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APPROACH TO THE SYNTHESIS OF AN APERIODIC ROBUST AUTOMATIC CONTROL SYSTEM BASED ON THE GRADIENT-SPEED METHOD OF LYAPUNOV VECTOR FUNCTIONS

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One of the actual problems for the theory and practice of control of dynamic objects is the development of methods for research and synthesis of control systems of multidimensional objects.

The paper proposes a universal approach to construct Lyapunov vector functions directly from the equation of state of control system and a new gradient-speed method of Lyapunov vector functions to study aperiodic robust stability of linear control system with m inputs and n outputs.

The study of aperiodic robust stability of automatic control systems is based on the construction of Lyapunov vector functions and gradient-speed dynamic control systems.

The basic statements of Lyapunov's theorem about asymptotic stability and notions of stability of dynamic systems are used. The representation of control systems as gradient systems and Lyapunov functions as potential functions of gradient systems from the catastrophe theory allow to construct the full-time derivative of Lyapunov vector functions always as a sign-negative function equal to the scalar product of the velocity vector on the gradient vector. The conditions of aperiodic robust stability are obtained as a system of inequalities on the uncertain parameters of the automatic control system, which are a condition for the existence of the Lyapunov vector-function.

A numerical example of synthesis of aperiodic robustness of a multidimensional control object is given. The example shows the main stages of the developed synthesis method, the study of the system stability at different values of the coefficients k, confirming the consistency of the proposed method. Transients in the system satisfy all requirements

Keywords: robust stability, linear multidimensional systems, Lyapunov vector-functions, aperiodic robustly stable systems

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1. Introduction

Actual automatic control systems are designed and operated under conditions of uncertainty. The uncertainty can be caused by not knowing the true values of the system parameters at the design stage and unpredictable change of them during operation. At the same time, the main property that characterizes the performance of an automatic control system under conditions of uncertainty is stability [1].

In modern automatic control theory, the main direction is to solve the problem of analysis and synthesis of control systems in conditions of uncertainty. Known methods of synthesis of automatic control system that use the state vector are based on modal control and do not allow to consider in a complex task of analysis and synthesis of automatic control system [2, 3]. Usually, direct, and inverse canonical transformations and complex ambiguous calculations of the roots of the characteristic equation of the closed system are required.

This study proposes and substantiates an approach to the analysis and synthesis of an aperiodic robustly stable linear automatic control system of desired quality based on the gradient-speed Lyapunov vector-function method. Methods based on the application of Lyapunov functions [4, 5] are universal methods for investigating stability and quality of linear and nonlinear automatic control systems under conditions of multidimensional control objects and uncertainties.

The well-known method of investigating the stability of linear control systems by the second Lyapunov method [6] involves solving Lyapunov matrix equations, which require complex and ambiguous calculations. Currently, Lyapunov function methods are mainly only a tool for theoretical research and cannot provide answers to many research questions of automatic control systems in real conditions. The main obstacle in this case is the lack of a universal approach to the construction of Lyapunov functions.

From the geometrical interpretation of Lyapunov's direct method theorems [7] one can clearly see that the dynamics of changes of phase trajectories in the phase space are everywhere determined by the velocity vector $\frac{dx}{dt}$, and the dynamics of changes of necessary Lyapunov functions along phase trajectories are everywhere determined by the gradient $\frac{\partial V(x)}{\partial x}$, vector from Lyapunov functions $V(x)$.

In the region of aperiodic robust stability parameters, these dynamical vectors have the same magnitude and opposite sign. This fact allows to consider the dynamical system as gradient systems and Lyapunov functions as potential functions of gradient systems from catastrophe theories [8, 9]. It is true for gradient systems:

$$\frac{dx_i}{dt} = -\frac{\partial V(x)}{\partial x_i}, i = 1, \ldots, n.$$

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On this basis the gradient-velocity method of Lyapunov vector functions is developed, which allows to solve the problem of investigation of aperiodic robust stability of the system [10]. The conditions of existence of Lyapunov vector functions are set in the form of a system of inequalities on uncertain parameters of the control system.

Thus, the solution of the problem to the synthesis of aperiodic robust-stable automatic control system based on the gradient-speed Lyapunov vector-functions method seems very relevant, both for the modern needs of science and its applications to practical problems, related to the design and simulation of control processes in engineering and, secondly, the presence of a large number of unresolved problems directly related to engineering practice.

2. Literature review and problem statement

In [1] the results of studies of real control systems, which are designed and operate under uncertainty, are presented. The uncertainty can be caused by the non-determinability of the true values of the system parameters at the design stage and their changes in the process of operation, which are difficult to predict.

Therefore, there remain unresolved issues related to the most pressing problem of creating a control system and such a problem, in a sense, provides the best protection from the conditions of uncertainty associated with the knowledge of the known properties of the control system. The paper [2] presents the results of the study of robust stability of the system, which is associated with the fact that the constraints on the change in the parameters of the control systems in the framework of the linear principle of stability are given. It should also be noted that instability in control systems arises as a result of the output of uncertain parameters of the system beyond the robust stability.

A sufficiently large number of works is devoted to the problem of robust stability of control systems. Such approach was used in [1, 2], but here the robust stability of polynomials and matrices is investigated mainly only within the linear principle of investigation of stability of control systems.

In [3, 4] the authors propose a new gradient-speed method of Lyapunov vector-functions. Although in itself the method of investigation of Lyapunov's direct method idea is universal. But the wide application of such method in practical research is restrained by the absence of a universal approach to the construction of Lyapunov functions.

It should be noted that geometrical interpretation of the asymptotic stability theorem makes it possible to find the possibility to choose in the equation of state the vector of anti-gradient of Lyapunov function equal to the velocity vector. This allows to consider the control system as a gradient system, and Lyapunov functions as potential functions of gradient systems, from the catastrophe theory [5]. Robust stability of the system is investigated by constructing a sign-negative function, which is equal to the scalar product of the gradient vector on the velocity vector.

The conditions of existence of Lyapunov vector functions are established from the conditions of positive definiteness of Lyapunov vector functions in the form of a system of inequalities on uncertain parameters of control objects.

In work [6] results of research of simple method of Lyapunov functions, represented by neural networks are presented. The obtained neural networks represent Lyapunov functions, on the basis of which the asymptotic stability or instability of the equilibrium point of a nonlinear system can be mathematically proved.

Let's note that one of the main advantages of this method is that it works for any nonlinear system, even if the number of state variables is quite large. The peculiarity of this method is that several different Lyapunov functions can be constructed for each system, including Lyapunov functions in quadratic form. This, in turn, makes it possible to choose the most appropriate function to solve a given problem.

The paper [7] describes a method to study the robustness of an aircraft landing control system constructed as a MIMO system. Here, the gradient-speed Lyapunov function method is used as the research apparatus.

In these studies, it is assumed that the wind gusts are zero and the controller is chosen in the form of one-parameter structurally stable mappings. As a result, the constructed function is investigated for stability in three stationary states.

In [8], asymptotically stable linear systems that are subject to unstructured time-varying perturbations are considered. Moreover, bounds for admissible perturbations are derived in such a way that the perturbed systems remain stable. Typically, these bounds are derived iteratively by adjusting the sequence of Lyapunov matrices.

Thus, the task of synthesizing control systems according to given quality indicators is to select, given a known dynamic description of the control object, the structure and parameters of the system that provide the required values of quality indicators.

3. The aim and objectives of the study

The aim of the study is to synthesize an aperiodic robustly stable automatic control system based on the gradient-speed Lyapunov vector function method to provide high quality control.

To achieve this aim, the following objectives are accomplished:

– investigate the robust stability of linear automatic control systems with an $m \times n$ control object matrix. To determine the Lyapunov function based on the gradient velocity method of Lyapunov vector functions from the equations of state and to determine the existence conditions of Lyapunov vector functions. From the coefficients of the Lyapunov vector function, determine the robust stability radius;

– investigate the system with control object matrices reduced to block diagonal form and determine the condition of robust stability. Using an example to investigate the stability and aperiodic transient response of a system, consider a fourth-order differential system and conduct simulation experiments using the Matlab Simulink software package.

4. Materials and methods

In this paper, using basic statements of Lyapunov theorem, the basic development of a new gradient-speed method of Lyapunov vector functions, which allows solving the problem of aperiodic robust stability of linear control systems with m inputs and cn outputs, is presented. The aperiodic robust stability of the system is investigated by the gradient-speed Lyapunov function vector method. This approach is presented by a mathematical apparatus based on gradient dynamical system and equivalence of Lyapunov and potential functions of gradient systems.

Both classical methods of stability theory and methods of robust stability theory are consistently applied in this work. At the same time, such methods and statements as the theory of automatic control, matrix theory, theory of differential equations, theory of stability, and theory of dynamical systems are used. Simulation software using the Simulink Matlab software package was used.

5. Results of a study on the aperiodic robust stability of linear multi-dimensional systems

5. 1. Outlining approaches to the study of a control system with m inputs and with n outputs

This section presents the course of the study of aperiodic robust stability of linear automatic control systems with a control object matrix of dimension $m*n$ by the gradient-speed method of Lyapunov vector functions:

$$\dot{x} = Ax + Bu, x \in R^n, u \in R^m, \tag{1}$$

$$y = Cx, y \in R^{\ell}.$$

The regulator is described by the equation:

$$u = -Kx. \tag{2}$$

Here

$$A \in R^{nxn}, \quad B \in R^{nxm}, \quad C \in R^{lxn}, \quad K \in R^{mxn},$$

$$u_i = -k_{i1}x_1 - k_{i2}x_2 - \ldots - k_{in}x_n, i = 1,2,\ldots,m.$$

Equations (1), taking into account the control law (2), can be presented in an expanded form:

$$\begin{cases} \dot{x}_1 = \left(a_{11} - \sum_{k=1}^{m} b_{1k}k_{k1} \right) x_1 + \\ + \left(a_{12} - \sum_{k=1}^{m} b_{1k}k_{k2} \right) x_2 + \ldots + \left(a_{1n} - \sum_{k=1}^{m} b_{1k}k_{kn} \right) x_n, \\ \dot{x}_2 = \left(a_{21} - \sum_{k=1}^{m} b_{2k}k_{k1} \right) x_1 + \\ + \left(a_{22} - \sum_{k=1}^{m} b_{2k}k_{k2} \right) x_2 + \ldots + \left(a_{2n} - \sum_{k=1}^{m} b_{2k}k_{kn} \right) x_n, \\ \ldots\ldots \\ \dot{x}_n = \left(a_{n1} - \sum_{k=1}^{m} b_{nk}k_{k1} \right) x_1 + \\ + \left(a_{n2} - \sum_{k=1}^{m} b_{nk}k_{k2} \right) x_2 + \ldots + \left(a_{nn} - \sum_{k=1}^{m} b_{nk}k_{kn} \right) x_n. \end{cases} \tag{3}$$

The Lyapunov function is constructed in the form of a vector function, based on the gradient-velocity method of Lyapunov vector functions. The components of the gradient vector are determined from the equations of state (3):

$$\frac{\partial V_i(x)}{\partial x_j} = -\left(a_{ij} - \sum_{k=1}^{m} b_{ik}k_{kj} \right) x_j, \quad i = 1,\ldots,n; \quad j = 1,\ldots,n. \tag{4}$$

Decomposing the components of the velocity vector by the coordinates of the equation of state (3) can be represented as

$$\left(\frac{dx_i}{dt} \right)_{x_j} = \left(a_{ij} - \sum_{k=1}^{m} b_{ik}k_{kj} \right) x_j, \quad i = 1,\ldots,n; \quad j = 1,\ldots,n. \tag{5}$$

The total time derivative of the Lyapunov vector function $V(x)$ considering the equations of motion (3), is defined as the scalar product of the gradient vector from the Lyapunov vector function (4), and the velocity vector (5), i.e.:

$$\frac{dV(x)}{dt} = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial V_i(x)}{\partial x_j} \left(\frac{dx_i}{dt} \right)_{x_j} =$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \left(-\left(a_{ij} - \sum_{k=1}^{m} b_{ik}k_{kj} \right)^2 x_j^2 \right). \tag{6}$$

It follows from expression (6) that the total time derivative of the Lyapunov vector function will be a sign-negative function. Using the components of the gradient vector, let's construct the vector of the Lyapunov function in scalar form:

$$V(x) = \frac{1}{2} \sum_{j=1}^{n} \begin{pmatrix} \sum_{k=1}^{m} b_{1k}k_{kj} - a_{1j} + \sum_{k=1}^{m} b_{2k}k_{kj} - \\ -a_{2j} + \ldots + \sum_{k=1}^{m} b_{nk}k_{kj} - a_{nj} \end{pmatrix} x_j^2. \tag{7}$$

The conditions for the existence of Lyapunov vector-functions, i.e., the positive definiteness of the Lyapunov vector function will be expressed:

$$\sum_{k=1}^{m} b_{1k}k_{kj} - a_{1j} +$$

$$+ \sum_{k=1}^{m} b_{2k}k_{kj} - a_{2j} + \ldots + \sum_{k=1}^{m} b_{nk}k_{kj} - a_{nj} > 0, \quad j = 1,\ldots,n. \tag{8}$$

Let's find the robust stability radius by the coefficients of the Lyapunov vector function. To do this it is possible to refer to parametric families of coefficients of components of the Lyapunov vector function, such as the interval family given in the form (9):

$$d_{ij} = d_{ij}^0 + \Delta_{ij}, \left| \Delta_{ij} \right| \le \gamma\, m_{ij}, i, j = 1,2,\ldots,n, \tag{9}$$

where the nominal coefficients $d_{ij}^0 = -\left(a_{ij}^0 - \sum_{k=1}^{m} b_{ik}^0 k_{kj}^0 \right)$ correspond to a positively defined Lyapunov function, i.e.:

$$\sigma(D_0) = \min_i \min_j -\left(a_{ij}^0 - \sum_{k=1}^{m} b_{ik}^0 k_{kj}^0 \right) > 0.$$

Let's require that the condition of positivity of coefficients holds for all functions of the family (10):

$$-\left(a_{ij}^0 - \sum_{k=1}^{m} b_{ik}^0 k_{kj}^0 \right) + \Delta_{ij} > 0, i = 1,2,\ldots,n; j = 1,2,\ldots,n. \tag{10}$$

It is clear that this inequality will be satisfied for all Δ_{ij} admissible if and only if $-\left(a_{ij}^0 - \sum_{k=1}^{m} b_{ik}^0 k_{kj}^0 \right) + \gamma\, m_{ij} > 0$, $i = 1,2,\ldots,n;$ $j = 1,2,\ldots,n$, i.e., at:

$$\gamma < \gamma^* = \min_i \min_j \frac{-\left(a_{ij}^0 - \sum_{k=1}^{m} b_{ik}^0 k_{kj}^0 \right)}{m_{ij}}.$$

In particular, if m_{ij}=1 (the scales of all coefficients of the components of the Lyapunov function term are the same), then $\gamma^* = \sigma(D_0)$.

Thus, the aperiodic robust stability radius of the interval family of positively defined functions is equal to the smallest value of the coefficients of the components of the Lyapunov vector-functions.

5. 2. Experimental results on aperiodic transients using the gradient velocity Lyapunov vector function method

Multidimensional systems in the canonical form of equations of state. Let the control system be described by the equation of state (1). The matrix of the control object A can be reduced to the block-diagonal form with the help of the nonspecific matrix P, the columns of which are the eigenfunctions of the matrix A:

$$\tilde{A} = P^{-1}AP = \text{diag}\{\Lambda, J_1,...,J_m, J_1',...,J_k'\}, \tag{11}$$

with diagonal quadratic blocks of the form:

$$\Lambda = \text{diag}\{\lambda_1,...,\lambda_l\}; \tag{12}$$

$$J_j = \begin{Vmatrix} \lambda_j & 1 & ... & 0 & 0 \\ 0 & \lambda_j & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & ... & \lambda_j & 1 \\ 0 & 0 & ... & 0 & \lambda_j \end{Vmatrix}, \quad N_j \times N_j, j = 1,...,m, \tag{13}$$

$$J_j' = \begin{Vmatrix} \alpha_j & -\beta_j \\ \beta_j & \alpha_j \end{Vmatrix}, \quad j = 1,...,k, \tag{14}$$

where λ_1, ..., λ_l – real simple, λ_i – real, N_j – multiples, $\lambda_j = \alpha_j \pm j\beta_j$ – are complexly conjugate eigenvalues of matrix A, and obviously $l + N_1 + ... + N_m + 2k = n$.

Let's show that the adopted structure (11) allows separate control and study by canonical representations of the object (12)–(14) corresponding to any diagonal block of the matrix $\tilde{A}$.

For this purpose, similarly to (1), write:

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u} = \begin{Vmatrix} \Lambda & & 0 \\ & J & \\ 0 & & J' \end{Vmatrix} \tilde{x} + \begin{Vmatrix} \tilde{B}_1 \\ \tilde{B}_2 \\ \tilde{B}_3 \end{Vmatrix} \tilde{u}; \tag{15}$$

$$\tilde{u} = -\tilde{k}^T\tilde{x} = -\begin{Vmatrix} \tilde{k}_1^T \tilde{k}_2^T \tilde{k}_3^T \end{Vmatrix} \tilde{x}, \tag{16}$$

where $\tilde{x} = P^{-1}x$, $\tilde{A} = P^{-1}AP$, $\tilde{B} = P^{-1}B$, $\tilde{k}^T = k^T P$, and the dimensions of the matrices $\tilde{B}_1$, $\tilde{B}_2$ and $\tilde{B}_3$ the control vector U correspond to the dimensions of the square matrices Λ, J, J'.

On the basis of (15) it is easy $\tilde{B}_2 = 0$, $\tilde{B}_3 = 0$ to see that it is possible to influence the system coordinates corresponding to the matrix Λ, keeping unchanged the system coordinates determined by matrices J, J', respectively by assuming $\tilde{B}_1 = 0$, $\tilde{B}_3 = 0$ or $\tilde{B}_1 = 0$, $\tilde{B}_2 = 0$. Thus, the further problem is reduced to the sequential study of aperiodic robust stability of linear control systems for canonical objects:

$$\dot{x}_1 = \Lambda\tilde{x}_1 + \tilde{B}_1 u, \tag{17}$$

$$\dot{x}_2 = J\tilde{x}_2 + \tilde{B}_2 u, \tag{18}$$

$$\dot{x}_3 = J'\,\tilde{x}_3 + \tilde{B}_3 u, \tag{19}$$

where

$$\tilde{x}_1 = \begin{Vmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_l \end{Vmatrix}, \quad \tilde{x}_2 = \begin{Vmatrix} \tilde{x}_{l+1} \\ \tilde{x}_{l+2} \\ \vdots \\ \tilde{x}_{l+L} \end{Vmatrix},$$

$$L = N_1 + ... + N_m, \quad \tilde{x}_3 = \begin{Vmatrix} \tilde{x}_{l+L+1} \\ \tilde{x}_{l+L+2} \\ \vdots \\ \tilde{x}_n \end{Vmatrix},$$

with matrices of the form (12)–(14). Let's consider in turn the problem of aperiodic robust stability (17)–(19) by the gradient-speed method of Lyapunov vector-functions.

Let's assume, for simplicity and clarity of writing, that:

$$b = \begin{Vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Vmatrix}, \quad u \in R, \quad u = -k^T x, \quad k = \begin{Vmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{Vmatrix}.$$

Similarly, to (11) write:

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{b}u = \begin{Vmatrix} \Lambda & & 0 \\ & J & \\ 0 & & J' \end{Vmatrix} \tilde{x} + \begin{Vmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \tilde{b}_3 \end{Vmatrix}, \tilde{u}, \tag{20}$$

$$\tilde{u} = -\tilde{k}^T\tilde{x} = -\begin{Vmatrix} \tilde{k}_1^T \tilde{k}_2^T \tilde{k}_3^T \end{Vmatrix} \tilde{x}, \tag{21}$$

where $\tilde{x} = P^{-1}x$, $\tilde{A} = P^{-1}A\,P$, $b = P^{-1}b$, $\tilde{k}^T = k^T P$, with the dimensionality of the column matrixes $\tilde{b}_1$, $\tilde{b}_2$, $\tilde{b}_3$ and matrix-strings $\tilde{k}_1^T$, $\tilde{k}_2^T$, $\tilde{k}_3^T$ correspond to the dimensions of square matrices Λ, J, J'. Based on (20), (21), by accepting $\tilde{k}_2^T = 0$, $\tilde{k}_3^T = 0$, it is not difficult to obtain the characteristic determinant of a closed system:

$$\left|\lambda I - \left(\tilde{A} - \tilde{b}\tilde{k}^T\right)\right| =$$
$$= \left|\lambda I_1 - \left(\Lambda - \tilde{b}_1\tilde{k}_1^T\right)\right| \cdot \left|\lambda I_2 - \left(J - \tilde{b}_2\tilde{k}_2^T\right)\right| \lambda I_3 -$$
$$-\left(J' - -\tilde{b}_3\tilde{k}_3^T\right),$$

from which it is obvious that by changing the coefficients of the matrix of the regulator $\tilde{k}_1^T$ one can influence the eigenvalues of the matrix $G_1 = \left(\Lambda - \tilde{b}_1\tilde{k}_1^T\right)$, while keeping the matrix eigenvalues unchanged J or J' respectively by taking $\tilde{k}_1^T = 0$, $\tilde{k}_3^T = 0$ or $\tilde{k}_1^T = 0$, $\tilde{k}_2^T = 0$. Thus, the sequential consideration of canonical objects similar to (17)–(19) becomes possible:

$$\dot{\tilde{x}} = \Lambda\tilde{x} + \tilde{b}_1 u, \tag{22}$$

$$\dot{\tilde{x}} = J\tilde{x} + \tilde{b}_2 u, \tag{23}$$

$$\dot{\tilde{x}} = J'\,\tilde{x} + \tilde{b}_3 u. \tag{24}$$

The system (22) is investigated by the gradient-velocity method of the Lyapunov function vector [4, 5].

$$\begin{cases} \dot{\tilde{x}}_1 = \left(\lambda_1 - \tilde{b}_1\tilde{k}_1\right)\tilde{x}_1, \\ \dot{\tilde{x}}_2 = \left(\lambda_2 - \tilde{b}_2\tilde{k}_2\right)\tilde{x}_2, \\ \vdots \\ \dot{\tilde{x}}_l = \left(\lambda_l - \tilde{b}_l\tilde{k}_l\right)\tilde{x}_l. \end{cases} \tag{25}$$

From (25) for the components of the gradient vector of the Lyapunov function vector:

$$V\left(\tilde{x}_1,...,\tilde{x}_l\right) = V_1\left(\tilde{x}_1,...,\tilde{x}_l\right) + ... + V_l\left(\tilde{x}_1,...,\tilde{x}_l\right)$$

let's obtain:

$$\begin{cases} \dfrac{\partial V_1(\tilde{x})}{\partial \tilde{x}_1} = -\left(\lambda_1 - \tilde{b}_1\tilde{k}_1\right)\tilde{x}_1, \dfrac{\partial V_1(\tilde{x})}{\partial \tilde{x}_2} = 0,..., \dfrac{\partial V_1(\tilde{x})}{\partial \tilde{x}_l} = 0, \\ \dfrac{\partial V_2(\tilde{x})}{\partial \tilde{x}_1} = 0, \dfrac{\partial V_2(\tilde{x})}{\partial \tilde{x}_2} = \\ = -\left(\lambda_2 - \tilde{b}_2\tilde{k}_2\right)\tilde{x}_2,..., \dfrac{\partial V_2(\tilde{x})}{\partial \tilde{x}_l} = 0, \\ ... \quad ... \quad ... \\ \dfrac{\partial V_l(\tilde{x})}{\partial \tilde{x}_1} = 0, \dfrac{\partial V_l(\tilde{x})}{\partial \tilde{x}_2} = 0,..., \dfrac{\partial V_l(\tilde{x})}{\partial \tilde{x}_l} = -\left(\lambda_l - \tilde{b}_l\tilde{k}_l\right)\tilde{x}_l. \end{cases} \tag{26}$$

From (25) the expansion of the velocity vector components on the system coordinates will be represented as:

$$\begin{cases} \left(\dfrac{d\tilde{x}_1}{dt}\right)_{x_1} = \left(\lambda_1 - \tilde{b}_1\tilde{k}_1\right)\tilde{x}_1, \left(\dfrac{d\tilde{x}_1}{dt}\right)_{x_2} = 0,..., \left(\dfrac{d\tilde{x}_1}{dt}\right)_{x_l} = 0, \\ \left(\dfrac{d\tilde{x}_2}{dt}\right)_{x_1} = 0, \left(\dfrac{d\tilde{x}_2}{dt}\right)_{x_2} = \\ = -\left(\lambda_2 - \tilde{b}_2\tilde{k}_2\right)\tilde{x}_2,..., \left(\dfrac{d\tilde{x}_2}{dt}\right)_{x_l} = 0, \\ ... \quad ... \quad ... \\ \left(\dfrac{d\tilde{x}_l}{dt}\right)_{x_1} = 0, \left(\dfrac{d\tilde{x}_l}{dt}\right)_{x_2} = 0,..., \left(\dfrac{d\tilde{x}_l}{dt}\right)_{x_l} = \\ = -\left(\lambda_l - \tilde{b}_l\tilde{k}_l\right)\tilde{x}_l. \end{cases} \tag{27}$$

The total time derivative of the Lyapunov function vector is calculated as the scalar product of the gradient vector (26) by the velocity vector (27):

$$\frac{dV(\tilde{x})}{dt} = \sum_{i=1}^{l} \frac{\partial V(\tilde{x})}{\partial \tilde{x}_i}\frac{d\tilde{x}_i}{dt} = -\sum_{i=1}^{l}\left(\lambda_i - \tilde{b}_i\tilde{k}_i\right)^2 \tilde{x}_i^2,$$

and will be a sign-negative function. Let's obtain the Lyapunov function in the form:

$$V(\tilde{x}) = -\left(\lambda_1 - \tilde{b}_1\tilde{k}_1\right)\tilde{x}_1^2 - \\ -\left(\lambda_2 - \tilde{b}_2\tilde{k}_2\right)\tilde{x}_2^2 - ... - \left(\lambda_l - \tilde{b}_l\tilde{k}_l\right)\tilde{x}_l^2. \tag{28}$$

The positive definiteness of the Lyapunov function (22) is given by the inequalities:

$$\lambda_1 - \tilde{b}_1\tilde{k}_1 < 0, \lambda_2 - \tilde{b}_2\tilde{k}_2 < 0, ..., \lambda_l - \tilde{b}_l\tilde{k}_l < 0. \tag{29}$$

Here are $\lambda_i - \tilde{b}_i\tilde{k}_i = \mu_i, i = 1,...,l$ the eigenvalues of the matrix of the closed system, and let's obtain the well-known result of the linear principle of stability $\mu_i = \lambda_i - \tilde{b}_i\tilde{k}_i < 0, i = 1,...,l$.

The system (23) is investigated by the gradient-velocity method of the vector of Lyapunov functions [1, 7]. Let's present the system of equations (23) in an expanded form for a single Jordanian block:

$$\begin{cases} \dot{\tilde{x}}_i = \lambda_i\tilde{x}_i + \tilde{x}_{i+1} - \tilde{b}_i\tilde{k}_i\tilde{x}_i, \\ \dot{\tilde{x}}_{i+1} = \lambda_i\tilde{x}_{i+1} + \tilde{x}_{i+2} - \tilde{b}_{i+1}\tilde{k}_{i+1}\tilde{x}_{i+1}, \\ \vdots \\ \dot{\tilde{x}}_{i+N_i} = \lambda_i\tilde{x}_{i+N_i} + \tilde{x}_{i+N_i} - \tilde{b}_{i+N_i}\tilde{k}_{i+N_i}\tilde{x}_{i+N_i}, \end{cases}$$

$$i = 1,...m. \tag{30}$$

From (30) the components of the gradient vector from the Lyapunov vector function will be equal:

$$\begin{cases} \dfrac{\partial V_i(\tilde{x})}{\partial \tilde{x}_i} = -\left(\lambda_i - \tilde{b}_i\tilde{k}_i\right)\tilde{x}_i, \dfrac{\partial V_i(\tilde{x})}{\partial \tilde{x}_{i+1}} = -\tilde{x}_{i+1}, \\ \dfrac{\partial V_{i+1}(\tilde{x})}{\partial \tilde{x}_{i+1}} = -\left(\lambda_i - \tilde{b}_{i+1}\tilde{k}_{i+1}\right)\tilde{x}_{i+1}, \dfrac{\partial V_{i+1}(\tilde{x})}{\partial \tilde{x}_{i+2}} = -\tilde{x}_{i+2}, \\ ... \quad ... \quad ... \\ \dfrac{\partial V_{i+N_i}(\tilde{x})}{\partial \tilde{x}_{i+N_i}} = -\left(\lambda_i - \tilde{b}_{i+N_i}\tilde{k}_{i+N_i}\right)\tilde{x}_{i+N_i}. \end{cases} \tag{31}$$

Let's decompose the components of the velocity vector by the system coordinates in the form:

$$\begin{cases} \left(\dfrac{d\tilde{x}_i}{dt}\right)_{\tilde{x}_i} = \left(\lambda_i - \tilde{b}_i\tilde{k}_i\right)\tilde{x}_i, \left(\dfrac{d\tilde{x}_i}{dt}\right)_{\tilde{x}_{i+1}} = \tilde{x}_{i+1}, \\ \left(\dfrac{d\tilde{x}_{i+1}}{dt}\right)_{\tilde{x}_{i+1}} = \left(\lambda_i - \tilde{b}_{i+1}\tilde{k}_{i+1}\right)\tilde{x}_{i+1}, \left(\dfrac{d\tilde{x}_{i+1}}{dt}\right)_{\tilde{x}_{i+2}} = \tilde{x}_{i+2}, \\ ... \quad ... \quad ... \\ \left(\dfrac{d\tilde{x}_{i+N_i}}{dt}\right)_{\tilde{x}_{i+N_i}} = \left(\lambda_i - \tilde{b}_{i+N_i}\tilde{k}_{i+N_i}\right)\tilde{x}_{i+N_i}. \end{cases} \tag{32}$$

The full-time derivatives of the Lyapunov vector function are calculated as the scalar product of the gradient vector (31) by the velocity vector (32):

$$\frac{dV(\tilde{x}_i)}{dt} = \sum\nolimits_{j=1}^{i+N_i} -\left[\left(\lambda_j - \tilde{b}_j\tilde{k}_j\right)^2 - \tilde{x}_{j+1}^2\right]. \tag{33}$$

From (33) the full-time derivatives are sign-negative functions and satisfy the condition.

The Lyapunov vector function in scalar form will be equal:

$$V(\tilde{x}) = \sum\nolimits_{j=1}^{i+N_i} -\frac{1}{2}\left[\left(\lambda_j - \tilde{b}_j\tilde{k}_j\right)\tilde{x}_j^2 - \tilde{x}_{j+1}^2\right]. \tag{34}$$

From (34) the positive definiteness condition, i.e., the existence of the Lyapunov function for system (23) let's obtain in the form:

$$\lambda_i - \tilde{b}_i\tilde{k}_i < 0, \lambda_i + 1 - \tilde{b}_{i+1}\tilde{k}_{i+1} < 0, ..., \lambda_i + 1 - \tilde{b}_{i+N_i}\tilde{k}_{i+N_i} < 0,$$

$$i = 1,...,m. \tag{35}$$

The system of inequalities (3) also expresses the condition of negativity of real multiple roots of the characteristic equation of the closed system.

The system (24) is investigated by the gradient-speed method of Lyapunov vector functions.

Let's consider the system (24) in expanded form for one block:

$$\begin{cases}\dot{\tilde{x}}_i = \alpha_i\tilde{x}_i + \beta_i\tilde{x}_{i+1} - \tilde{b}_i\tilde{k}_i\tilde{x}_i,\\ \dot{\tilde{x}}_{i+1} = -\beta_i\tilde{x}_i + \alpha_i\tilde{x}_{i+1} - \tilde{b}_{i+1}\tilde{k}_{i+1}\tilde{x}_{i+1},\end{cases}$$

$$i = 1,\ldots k. \tag{36}$$

Let's construct the Lyapunov function in the form of a vector-function with components $V_i(\tilde{x})$ and $V_{i+1}(\tilde{x})$, and for components of the gradient vector of the Lyapunov function (36) let's obtain:

$$\begin{cases}\left(\dfrac{d\tilde{x}_i}{dt}\right)_{\tilde{x}_i} = \left(\alpha_i - \tilde{b}_i\tilde{k}_i\right)\tilde{x}_i, \left(\dfrac{d\tilde{x}_i}{dt}\right)_{\tilde{x}_{i+1}} = \beta_i\tilde{x}_{i+1},\\ \left(\dfrac{d\tilde{x}_{i+1}}{dt}\right)_{\tilde{x}_i} = -\beta_i\tilde{x}_i, \left(\dfrac{d\tilde{x}_{i+1}}{dt}\right)_{\tilde{x}_{i+1}} = \left(\alpha_i - \tilde{b}_{i+1}\tilde{k}_{i+1}\right)\tilde{x}_{i+1},\end{cases}$$

$$\begin{cases}\dfrac{\partial V_i(\tilde{x})}{\partial \tilde{x}_i} = -\left(\alpha_i - \tilde{b}_i\tilde{k}_i\right)\tilde{x}_i, \dfrac{\partial V_i(\tilde{x})}{\partial \tilde{x}_{i+1}} = -\beta_i\tilde{x}_{i+1},\\ \dfrac{\partial V_{i+1}(\tilde{x})}{\partial \tilde{x}_i} = \beta_i\tilde{x}_i, \dfrac{\partial V_i(\tilde{x})}{\partial \tilde{x}_{i+1}} = -\left(\alpha_i - \tilde{b}_{i+1}\tilde{k}_{i+1}\right)\tilde{x}_{i+1}.\end{cases} \tag{37}$$

From (36) decomposition of the velocity vector components by coordinates let's obtain:

$$\begin{cases}\left(\dfrac{d\tilde{x}_i}{dt}\right)_{\tilde{x}_i} = \left(\alpha_i - \tilde{b}_i\tilde{k}_i\right)\tilde{x}_i, \left(\dfrac{d\tilde{x}_i}{dt}\right)_{\tilde{x}_{i+1}} = \beta_i\tilde{x}_{i+1},\\ \left(\dfrac{d\tilde{x}_{i+1}}{dt}\right)_{\tilde{x}_i} = -\beta_i\tilde{x}_i, \left(\dfrac{d\tilde{x}_{i+1}}{dt}\right)_{\tilde{x}_{i+1}} = \left(\alpha_i - \tilde{b}_{i+1}\tilde{k}_{i+1}\right)\tilde{x}_{i+1}.\end{cases} \tag{38}$$

The full-time derivatives of the Lyapunov function vector are calculated as the scalar product of the gradient vector (37) by the velocity vector (38):

$$\frac{dV(\tilde{x}_i)}{dt} = -\left(\alpha_i - \tilde{b}_i\tilde{k}_i\right)^2\tilde{x}_i^2 - \beta_i^2\tilde{x}_{i+1}^2 - -\left(\alpha_i - \tilde{b}_{i+1}\tilde{k}_{i+1}\right)^2\tilde{x}_{i+1}^2 - \beta_i^2\tilde{x}_i^2. \tag{39}$$

The function (39) is a guaranteed sign-negative function. From the components of the gradient vector (37) let's construct the vector Lyapunov functions in scalar form (with $\tilde{b}_i\tilde{k}_i = \tilde{b}_{i+1}\tilde{k}_{i+1}$ and $\tilde{x}_i = \tilde{x}_{i+1}$) let's obtain:

$$V(\tilde{x}) = -\left(\alpha_i - \tilde{b}_i\tilde{k}_i\right)\tilde{x}_i^2, i = 1,\ldots,k. \tag{40}$$

The conditions of positive definiteness, i.e., the existence of the Lyapunov function are written:

$$\alpha_i - \tilde{b}_i\tilde{k}_i < 0, i = 1,\ldots k. \tag{41}$$

Condition (41) also expresses the negativity of the real part of the eigenvalues $\mu_i = \alpha_i - \tilde{b}_i\tilde{k}_i < 0, i = 1,\ldots,k$ of the matrix of the closed system.

Let the dynamics of a stationary linear system be described by the equation.

$$\begin{cases}\dot{x}_1 = -11x_1 + 2x_2 + 5x_3 - 2x_4,\\ \dot{x}_2 = 2x_1 - 10x_2 + 5x_3 - 2x_4,\\ \dot{x}_3 = 2x_2 - 8x_3 + 2x_4,\\ \dot{x}_4 = x_1 - 4x_2 - 5x_3 - 11x_4.\end{cases} \tag{42}$$

It is required to investigate the aperiodic robust stability, i.e., the stability of the system with aperiodic transient process by the gradient-speed method of Lyapunov vector-functions. From (11) determine the components of the gradient vector from the Lyapunov vector-function $V(x)=(V_1(x), V_2(x), V_3(x), V_4(x))$.

From (42) the components of the vector of gradients from the vector of Lyapunov functions are determined:

$$\begin{cases}\dfrac{\partial V_1(x)}{\partial x_1} = 11x_1, \dfrac{\partial V_1(x)}{\partial x_2} = -2x_2,\\ \dfrac{\partial V_1(x)}{\partial x_3} = -5x_3, \dfrac{\partial V_1(x)}{\partial x_4} = 2x_4,\\ \dfrac{\partial V_2(x)}{\partial x_1} = -2x_1, \dfrac{\partial V_2(x)}{\partial x_2} = 10x_2,\\ \dfrac{\partial V_2(x)}{\partial x_3} = -5x_3, \dfrac{\partial V_2(x)}{\partial x_4} = 2x_4,\\ \dfrac{\partial V_3(x)}{\partial x_2} = -2x_2, \dfrac{\partial V_3(x)}{\partial x_3} = 8x_3,\\ \dfrac{\partial V_3(x)}{\partial x_4} = -2x_4,\\ \dfrac{\partial V_4(x)}{\partial x_1} = -x_1, \dfrac{\partial V_4(x)}{\partial x_2} = 4x_2,\\ \dfrac{\partial V_4(x)}{\partial x_3} = 5x_3, \dfrac{\partial V_4(x)}{\partial x_4} = 11x_4.\end{cases} \tag{43}$$

From (43) let's determine the expansion of the velocity vector components by coordinates (x_1, x_2, x_3, x_4):

$$\begin{cases}\left(\dfrac{dx_1}{dt}\right)_{x_1} = -11x_1, \left(\dfrac{dx_1}{dt}\right)_{x_2} = 2x_2,\\ \left(\dfrac{dx_1}{dt}\right)_{x_3} = 5x_3, \left(\dfrac{dx_1}{dt}\right)_{x_4} = -2x_4,\\ \left(\dfrac{dx_2}{dt}\right)_{x_1} = 2x_1, \left(\dfrac{dx_2}{dt}\right)_{x_2} = -10x_2,\\ \left(\dfrac{dx_2}{dt}\right)_{x_3} = 5x_3, \left(\dfrac{dx_2}{dt}\right)_{x_4} = -2x_4,\\ \left(\dfrac{dx_3}{dt}\right)_{x_1} = 0, \left(\dfrac{dx_3}{dt}\right)_{x_2} = 2x_2,\\ \left(\dfrac{dx_3}{dt}\right)_{x_3} = -8x_3, \left(\dfrac{dx_3}{dt}\right)_{x_4} = 2x_4,\\ \left(\dfrac{dx_4}{dt}\right)_{x_1} = x_1, \left(\dfrac{dx_4}{dt}\right)_{x_2} = -4x_2,\\ \left(\dfrac{dx_4}{dt}\right)_{x_3} = -5x_3, \left(\dfrac{dx_4}{dt}\right)_{x_4} = -11x_4.\end{cases} \tag{44}$$

The full-time derivative of the Lyapunov vector function is defined as the scalar product of the gradient vector (43) by the velocity vector (44):

$$\frac{dV(x)}{dt}=\sum_{i=1}^{4}\sum_{j=1}^{4}\frac{\partial V_i(x)}{\partial x_j}\left(\frac{dx_i}{dt}\right)_{x_j}=$$
$$=-130x_1^2-124x_2^2-139x_3^2-133x_4^2. \quad (45)$$

The quadratic form (45) is a sign-negative function. From (43) let's calculate the Lyapunov vector function in scalar form:

$$V(x)=4x_1^2+10x_2^2+1.5x_3^2+6.5x_4^2. \quad (46)$$

The existence conditions, i.e., positive definiteness of the Lyapunov function (46) are satisfied:

$$4>0,\ 10>0,\ 1.5>0,\ 6.5>0. \quad (47)$$

Thus, the system (40) is aperiodic robust stable. This is confirmed by conditions (45) and (47). The theoretical positions obtained will be confirmed by the results of simulation experiments carried out with the Simulink Matlab Software package, the model in Simulink is shown in Fig. 1 and the aperiodic robustness of the system is confirmed in Fig. 2.

Thus, system (1) is superstable, i.e., an aperiodic transient.

The conditions of aperiodic robust stability are obtained by us in the form of a system of inequalities on the uncertain parameters of the automatic control system, which is itself a condition for the existence of the Lyapunov vector-function.

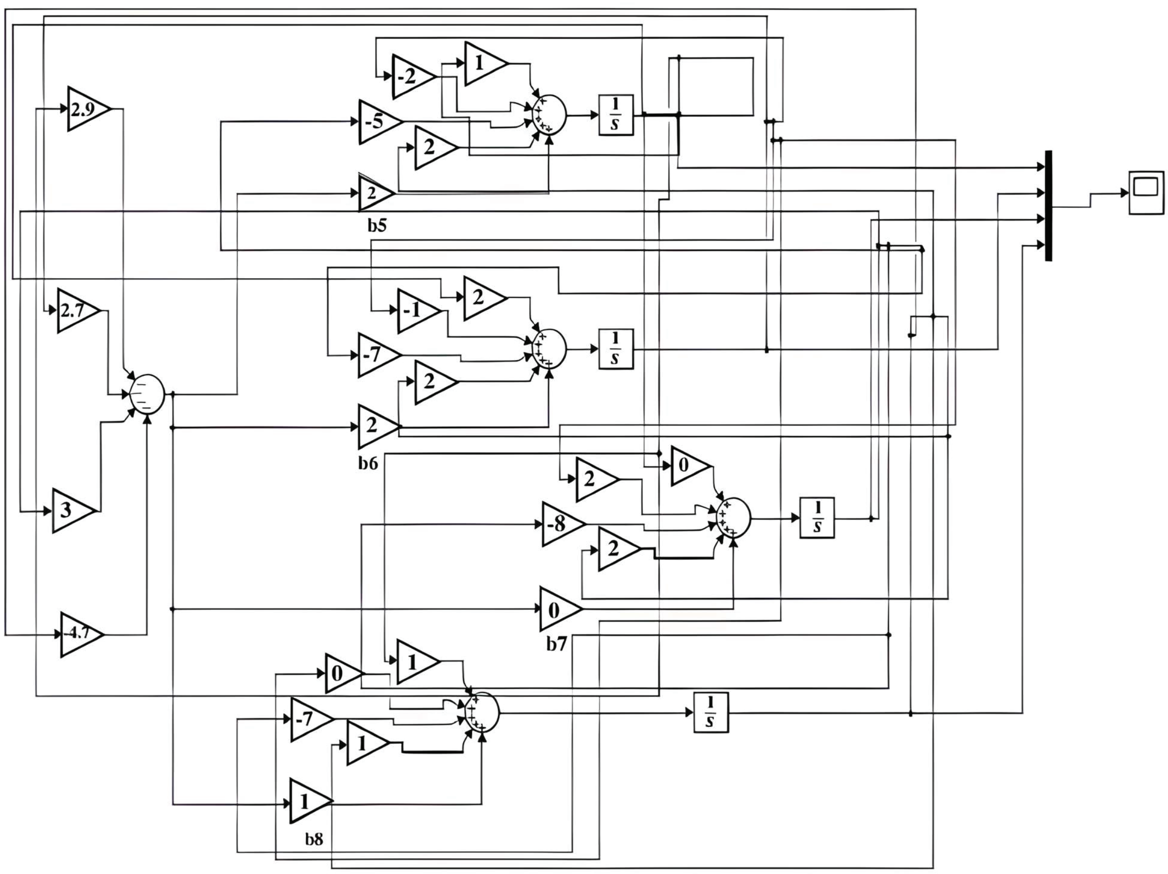

Fig. 1. Simulation model of the system transients

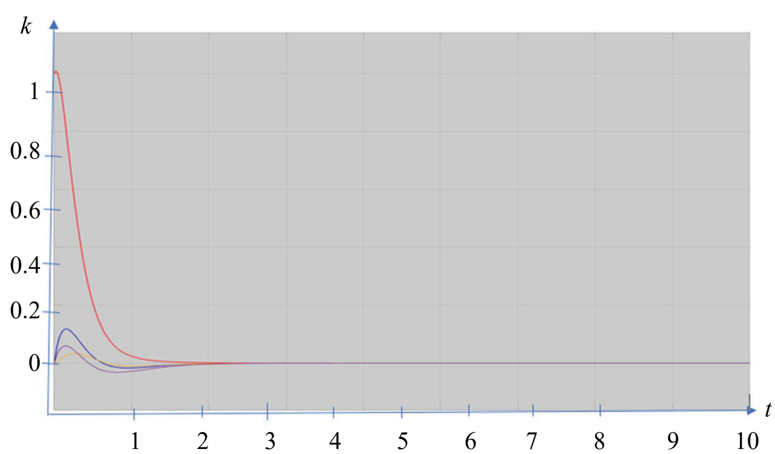

Fig. 2. Function graphs $x_i(t)$, (i=1,...,4)

Application of the method of gradient-speed Lyapunov vector-function to study aperiodic robust stability of multidimensional linear automatic control systems in the canonical representation and verification of the results of the study of linear stability principles shows the consistency of the proposed method.

6. Discussion and interpretation of the obtained stability conditions

In this paper, using basic statements of Lyapunov theorem, the basic development of a new gradient-speed Lyapunov vector function method, which allows to solve the problem of investigation of robust stability of linear control systems with m inputs and c n outputs is presented. The results of the study of aperiodic robust stability of linear automatic control systems with control object matrix of dimension $m \times n$, by gradient-speed Lyapunov vector function of the equation of state of the system (1). The Lyapunov function based on the gradient-speed Lyapunov vector functions method from the equations of state (1) and the condition of existence of positive definiteness of the vector Lyapunov functions in the form (8) are determined. To determine the robust stability radius from the coefficients of the Lyapunov vector function, let's turn to parametric families of coefficients of Lyapunov vector function components given in the form (9) and the function equal to the smallest value of coefficients of Lyapunov vector function components (10).

Thus the aperiodic robust stability radius of interval family of positively definite functions (10) is equal to the smallest value of coefficients of Lyapunov vector functions components.

A system with control object matrices reduced to the block-diagonal form is investigated (11). The problem is reduced to the sequential study of aperiodic robust stability of linear control systems for canonical objects (17)–(19). Let's define the robust stability condition, i.e. the existence of the Lyapunov function (31). The stability condition of the system (42) is obtained from the condition of positive definiteness of the Lyapunov function (46), in the form of a system of inequalities on the uncertain parameters of the control objects and the given parameters of the regulator (47). The system (42) is an aperiodically stable system. This is confirmed by conditions (46) and (47), and experimental results, the model in Simulink are shown in Fig. 1 and the graph of aperiodic robust stability of the system in Fig. 2.

The aperiodic robust automatic control system synthesized by the gradient-speed Lyapunov vector-function method obtains all direct measures of transient quality: shape of the transient curve, Fig. 2, absence of transient surge in the initial time period, control time, overshoot, transient oscillation, static control errors, stability and robustness.

Thus, for a broad class of systems, let's believe the theory is sufficiently well developed that work can begin to develop an effective approach to assist control engineers in incorporating the parametric approach into their analysis and design toolkits.

The gradient-speed method of vector Lyapunov functions considered in this paper allows to analytically synthesize an effective, physically realizable multidimensional control system under uncertainty in the transient mode. It can be applied to create control systems for various technical objects.

The practical relevance of these results should motivate new theoretical research into typical application methods, the zone of clarification of the robust control-design complex automated system.

Finally, these are the main results which theoretically represent the most promising direction. These studies are particularly important for the design of more efficient automated control systems.

This method of investigation does not establish the boundary of aperiodic stability, but only indicates the fact of existence of stability in linear stationary control systems. The problems of stability research are not solved by traditional methods and they are not suitable for investigating the stability of a large-dimension system. Therefore, a new universal method for investigating the stability of a control system based on the second Lyapunov method is proposed.

On this basis the gradient-velocity method of vector-functions A. M. Lyapunov for the study of control systems of aperiodic robust stability is proposed.

This method does not establish the boundary of aperiodic stability, but points only to the fact of existence of stability in linear stationary control systems. The problems of stability research are not solved by traditional methods, and they are not suitable for investigating the stability of a large-dimension system. Therefore, a new universal method for investigating the stability of a control system based on the second Lyapunov method is proposed.

7. Conclusions

1. In this paper, using basic statements of Lyapunov theorem, the basic development of new gradient-speed Lyapunov vector-function method, which allows to solve the problem of investigation of robust stability of linear control systems with m-inputs and cn-outputs, is presented. And also let's present the control systems – both gradient systems and Lyapunov functions, and potential functions of gradient systems from the catastrophe theory. These functions which are derived from the catastrophe theory allow one to construct the full time derivative of the Lyapunov vector functions to a sign-negative function equal to the scalar product of the velocity vector on the gradient vector.

The conditions of robust stability are obtained as system of inequalities on uncertain parameters of the automatic control system, which are the condition of existence of Lyapunov vector-function.

2. The study of aperiodic robust stability of automatic control systems was based on the construction of Lyapunov vector functions and gradient-speed dynamic control systems. Therefore, the new gradient-speed Lyapunov vector function method solves the problem of investigating the robustness of linear control systems with m-inputs and c-n-outputs. Aperiodic robust automatic control system synthesized by gradient-speed Lyapunov function vector method obtains all direct measures of transient quality: the shape of transient curve, the absence of spike in transient process in initial period, control time, overshooting, oscillation of transients, static control errors, stability and robustness.

Conflict of interest

The authors declare that they have no conflicts of interest in connection with this study, including financial, personal, authorial, or any other that might affect the research and results presented in this paper.

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Data availability

Manuscript has no associated data.

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