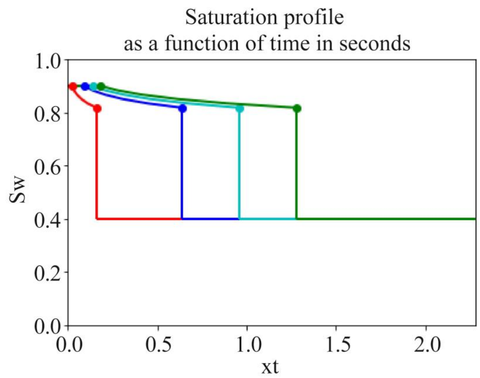


Fig. 2. Water saturation profile as a function of time «t» and distance «x»

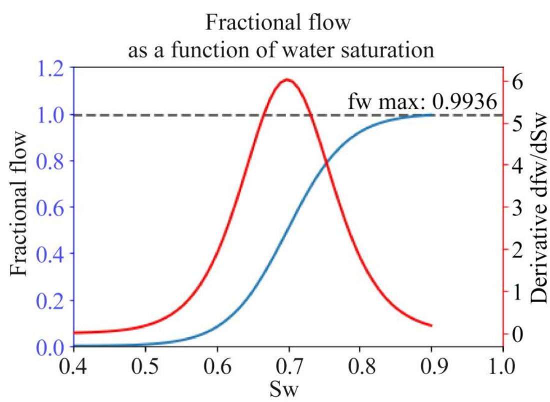


Fig. 3. Fractional flow curve in terms of water saturation S_w: red – derivative df_w/dS_w; blue – fractional flow f_w

The gap tolerance conditions obtained by the «vanishing viscosity» method are in perfect agreement with the forecast calculations. Indeed, the convexity property of the function $F(s)$ in the Buckley-Leverett mathematical model (up) down by definition means that any chord connecting points in a straight line shows the validity of the Buckley-Leverett mathematical model itself.

5. 3. Present a numerical solution of the problem under consideration to confirm or refute the convergence of the found solution to a function well approximated by the solution of the corresponding Riemann problem

It is rather well-known that the difficulty resides in the fact that the typical water fractional flow curve has an inflection point, which provides two values of water saturation at the same time since df_w/dS_w will reach a maximum value (Fig. 4) for typical examples with capillary pressure.

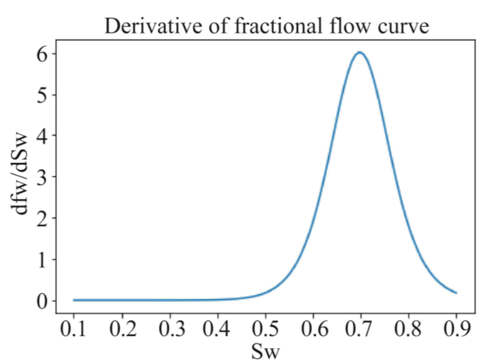


Fig. 4. Derivative of fractional flow df_w/dS_w

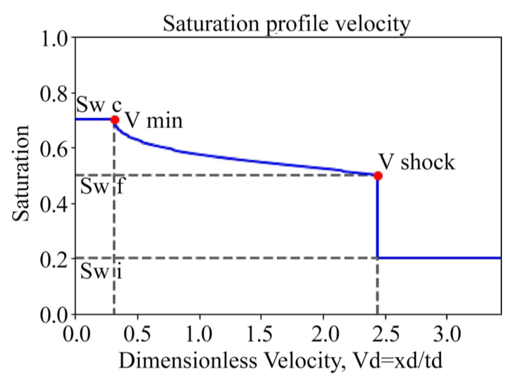
The numerical experiments suggest convergence in the local limit and convergence can be recovered provided viscosity is added to both the local and the nonlocal equations. Motivated by these analytic results, we investigate the role of numerical viscosity in the numerical study of the local limit of nonlocal conservation laws (Table 2).

Table 2

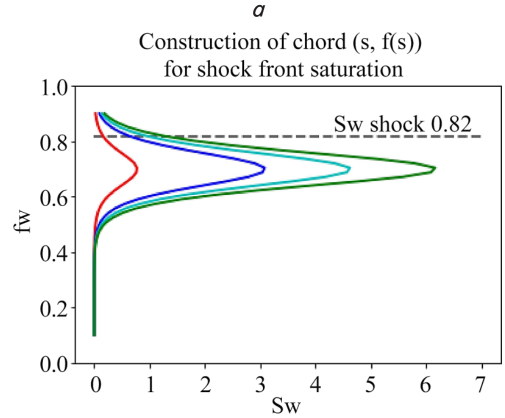
Results of calculations of experimental data

S_w	df_s/dS_w
0.500	0.152
0.525	0.313
0.550	0.487
0.575	0.889
0.600	1.519
0.625	2.721
0.650	4.219
0.675	5.817
0.700	6.613

The model of nonlinear wave propagation and how the use of the method allows you to cope with sharp fronts (or discontinuities) and develop them correctly, as well as to follow the formation of a jump and rarefaction (Fig. 5, *a*, *b*), is presented. The formation of an abrupt jump (jump) is observed.



a



b

Fig. 5. Shock and rarefaction formation: *a* – fractional flow graph; *b* – classical graph of oil and water phases

Fig. 5 shows the fractional flow graph associated with the classical Darcy's system together with the one associated with the coupled system for three values of tolerances as indicated above. For instance, if one chooses to use the Welge approach to locate the front, that is, if we trace a line passing through the saturation S_{wc}, corresponding to the beginning of the water injection, and tangent to the fractional flow, we observe different values of S_{wf} affecting the waterfront position and therefore it is advance. Fig. 5, *b* shows the classical graph of oil and water phases, respectively, corresponding to the tolerances above.

6. Discussion of the results, the main result of the limits of vanishing viscosity approximations

Our result is based on the fact that the entropy solutions of the hyperbolic system (2), (3) indeed coincide with the exceptions of the parabolic system (10). In our point of view, to develop a satisfactory theory of vanishing viscosity limits, the essence of the question is a good approximation of the limits of the solution of a parabolic system. Roughly speaking, a function u is a viscosity solution if: in a forward neighborhood of each point of the jump, the function u is well approximated by the self-similar solution of the corresponding Riemann problem.

In principle, different subsequences $\varepsilon \to 0$ may yield different limits. To achieve uniqueness, it is sufficient to define a viscous solution for a hyperbolic system of conservation laws (2), (3) based on local integral estimates (7), (8). By prioritizing in this way, we can also visualize the results of the shock position estimation as it develops at different time steps (Fig. 2).

In other early works, the results of numerical simulation of the transition flow of saturated-unsaturated water [2, 4, 13–15] are presented, which complicates the analysis of the viscous solution. We provide an improved Buckley-Leverett theory model for each phase of a two-phase fluid flow depending on saturation, capillary pressure, permeability and porosity of various phases, and initial and boundary conditions. The improvement of the model is carried out by calculating the mass conservation coupling equation for each phase. Here, we presented a method allowing any classical existing code or software based on the classical Darcy's approach taking into account the coupling between phases.

We agree that most industrial regulations use the classical Darcy approach for modeling multiphase flows, and any deviation from this scenario will be called a «Darcy-free flow». In our case, we can avoid this because the system (2), (3) is not a physical, but rather an artificial mathematical system that can be solved using standard petroleum solutions, and their artificial solution can be manipulated to obtain a solution to the physical system (1).

The considered method, based on the Buckley-Leverett theory, uses vanishing viscosity for frontal advance, but in general, it can be applied to a variety of systems that use different technological approaches and opens the way for further research. In particular, stochastic analysis of two-phase flow in stratified porous media seems promising [13]. Stochastic models, which include some assumptions about porous media, simplify and stabilize fuzzy information.

We neglect the effects of capillarity and the medium is originally saturated with oil. This should be taken into account in practice. Since growing experimental, computational and theoretical evidence should be noted that the constitutive equation for the average seepage velocity has the form of a power law in the pressure gradient over a wide range of capillary numbers.

In the future, we plan to use stochastic data and analyze them.

7. Conclusions

1. The main step is to analyze the solution with vanishing viscosity of the general Riemann problem, whose actions are theoretically explained by the fact that for a given left state u^- there is a continuous Lipschitz curve of right states u^+:

– in a forward neighborhood of each point of the jump, the function u is well approximated by the self-similar solution of the corresponding Riemann problem;

– on a region where its total variation is small, u can be accurately approximated by the solution of a linear system with a constant coefficient.

2. Solutions are derived for the horizontal, steady flow of two viscous incompressible fluids, including the permeability endpoints, which can be used to generate the relative permeability curves of the Buckley-Leverett class. From a qualitative point of view, we find an important result. The relative permeabilities depend on the fluid pressure and this occurs also when we model the solid matrix as a rigid medium. This fact helps us to understand some remarkable quantitative differences of experimental data with the classical model. Moreover, the mathematical analysis we have provided allows us to gain insights to perform rational experiments to validate the theoretical models.

3. The numerical solution of the model under consideration confirms the convergence of the left and right limits of the found solution to a function that well approximates the solution of the original corresponding Riemann problem:

– depending on the wettability, capillary action, and pore-size distribution, low-permeability media also favor the appearance of preferential flow paths for the phases and therefore further limit momentum exchange by reducing the fluid-fluid interfacial area;

– highly permeable media may yield large fluid-fluid interfaces relative to the fluid-solid ones, therefore maximizing exchanges.

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