Using an asymptotic integration method, this paper investigates the axisymmetric problem of elasticity theory for an inhomogeneous transversal-isotropic truncated cone of small thickness. It is believed that the moduli of elasticity are arbitrary continuous functions of the cone opening angle. It is assumed that the lateral part of the cone is free of stresses, and arbitrary boundary conditions are set at the ends of the cone, leaving it in equilibrium. Homogeneous solutions have been constructed, that is, all solutions to equilibrium equations that satisfy the condition of absence of stresses on the lateral surfaces of the cone. Three groups of solutions were derived: a penetrating solution, solutions such as a simple edge effect, as well as the boundary layer solutions. An analysis of the stressed-strained state was carried out. It is shown that the penetrating solution and solutions having the character of a boundary effect determine the internal stressed-strained state of the cone. Solutions having the character of a boundary layer are localized at the ends of the cone and its first terms are equivalent to the edge effect of the Saint-Venant inhomogeneous plate.

A particular type of inhomogeneous transversal-isotropic cone of small thickness with the degeneration of its median surface into a plane has been studied. It is shown that this case of degeneration is special, and the solutions consist of a penetrating solution and a solution of the nature of the boundary layer.

Asymptotic formulas have been derived that make it possible to calculate the stressed-strained state of an inhomogeneous transversal-isotropic cone of small thickness. On the basis of the obtained solutions, it is possible to build a new refined applied theory and determine the applicability of existing applied theories for conical shells. New classes of solutions have been identified that no applied theory can describe

Keywords: inhomogeneous transversal-isotropic truncated cone, homogeneous solutions, opening angle, median surface, boundary layer

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CONSTRUCTION OF A HOMOGENEOUS SOLUTION TO THE ELASTICITY THEORY PROBLEM FOR AN INHOMOGENEOUS TRUNCATED TRANSVERSALLY ISOTROPIC CONE

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1. Introduction

One of the properties of materials that affect the stressed-strained state of elastic bodies is their heterogeneity. Heterogeneous shells are widely used in various fields of technology. Many tasks related to the study of heterogeneous shells are correctly resolved only within the framework of the theory of elasticity. The complexity of the phenomena arising from the deformation of heterogeneous shells has given rise to a number of applied theories based on various hypotheses. To assess the applicability of existing applied theories and to build new refined applied theories, it is relevant to investigate the stressed-strained state of inhomogeneous shells based on three-dimensional equations of the theory of elasticity. Despite the fact that this is associated with significant mathematical difficulties in the study of heterogeneous shells from the standpoint of three-dimensional equations of the theory of elasticity, their mechanical, geometric structure is more adequately taken into account and new qualitative and quantitative effects arise. Therefore, research into the stressed-strained state of a heterogeneous shell based on the equations of the theory of elasticity is relevant.

2. Literature review and problem statement

The cone is one of the main elements of the structures. Structural elements in the form of circular conical shells are widely used in aviation, rocket and space technology, chemical and oil refining engineering. These include transition compartments and head parts of rockets, turbine and jet engine housings, fuel tank bottoms. Among the classical elastic shells, conical shells are more complex. This is explained by the mathematical complexities caused by the geometry of the problem. The problems of the elasticity theory regarding the cone are addressed in a number of studies. In [1], a solution to the problem of equilibrium of an isotropic homogeneous cone under the action of arbitrary loads is considered. In [2], an exact solution to the problem of torsion of an elastic truncated heterogeneous cone in a static formulation is constructed. In [3, 4], solutions to boundary problems for cones under various boundary conditions on the surface of the cone are given. In [5], the axisymmetric problems for the truncated cone are investigated. In [6], an exact solution to the axisymmetric mixed problem of elasticity theory for a truncated circular hollow cone is constructed, taking into account its natural weight. In [7], with the help of an integral transformation, the established torsional oscillation of an elastic truncated cone is considered and the dependence of eigenfrequencies on the geometric parameters of the cone is studied.

Asymptotic methods have made a significant contribution to solving the three-dimensional problems of the theory of elasticity for the cone. In [8–10], a method of homogeneous solutions was used to study the axisymmetric problem of the theory of elasticity for a truncated isotropic cone, when the lateral part of the cone is free of stresses. However, the system of homogeneous solutions constructed in [8] is not complete and therefore does not make it possible to meet arbitrary boundary conditions at the ends of the cone. In [9, 10], the roots of the characteristic equation are classified with a relatively small parameter characterizing the thinness of the cone; solutions were constructed depending on the roots of the characteristic equation. It is shown that the stress-deformed state in a cone of small thickness consists of three types: a penetrating stressed state, a simple edge effect, and a boundary layer. In [11], the problem of elasticity theory for an isotropic hollow cone with a fixed lateral surface is studied. It is shown that, in this case, the stressed-strained state has the character of a boundary layer. Paper [12] outlines the general asymptotic theory of a truncated isotropic hollow cone. The spatial stressed-strained state of a truncated cone of small thickness under various boundary conditions on the lateral surface is investigated. Asymptotic decompositions of homogeneous solutions are derived, making it possible to calculate the stressed-strained state under various boundary conditions at the ends of the cone. On the basis of the qualitative analysis, the nature of the stressed-strained state was explained. An asymptotic analysis of the problems of propagation of harmonic torsional waves in a cone is carried out. In [13], a three-dimensional asymptotic theory of a transversal-isotropic homogeneous cone of small thickness is constructed, which includes methods for building homogeneous and inhomogeneous solutions. For a transversal-isotropic homogeneous cone, characteristic new groups of solutions are derived. A comparison of constructed solutions with solutions obtained from applied theories is given.

In [14, 15], based on the method of asymptotic integration of the equations of the theory of elasticity, the axisymmetric problem of the theory of elasticity and the torsional problem for an inhomogeneous isotropic cone of small thickness, when the side surfaces of the cone are free of stresses, were studied. The nature of the constructed solutions is explained. Paper [16] shows that when homogeneous mixed boundary conditions are set on the lateral surfaces of a heterogeneous cone, the stressed-strained state is composed only of a solution having the character of a boundary layer. In [17], the axisymmetric problems of the theory of elasticity for a transversal-isotropic cone of variable thickness are studied. Homogeneous solutions have been built and their classification has been made.

In [18–22], a particular type of cone was studied during the degeneration of its median surface into a plane, which corresponds to a plate of variable thickness. In [18], based on the method of homogeneous solutions, the asymptotic behavior of the axisymmetric stressed-strained state of a plate of variable thickness is investigated. In [19], the stressed-strained state of an inhomogeneous plate of variable thickness is analyzed. Asymptotic formulas are derived for movements and stresses. In [20, 21], homogeneous solutions are constructed for the problem of bending a transversal-isotropic plate of variable thickness. An asymptotic analysis of homogeneous solutions is carried out and the nature of the stressed-strained state is established. In [22], the non-axisymmetric problem of the elasticity theory for a plate of variable thickness is divided into two independent tasks: the tensile problem and the plate bending problem. Homogeneous solutions to the neo-asymmetric stretching problem of the elasticity theory for a transversal-isotropic plate are constructed.

The study of the stressed-strained state of the cone, like any theory, proceeds from simple models to more complex ones. Therefore, the logic of the development of research requires the analysis of the cone from more general positions. In the study of transversal-isotropic inhomogeneous cones, all real properties of materials are taken into account. However, the problem of elasticity theory for a transversal-isotropic heterogeneous cone has not been studied. The question of the correlation of applied theories for inhomogeneous transversal-isotropic conic shells and the corresponding three-dimensional problems of elasticity theory remains open.

The analysis of three-dimensional problems of elasticity theory for transversal-isotropic inhomogeneous cones is associated with complex mathematical difficulties. This more adequately takes into account their mechanical, geometric structure and leads to the emergence of new effects. The analysis of the stressed-strained state of a heterogeneous cone based on three-dimensional equations of the theory of elasticity is reduced to the study of boundary problems for systems of linear differential equations of the second order in partial derivatives with variable coefficients. These coefficients include moduli of elasticity, which are arbitrary positive continuous functions of the cone opening angle, which significantly complicates the construction of solutions to problems.

3. The aim and objectives of the study

The aim of this work is to study the behavior of a solution to the problem of elasticity theory and to reveal the features of the stressed-strained state for a transversal-isotropic inhomogeneous cone of small thickness. This will make it possible to derive asymptotic formulas for calculating the three-dimensional stressed-strained state of a transversal-isotropic inhomogeneous cone. Based on the analysis, it is possible to assess the applicability of various applied theories and build a more refined applied theory for heterogeneous conical shells.

To accomplish the aim, the following tasks have been set:

– to build homogeneous solutions for a transversal-isotropic heterogeneous cone and derive asymptotic formulas for displacements, stresses;

– to study the nature of stressed-strained states corresponding to different types of homogeneous solutions;

– to investigate a particular kind of heterogeneous transversal-isotropic cone during the degeneration of its median surface into a plane.

4. The study materials and methods

Based on the equations of the theory of elasticity, the three-dimensional stressed-strained state of a transversal-isotropic inhomogeneous cone of small thickness is studied. A complete system of elasticity theory equations for an inhomogeneous transversal-isotropic cone in a spherical coordinate system is given and the boundary problem is stated. Given that the stated boundary value problems include a small parameter characterizing the thickness of the cone, the method of asymptotic integration of the equations of the theory of elasticity is used to construct the solution. This method is one of the effective methods for studying the three-dimensional stressed state of elastic inhomogeneous bodies.

5. Results of investigating the construction of a homogeneous solution for a heterogeneous transversal-isotropic cone of small thickness

5. 1. Construction of homogeneous solutions for truncated heterogeneous transversal-isotropic cone of small thickness

The axisymmetric problem of the elasticity theory for a heterogeneous transversal-isotropic truncated hollow cone of variable thickness, which is a body with two conical and two spherical boundaries, is considered (Fig. 1). In a spherical coordinate system r, ϕ, θ, the region occupied by a cone is denoted by

$$\Gamma = \{ r \in [r_1, r_2], \theta \in [\theta_1, \theta_2], \phi \in [0, 2\pi] \}.$$

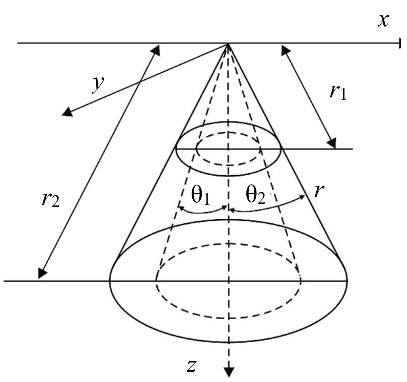

Fig. 1. Truncated cone in a spherical coordinate system: $\theta=\theta_s$ – conical part of the boundary; $r=r_s$ – spherical parts of the boundary (s=1, 2)

The spherical parts of the boundary ($r=r_s$) are called the ends of the cone, and the rest of the boundary ($\theta=\theta_s$) is termed the lateral surface (s=1, 2).

The system of equilibrium equations in the absence of mass forces in a spherical coordinate system r, θ, ϕ takes the form [23]:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{2\sigma_{rr} - \sigma_{\phi\phi} - \sigma_{\theta\theta} + \sigma_{r\theta}\operatorname{ctg}\theta}{r} = 0, \quad (1)$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{3\sigma_{r\theta} + \left(\sigma_{\theta\theta} - \sigma_{\phi\phi}\right)\operatorname{ctg}\theta}{r} = 0, \quad (2)$$

where σ_{rr}, $\sigma_{r\theta}$, $\sigma_{\phi\phi}$, $\sigma_{\theta\theta}$ are the components of the stress tensor, which are expressed in terms of the components of the displacement vector $u_r=u_r(r;\theta)$, $u_\theta=u_\theta(r;\theta)$ as follows [13]:

$$\sigma_{rr} = A_{11}\frac{\partial u_r}{\partial r} + A_{12}\left(\frac{u_\theta}{r}\operatorname{ctg}\theta + 2\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right), \quad (3)$$

$$\sigma_{\phi\phi} = A_{12}\frac{\partial u_r}{\partial r} + \left(A_{22} + A_{23}\right)\frac{u_r}{r} + A_{22}\frac{u_\theta}{r}\operatorname{ctg}\theta + A_{23}\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}, \quad (4)$$

$$\sigma_{\theta\theta} = A_{12}\frac{\partial u_r}{\partial r} + \left(A_{22} + A_{23}\right)\frac{u_r}{r} + A_{23}\frac{u_\theta}{r}ctg\theta + A_{22}\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}, \quad (5)$$

$$\sigma_{r\theta} = A_{44}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right). \quad (6)$$

It is assumed that the side part of the cone is free of stresses:

$$\sigma_{\theta\theta}\big|_{\theta=\theta_s} = 0, \quad \sigma_{\rho\theta}\big|_{\theta=\theta_s} = 0, \quad (s=1, 2), \quad (7)$$

and at the ends of the cone the arbitrary boundary conditions are set that leave it in equilibrium.

Substituting (3) to (6) in (1), (2), and (7), the following is obtained

$$\left(L_0 + \varepsilon\partial_1 L_1 + \varepsilon^2\partial_1^2 L_2\right)\bar{u} = \bar{0}, \quad (8)$$

$$\left(M_0 + \varepsilon\partial_1 M_1\right)\bar{u}\big|_{\eta=\pm 1} = \bar{0}, \quad (9)$$

where

$$L_0 = \left\| \begin{matrix} \begin{gathered}\partial\left(b_{44}\partial\right) + 2\varepsilon^2 \times \\ \times\left(b_{12} - b_{22} - b_{23}\right) + \\ +\varepsilon b_{44}\operatorname{ctg}\left(\theta_0 + \varepsilon\eta\right)\partial\end{gathered} & \begin{gathered}\varepsilon\left(b_{12} - b_{22} - b_{23}\right)\partial + \\ +\varepsilon^2\begin{pmatrix} b_{12} - b_{23} - \\ -b_{44} - b_{22}\end{pmatrix}\times \\ \times\operatorname{ctg}\left(\theta_0 + \varepsilon\eta\right) - \varepsilon\partial\left(b_{44}\right)\end{gathered} \\ \varepsilon\partial\left(b_{22} + b_{23}\right) + 2\varepsilon b_{44}\partial & \begin{gathered}\partial\left(b_{22}\partial\right) + \\ +\varepsilon\partial\left(b_{23}\operatorname{ctg}\left(\theta_0 + \varepsilon\eta\right)\right) - \\ -2\varepsilon^2 b_{44} + \left(b_{22} - b_{23}\right)\times \\ \times\varepsilon\operatorname{ctg}\left(\theta_0 + \varepsilon\eta\right)\partial - \\ -\left(b_{22} - b_{23}\right)\varepsilon^2 \times \\ \times\operatorname{cgt}^2\left(\theta_0 + \varepsilon\eta\right)\end{gathered}\end{matrix} \right\|,$$

$$L_1 = \left\| \begin{matrix} 2\varepsilon b_{11} & \begin{gathered}\partial\left(b_{44}\right) + b_{12}\partial + \\ +\varepsilon\left(b_{44} + b_{12}\right)\operatorname{ctg}\left(\theta_0 + \varepsilon\eta\right)\end{gathered} \\ \partial\left(b_{12}\right) + b_{44}\partial & 2\varepsilon b_{44}\end{matrix} \right\|,$$

$$L_2 = \left\| \begin{matrix} b_{11} & 0 \\ 0 & b_{44}\end{matrix} \right\|,$$

$$M_0 = \left\| \begin{matrix} b_{44}\partial & -\varepsilon b_{44} \\ \left(b_{22} + b_{23}\right)\varepsilon & b_{22}\partial + \varepsilon b_{23}\operatorname{cgt}\left(\theta_0 + \varepsilon\eta\right)\end{matrix} \right\|,$$

$$M_1 = \left\| \begin{matrix} 0 & b_{44} \\ b_{12} & 0\end{matrix} \right\|,$$

$$\partial = \frac{\partial}{\partial \eta}, \quad \partial_1 = \rho\frac{\partial}{\partial \rho}, \quad \partial_1^2 = \rho^2\frac{\partial^2}{\partial \rho^2}, \quad \bar{u} = \left(u_\rho, u_\theta\right)^T;$$

$\eta = \dfrac{\theta - \theta_0}{\varepsilon}$ – new dimensionless variables, $\eta \in [-1; 1]$;

$\theta_0 = \dfrac{\theta_1 + \theta_2}{2}$ – the opening angle of the median surface of the cone;

$\varepsilon=\dfrac{\theta_2-\theta_1}{2}$ – a small parameter characterizing the thickness of the cone;

$r_0=\sqrt{r_1r_2}$; $u_\rho=\dfrac{u_r}{r_0}$, $u_\theta=\dfrac{u_\theta}{r_0}$, $b_{ij}=\dfrac{A_{ij}}{G_0}$ – dimensionless quantities, G_0 is some characteristic modulus, e.g. $G_0=\max|A_{ij}|$.

It is assumed that the moduli of elasticity $b_{ij}=b_{ij}(\eta)$ are arbitrary continuous functions of the variable η, the values of which can change within the same order.

Here, $\theta_0\neq\pi/2$. $\theta_0=\pi/2$ is a special case and corresponds to a heterogeneous transversal-isotropic plate of variable thickness, which will be studied separately.

A homogeneous solution is any solution to equations (8) that satisfies the condition of absence of stresses on the lateral surfaces, that is, (9).

A solution to (8), (9) is sought in the form:

$$u_\rho(\rho,\eta)=\rho^{z-\frac{1}{2}}\upsilon(\eta),\ \ u_\theta(\rho,\eta)=\rho^{z-\frac{1}{2}}w(\eta). \tag{10}$$

Substituting (10) in (8), (9), a non-self-adjoint problem with a spectral parameter z is obtained:

$$\left(L_0+\varepsilon\left(z-\frac{1}{2}\right)(L_1-\varepsilon L_2)+\varepsilon^2\left(z-\frac{1}{2}\right)^2L_2\right)\bar{c}=\bar{0}, \tag{11}$$

$$\left.\left(M_0+\varepsilon\left(z-\frac{1}{2}\right)M_1\right)\bar{c}\right|_{\eta=\pm1}=\bar{0}, \tag{12}$$

where $\bar{c}(\eta)=(\upsilon(\eta),w(\eta))^T$.

At $\varepsilon\to0$, an asymptotic method based on three iterative processes was used to solve (11), (12) [14–16, 19, 24–27].

$z_0=-\dfrac{1}{2}$ – is the eigenvalue (11), (12) and a homogeneous solution corresponding to the first iterative process, which takes the form:

$$u_\rho^{(1)}=\frac{A}{\rho}\times\left\{\begin{array}{l}m_0\operatorname{ctg}\theta_0+\\ +\varepsilon\left[\begin{array}{l}-2\eta(m_0+t_0)+m_1+2t_1+\\ +\operatorname{ctg}^2\theta_0\left(m_0\left(\begin{array}{l}d_2-d_1+\\ +d_3-t_1-m_1\end{array}\right)+\\ +t_0(d_3-d_0)\right)(m_0+t_0)^{-1}\end{array}\right]+\\ +O(\varepsilon^2)\end{array}\right\}, \tag{13}$$

$$u_\theta^{(1)}=\frac{A}{\rho}\left\{\begin{array}{l}-(m_0+t_0)+\varepsilon\operatorname{ctg}\theta_0\times\\ \times\left[\begin{array}{l}(m_0+t_0)\int_{-1}^{\eta}\frac{b_{23}}{b_{22}}dx+m_0(\eta+1)+\\ +m_0\int_{-1}^{\eta}\frac{(b_{12}-b_{23})}{b_{22}}dx\end{array}\right]+\\ +O(\varepsilon^2)\end{array}\right\}. \tag{14}$$

The stresses corresponding to the solutions (13), (14) are as follows:

$$\sigma_{\theta\theta}^{(1)}=\frac{A\varepsilon}{\rho^2}\operatorname{ctg}\theta_0\left[\begin{array}{l}m_0\int_{-1}^{\eta}\frac{b_{12}(b_{23}-b_{22})}{b_{22}}dx-\\ -t_0\int_{-1}^{\eta}\frac{b_{22}^2-b_{23}^2}{b_{22}}dx+O(\varepsilon)\end{array}\right], \tag{15}$$

$$\sigma_{\rho\rho}^{(1)}=\frac{A}{\rho^2}\operatorname{ctg}\theta_0\left[\frac{m_0}{g_0}+t_0\frac{b_{12}(b_{23}-b_{22})}{b_{22}}+O(\varepsilon)\right], \tag{16}$$

$$\sigma_{\varphi\varphi}^{(1)}=\frac{A}{\rho^2}\operatorname{ctg}\theta_0\left[\begin{array}{l}m_0\frac{b_{12}(b_{23}-b_{22})}{b_{22}}-\\ -t_0\frac{(b_{22}^2-b_{23}^2)}{b_{22}}+O(\varepsilon)\end{array}\right], \tag{17}$$

$$\sigma_{\rho\theta}^{(1)}=\frac{A\varepsilon}{\rho^2}\operatorname{ctg}\theta_0\left[\begin{array}{l}m_0\int_{-1}^{\eta}\frac{b_{12}(b_{23}-b_{22})}{b_{22}}dx-\\ -t_0\int_{-1}^{\eta}\frac{(b_{22}^2-b_{23}^2)}{b_{22}}dx+O(\varepsilon)\end{array}\right], \tag{18}$$

where

$$d_0=\int_{-1}^{1}\frac{(b_{22}+b_{23})}{b_{22}}\left(\int_{-1}^{\eta}\frac{(b_{23}^2-b_{22}^2)}{b_{22}}dx\right)d\eta,$$

$$d_1=\int_{-1}^{1}\frac{(b_{22}+b_{23})}{b_{22}}\left(\int_{-1}^{\eta}\frac{b_{12}(b_{23}-b_{22})}{b_{22}}dx\right)d\eta,$$

$$d_2=\int_{-1}^{1}\frac{(b_{23}^2-b_{22}^2)}{b_{22}}\left(\int_{-1}^{\eta}\frac{b_{12}-b_{23}-b_{22}}{b_{22}}dx\right)d\eta;$$

$$d_3=\int_{-1}^{1}\frac{(b_{23}^2-b_{22}^2)}{b_{22}}\left(\int_{-1}^{\eta}\frac{b_{23}}{b_{22}}dx\right)d\eta;$$

$$m_k=\int_{-1}^{1}\frac{(b_{22}^2-b_{23}^2)}{b_{22}}\eta^k d\eta;$$

$$t_k=\int_{-1}^{1}\frac{b_{12}(b_{23}-b_{22})}{b_{22}}\eta^k d\eta;$$

$$g_0=\frac{b_{22}}{b_{12}^2-b_{11}b_{22}}.$$

Solutions that correspond to the second iterative process are found in the form:

$$\upsilon(\eta)=\varepsilon^{\frac{1}{2}}\left(\upsilon_{20}(\eta)+\varepsilon^{\frac{1}{2}}\upsilon_{21}(\eta)+....\right), \tag{19}$$

$$w(\eta)=w_{20}(\eta)+\varepsilon^{\frac{1}{2}}w_{21}(\eta)+...., \tag{20}$$

$$z=\varepsilon^{-\frac{1}{2}}(\alpha_0+\varepsilon\alpha_1+....). \tag{21}$$

Substituting (19) to (21) in (11), (12), after some transformations, the following is obtained:

$$u_\rho^{(2)}=\left(\frac{\varepsilon}{\rho}\right)^{\frac{1}{2}}\sum_{j=1}^{4}B_jU_{\rho j}^{(2)},$$

$$u_\theta^{(2)}=\rho^{-\frac{1}{2}}\sum_{j=1}^{4}B_jU_{\theta j}^{(2)}, \tag{22}$$

where

$$U_{\rho j}^{(2)}=\left\{\begin{array}{l}-(\eta+1)\alpha_{0j}^{2}\left(b_{11}^{(0)}-p_0\right)q_j+q_j^{2}+\\+\varepsilon^{\frac{1}{2}}\left[\begin{array}{l}(\eta+1)\left(\begin{array}{l}\alpha_{0j}\left(b_{11}^{(0)}-p_0\right)\times\\\times\left(m_0+2t_0\right)\times\\\times\mathrm{ctg}\theta_0-\alpha_{1j}\alpha_{0j}^{2}\times\\\times\left(b_{11}^{(0)}-p_0\right)q_j\ln\rho\end{array}\right)+\\+\alpha_{1j}q_j^{2}\ln\rho\end{array}\right]+O(\varepsilon)\end{array}\right\}\times$$
$$\times\exp\left(\frac{\alpha_{0j}}{\sqrt{\varepsilon}}\ln\rho\right),$$

$$U_{\theta j}^{(2)}=\left\{\begin{array}{l}\alpha_{0j}\left(b_{11}^{(0)}-p_0\right)q_j+\\+\varepsilon^{\frac{1}{2}}\left[\begin{array}{l}-\frac{3}{2}\alpha_{0j}^{2}\left(\begin{array}{l}p_1+p_0-\\-b_{11}^{(1)}-b_{11}^{(0)}\end{array}\right)\times\\\times\left(m_0+\frac{1}{2}t_0\right)\times\\\times\mathrm{ctg}\theta_0+\alpha_{0j}\alpha_{1j}q_j\ln\rho\end{array}\right]\times\\\times\left(b_{11}^{(0)}-p_0\right)+O(\varepsilon)\end{array}\right\}\times$$
$$\times\exp\left(\frac{\alpha_{0j}}{\sqrt{\varepsilon}}\ln\rho\right),$$

$$q_j=\alpha_{0j}^{2}\left(b_{11}^{(1)}+b_{11}^{(0)}-p_1-p_0\right)+t_0\mathrm{ctg}\theta_0;$$

$$b_{ij}^{(k)}=\int_{-1}^{1}b_{ij}\eta^{k}\mathrm{d}\eta;\quad p_k=\int_{-1}^{1}\frac{b_{12}^{2}}{b_{22}}\eta^{k}\mathrm{d}\eta.$$

A biquadrate equation is derived for determining α_{0j}

$$\left(p_0p_2-p_0b_{11}^{(2)}-p_2b_{11}^{(0)}+b_{11}^{(0)}b_{11}^{(2)}-\left(b_{11}^{(1)}-p_1\right)^{2}\right)\alpha_{0j}^{4}+$$
$$+2\mathrm{ctg}\theta_0\left(t_1b_{11}^{(0)}-t_1p_0-t_0b_{11}^{(1)}+t_0p_1\right)\alpha_{0j}^{2}+$$
$$+\left(m_0\left(b_{11}^{(0)}-p_0\right)-t_0^{2}\right)\mathrm{ctg}^{2}\theta_0=0. \qquad (23)$$

The stresses corresponding to the second iterative process are:

$$\sigma_{\varphi\varphi}^{(2)}=\rho^{-3/2}\sum_{j=1}^{4}B_jQ_{\varphi j}^{(2)},\quad \sigma_{\theta\theta}^{(2)}=\varepsilon\rho^{-3/2}\sum_{j=1}^{4}B_jQ_{\theta j}^{(2)}, \qquad (24)$$

$$\sigma_{\rho\theta}^{(2)}=\varepsilon^{\frac{1}{2}}\rho^{-3/2}\sum_{j=1}^{4}B_jF_j^{(2)},\quad \sigma_{\rho\rho}^{(2)}=\rho^{-3/2}\sum_{j=1}^{4}B_jQ_{\rho j}^{(2)}, \qquad (25)$$

where

$$Q_{\varphi j}^{(2)}=\left\{\begin{array}{l}\alpha_{0j}\left(b_{11}^{(0)}-p_0\right)q_j\times\\\times\left[\begin{array}{l}\alpha_{0j}^{2}\frac{b_{12}\left(b_{23}-b_{22}\right)}{b_{22}}(\eta+1)+\\+\mathrm{ctg}\theta_0\frac{\left(b_{22}^{2}-b_{23}^{2}\right)}{b_{22}}\end{array}\right]-\\-\alpha_{0j}\frac{b_{12}\left(b_{23}-b_{22}\right)}{b_{22}}q_j^{2}+\\+O\left(\varepsilon^{1/2}\right)\end{array}\right\}\exp\left(\frac{\alpha_{0j}}{\sqrt{\varepsilon}}\ln\rho\right),$$

$$Q_{\rho j}^{(2)}=\left\{\begin{array}{l}\alpha_{0j}\left(b_{11}^{(0)}-p_0\right)q_j\times\\\times\left[\begin{array}{l}\alpha_{0j}^{2}\frac{\left(b_{12}^{2}-b_{11}b_{22}\right)}{b_{22}}(\eta+1)+\\+\frac{b_{12}\left(b_{22}-b_{23}\right)}{b_{22}}\mathrm{ctg}\theta_0\end{array}\right]+\\+\alpha_{0j}\frac{\left(b_{11}b_{22}-b_{12}^{2}\right)}{b_{22}}q_j^{2}+\\+O\left(\varepsilon^{1/2}\right)\end{array}\right\}\times$$
$$\times\exp\left(\frac{\alpha_{0j}}{\sqrt{\varepsilon}}\ln\rho\right),$$

$$Q_{\theta j}^{(2)}=$$
$$=\left\langle\begin{array}{l}\alpha_{0j}\left(b_{11}^{(0)}-p_0\right)q_j\times\\\times\left\{\begin{array}{l}\alpha_{0j}^{4}\left[\begin{array}{l}\int_{-1}^{\eta}\left(\int_{-1}^{y}\frac{b_{12}^{2}}{b_{22}}(x+1)\mathrm{d}x\right)\mathrm{d}y-\\-\int_{-1}^{\eta}\left(\int_{-1}^{y}b_{11}(x+1)\mathrm{d}x\right)\mathrm{d}y\end{array}\right]-\\-\alpha_{0j}^{2}\mathrm{ctg}\theta_0\int_{-1}^{\eta}\left(\int_{-1}^{y}\frac{b_{12}\left(b_{23}-b_{22}\right)}{b_{22}}\mathrm{d}x\right)\mathrm{d}y+\\+\alpha_{0j}^{2}\mathrm{ctg}\theta_0\int_{-1}^{\eta}\frac{b_{12}\left(b_{23}-b_{22}\right)}{b_{22}}(x+1)\mathrm{d}x+\\+\int_{-1}^{\eta}\frac{\left(b_{22}^{2}-b_{23}^{2}\right)}{b_{22}}\mathrm{d}x\,\mathrm{ctg}^{2}\theta_0\end{array}\right\}+\\+q_j^{2}\alpha_{0j}\left\{\begin{array}{l}\alpha_{0j}^{2}\left[\begin{array}{l}\int_{-1}^{\eta}\left(\int_{-1}^{y}b_{11}\mathrm{d}x\right)\mathrm{d}y-\\-\int_{-1}^{\eta}\left(\int_{-1}^{y}\frac{b_{12}^{2}}{b_{22}}\mathrm{d}x\right)\mathrm{d}y\end{array}\right]-\\-\alpha_{0j}\int_{-1}^{\eta}\frac{b_{12}\left(b_{23}-b_{22}\right)}{b_{22}}\mathrm{d}x\,\mathrm{ctg}\theta_0\end{array}\right\}+\\+O\left(\varepsilon^{1/2}\right)\end{array}\right\rangle\times$$
$$\times\exp\left(\frac{\alpha_{0j}}{\sqrt{\varepsilon}}\ln\rho\right),$$

$$F_j^{(2)}=\left\{\begin{array}{l}\alpha_{0j}\left(b_{11}^{(0)}-p_0\right)q_j\times\\\times\left[\begin{array}{l}\alpha_{0j}^{3}\left(\begin{array}{l}\int_{-1}^{\eta}b_{11}(x+1)\mathrm{d}x-\\-\int_{-1}^{\eta}\frac{b_{12}^{2}}{b_{22}}(x+1)\mathrm{d}x\end{array}\right)+\\+\alpha_{0j}\mathrm{ctg}\theta_0\int_{-1}^{\eta}\frac{b_{12}\left(b_{23}-b_{22}\right)}{b_{22}}\mathrm{d}x\end{array}\right]+\\+\alpha_{0j}^{2}\cdot q_j^{2}\left(\int_{-1}^{\eta}\frac{b_{12}^{2}}{b_{22}}\mathrm{d}x-\int_{-1}^{\eta}b_{11}\mathrm{d}x\right)+\\+O\left(\varepsilon^{\frac{1}{2}}\right)\end{array}\right\}\times$$
$$\times\exp\left(\frac{\alpha_{0j}}{\sqrt{\varepsilon}}\ln\rho\right).$$

Solutions (11), (12) corresponding to the third iterative process are sought in the form:

$$\upsilon(\eta)=\varepsilon\left(\upsilon_{30}(\eta)+\varepsilon\upsilon_{31}(\eta)+\ldots\right), \quad (26)$$

$$w(\eta)=\varepsilon\left(w_{30}(\eta)+\varepsilon w_{31}(\eta)+\ldots\right), \quad (27)$$

$$z=\varepsilon^{-1}\left(\beta_0+\varepsilon\beta_1+\ldots.\right). \quad (28)$$

After substitution (26) to (28) in (11), (12) for the first decomposition terms a spectral problem describing a potential solution to a transversal-isotropic plate heterogeneous in thickness is derived [28]:

$$\left(b_{44}\upsilon_{30}'(\eta)\right)'+\beta_0\left[\left(b_{44}w_{30}(\eta)\right)'+b_{12}w_{30}'(\eta)\right]+$$
$$+\beta_0^2 b_{11}\upsilon_{30}(\eta)=0, \quad (29)$$

$$\left(b_{22}w_{30}'(\eta)\right)'+\beta_0\left[\left(b_{12}\upsilon_{30}(\eta)\right)'+b_{44}\upsilon_{30}'(\eta)\right]+$$
$$+\beta_0^2 b_{44}w_{30}(\eta)=0, \quad (30)$$

$$b_{44}\left(\upsilon_{30}'(\eta)+\beta_0 w_{30}(\eta)\right)=0,\ \text{at}\ \eta=\pm1, \quad (31)$$

$$b_{22}w_{30}'(\eta)+\beta_0 b_{12}\upsilon_{30}(\eta)=0,\ \text{at}\ \eta=\pm1. \quad (32)$$

By replacing

$$\upsilon_{30}(\eta)=\beta_0^{-2}g_0 f''(\eta)-g_1 f(\eta), \quad (33)$$

$$w_{30}(\eta)=-\beta_0^{-3}\left(g_0 f''(\eta)\right)'+$$
$$+\beta_0^{-1}g_2 f'(\eta)+\beta_0^{-1}\left(g_1 f(\eta)\right)', \quad (34)$$

the spectral problem (29) to (32) is reduced to the following:

$$\left(g_0 f''(\eta)\right)''-\beta_0^2\begin{bmatrix}\left(g_1 f(\eta)\right)''+\\+g_1 f''(\eta)+\\+\left(g_2 f'(\eta)\right)'\end{bmatrix}+\beta_0^4 g_3 f(\eta)=0, \quad (35)$$

$$f'(\eta)\big|_{\eta=\pm1}=0, \quad (36)$$

$$\beta_0 f(\eta)\big|_{\eta=\pm1}=0, \quad (37)$$

where

$$g_1=\frac{b_{12}}{b_{12}^2-b_{11}b_{22}};\ g_2=b_{44}^{-1};\ g_3=\frac{b_{11}}{b_{12}^2-b_{11}b_{22}}.$$

(35) to (37) is a generalization of the Papkovic spectral problem for the heterogeneous transversal-isotropic case [25].

Solutions corresponding to the third iterative process are as follows:

$$u_\rho^{(3)}(\rho,\eta)=\varepsilon\rho^{-\frac{1}{2}}\times$$
$$\times\sum_{k=1}^{\infty}D_k\begin{bmatrix}\beta_{0k}^{-2}g_0 f_k''(\eta)-\\-g_1 f_k(\eta)+O(\varepsilon)\end{bmatrix}\exp\left(\frac{\beta_{0k}}{\varepsilon}\ln\rho\right), \quad (38)$$

$$u_\theta^{(3)}(\rho,\eta)=\varepsilon\rho^{-\frac{1}{2}}\times$$
$$\times\sum_{k=1}^{\infty}D_k\begin{bmatrix}-\beta_{0k}^{-3}\left(g_0 f_k''(\eta)\right)'+\\+\beta_{0k}^{-1}g_2 f_k'(\eta)+\\+\beta_{0k}^{-3}\left(g_1 f_k(\eta)\right)'+O(\varepsilon)\end{bmatrix}\exp\left(\frac{\beta_{0k}}{\varepsilon}\ln\rho\right). \quad (39)$$

The stresses corresponding to the third iterative process are:

$$\sigma_{\theta\theta}^{(3)}=\rho^{-\frac{3}{2}}\sum_{k=1}^{\infty}D_k\begin{bmatrix}-\beta_{0k}b_{22}f_k(\eta)+\\+O(\varepsilon)\end{bmatrix}\exp\left(\frac{\beta_{0k}}{\varepsilon}\ln\rho\right), \quad (40)$$

$$\sigma_{\rho\theta}^{(3)}=\rho^{-\frac{3}{2}}\sum_{k=1}^{\infty}D_k\left[f_k'(\eta)++O(\varepsilon)\right]\exp\left(\frac{\beta_{0k}}{\varepsilon}\ln\rho\right), \quad (41)$$

$$\sigma_{\varphi\varphi}^{(3)}=\rho^{-\frac{3}{2}}\sum_{k=1}^{\infty}D_k\begin{bmatrix}\beta_{0k}^{-1}\dfrac{b_{12}\left(b_{22}-b_{23}\right)}{b_{12}^2-b_{11}b_{22}}f_k''(\eta)+\\+\beta_{0k}\dfrac{\left(b_{11}b_{23}-b_{12}^2\right)}{b_{12}^2-b_{11}b_{22}}f_k(\eta)+O(\varepsilon)\end{bmatrix}\times$$
$$\times\exp\left(\frac{\beta_{0k}}{\varepsilon}\ln\rho\right), \quad (42)$$

$$\sigma_{\rho\rho}^{(3)}=\rho^{-\frac{3}{2}}\sum_{k=1}^{\infty}D_k\left[-\beta_{0k}^{-1}f_k''(\eta)+O(\varepsilon)\right]\exp\left(\frac{\beta_{0k}}{\varepsilon}\ln\rho\right). \quad (43)$$

General solutions (11), (12) will be superpositions of solutions (13), (14), (22), (38), (39) corresponding to the above iterative processes:

$$u_\rho(\rho,\eta)=u_\rho^{(1)}+u_\rho^{(2)}+u_\rho^{(3)}, \quad (44)$$

$$u_\theta(\rho,\eta)=u_\theta^{(1)}+u_\theta^{(2)}+u_\theta^{(3)}. \quad (45)$$

Based on (15) to (18), (24), (25), (40) to (43) for stresses the following is obtained:

$$\sigma_{\varphi\varphi}=\sigma_{\varphi\varphi}^{(1)}+\sigma_{\varphi\varphi}^{(2)}+\sigma_{\varphi\varphi}^{(3)},\quad \sigma_{\theta\theta}=\sigma_{\theta\theta}^{(1)}+\sigma_{\theta\theta}^{(2)}+\sigma_{\theta\theta}^{(3)}, \quad (46)$$

$$\sigma_{\rho\rho}=\sigma_{\rho\rho}^{(1)}+\sigma_{\rho\rho}^{(2)}+\sigma_{\rho\rho}^{(3)},\quad \sigma_{\rho\theta}=\sigma_{\rho\theta}^{(1)}+\sigma_{\rho\theta}^{(2)}+\sigma_{\rho\theta}^{(3)}. \quad (47)$$

The derived asymptotic formulas (44) to (47) make it possible to calculate the stressed-strained state of a transversal-isotropic cone of small thickness with any predetermined accuracy.

5. 2. Analysis of the stressed-strained state determined by homogeneous solutions

An analysis of the stressed-strained state corresponding to various groups of constructed homogeneous solutions is carried out.

The relationship of homogeneous solutions with the main vector P of stresses acting in the cross-section ρ=const is considered.

Note that:

$$P=2\pi\rho^2\varepsilon\int_{-1}^{1}\begin{pmatrix}\sigma_{\rho\rho}\cos\left(\theta_0+\varepsilon\eta\right)-\\-\sigma_{\rho\theta}\sin\left(\theta_0+\varepsilon\eta\right)\end{pmatrix}\sin\left(\theta_0+\varepsilon\eta\right)d\eta. \quad (48)$$

Substituting (47) to (48), the following is derived:

$$P = 2\pi\varepsilon\omega_0 A, \tag{49}$$

where

$$\omega_0 = \left[m_0 \left(p_0 - b_{11}^{(0)} \right) + t_0^2 \right] \cos^2\theta_0 + O(\varepsilon).$$

The stressed state corresponding to the second and third iterative processes is self-balanced in each section ρ=const.

In the ρ=const cross-section, the bending moment and cutting force for the second and third groups of solutions are:

$$M_{bend}^{(2)} = O\left(\varepsilon^2\right), \; Q_{cut}^{(2)} = O\left(\varepsilon^{\frac{3}{2}}\right), \tag{50}$$

$$M_{bend}^{(3)} = O\left(\varepsilon^3\right), \; Q_{cut}^{(3)} = O\left(\varepsilon^2\right). \tag{51}$$

From (50), (51), it is derived that the main parts of the bending moment and the cutting force determine the solution to the corresponding second iterative process.

A solution to (13), (14), corresponding to the first iterative process, determines the internal stressed-strained state of the cone. The first terms of its decomposition by ε determine the momentless tense state.

The solution to (22) has the character of a boundary effect. The first terms of the decomposition by ε of the solution to (22), (24), (25) in conjunction with the first terms (13) to (18) can be considered solutions in applied shell theory.

Solutions to (38), (39) have the character of a boundary layer. The first terms (38), (39) are equivalents to the edge effect of Saint-Venant for a heterogeneous transversal-isotropic plate [25]. For imaginary β_{0k}, the boundary layer solution is attenuated very weakly. In this case, the stressed-deformed state of the transversal-isotropic inhomogeneous cone is qualitatively different from the state of the isotropic inhomogeneous cone. When β_{0k} are real or complex, the overall picture of the stressed-strained state is qualitatively similar to the corresponding pattern for an isotropic inhomogeneous cone [14], and they differ in the rate of attenuation of the Saint-Venant border layers.

It is assumed that at the ends of the cone (on the spherical part of the boundary) the the following boundary conditions are set:

$$\sigma_{\rho\rho}\big|_{\rho=\rho_j} = f_{1j}(\eta), \tag{52}$$

$$\sigma_{\rho\theta}\big|_{\rho=\rho_j} = f_{2j}(\eta), \tag{53}$$

where $f_{1j}(\eta)$, $f_{2j}(\eta)$ (j=1, 2) are fairly smooth functions that satisfy equilibrium conditions.

The relation of the constant A to the principal vector P is given by the equality:

$$A = \frac{P}{2\pi\varepsilon\omega_0}. \tag{54}$$

To determine the unknown constants B_j (j=1, ..., 4), D_k (k=1, 2, ...), the variational Lagrange principle [21–23] was applied. Since homogeneous solutions satisfy the equilibrium equation and boundary conditions on the lateral surface, the variational principle takes the following form:

$$\rho_1^2 \int_{-1}^{1} \left[\begin{gathered} \left(\sigma_{\rho\rho} - f_{11}(\eta)\right)\delta u_\rho + \\ + \left(\sigma_{\rho\theta} - f_{21}(\eta)\right)\delta u_\theta \end{gathered} \right]_{\rho=\rho_1} \cdot \sin\left(\theta_0 + \varepsilon\eta\right) d\eta +$$

$$+ \rho_2^2 \int_{-1}^{1} \left[\begin{gathered} \left(\sigma_{\rho\rho} - f_{12}(\eta)\right)\delta u_\rho + \\ + \left(\sigma_{\rho\theta} - f_{22}(\eta)\right)\delta u_\theta \end{gathered} \right]_{\rho=\rho_2} \cdot \sin\left(\theta_0 + \varepsilon\eta\right) d\eta = 0. \tag{55}$$

Bearing in mind that $\sigma_{\rho\rho}^{(2)} = O(1)$, $\sigma_{\rho\theta}^{(2)} = O\left(\varepsilon^{1/2}\right)$, the assumptions regarding the external load have been clarified. Let's say that $f_{1s}=O(1)$.

After certain transformations, the following is derived:

$$\int_{-1}^{1} \sigma_{\rho\theta} d\eta = \varepsilon^{\frac{1}{2}} \rho^{-\frac{3}{2}} \times$$

$$\times \sum_{j=1}^{4} B_j \left\{ \begin{gathered} \alpha_{0j}\left(b_{11}^{(0)} - p_0\right) q_j \alpha_{0j} \times \\ \times \left[\begin{gathered} \alpha_{0j}^2 \left(b_{11}^{(0)} - b_{11}^{(2)} - p_0 + p_2 \right) + \\ + \left(t_0 - t_1\right) \mathrm{ctg}\theta_0 \end{gathered} \right] + \\ + \alpha_{0j}^2 q_j^2 \left(p_0 - p_1 - b_{11}^{(0)} + b_{11}^{(1)} \right) + O\left(\varepsilon^{1/2}\right) \end{gathered} \right\} \times$$

$$\times \exp\left(\frac{\alpha_{0j}}{\sqrt{\varepsilon}} \ln\rho \right). \tag{56}$$

Tangential stresses specified on the spherical parts of the boundary are represented as:

$$f_{2s}(\eta) = f_{2s}^{(1)} + f_{2s}^{(2)},$$

where

$$f_{2s}^{(1)} = \frac{1}{2} \int_{-1}^{1} f_{2s}(\eta) d\eta; \; f_{2s}^{(2)} = f_{2s} - f_{2s}^{(1)}.$$

Based on (56), it is derived that $f_{2s}^{(1)} = O\left(\varepsilon^{1/2}\right)$. Then $f_{2s}^{(2)}$ can have the order $f_{2s}^{(2)}$ $O(1)$. Thus,

$$f_{1s} = O(1), \;\; f_{2s}^{(1)} = O\left(\varepsilon^{1/2}\right), \; f_{2s}^{(2)} = O(1). \tag{57}$$

Substituting (44), (45), (47) in (55) and considering δB_j, δD_j to be independent variations, systems of linear algebraic equations are derived from (55):

$$\sum_{j=1}^{4} m_{jk} B_{j0} = h_k^{'}; \;\; \left(k = \overline{1,4}\right), \tag{58}$$

$$\sum_{n=1}^{\infty} g_{kn} D_{n0} = h_k^{''}; \;\; \left(k = 1, 2, ...\right), \tag{59}$$

where

$$m_{jk} = \left[\begin{gathered} \alpha_{0j}\alpha_{0k}^2 \left(b_{11}^{(0)} - p_0\right) \left(\begin{gathered} p_0 + p_1 - \\ -b_{11}^{(0)} - b_{11}^{(1)} \end{gathered} \right) q_k q_j^2 + \\ + \alpha_{0k}\alpha_{0j}^2 \left(b_{11}^{(0)} - p_0\right)^2 \tau_{2j} q_k q_j + \\ + \alpha_{0k}\alpha_{0j}^2 \left(b_{11}^{(0)} - p_0\right) \left(\begin{gathered} p_0 - p_1 - \\ -b_{11}^{(0)} + b_{11}^{(1)} \end{gathered} \right) q_k q_j^2 - \\ - \alpha_{0j}\alpha_{0k}^2 \left(b_{11}^{(0)} - p_0\right)^2 \tau_{1j} q_k q_j \end{gathered} \right] \times$$

$$\times \sum_{s=1}^{2} \exp\left(\frac{\left(\alpha_{0k} + \alpha_{0j}\right)}{\sqrt{\varepsilon}} \ln\rho_s \right);$$

$$h_k^{'}=\sum_{s=1}^{2}\rho_s^{3/2}\int_{-1}^{1}\left\{\begin{array}{l}\gamma_{1s}^{(0)}(\eta)\left[\begin{array}{l}\alpha_{0k}^2 q_k\left(p_0-b_{11}^{(0)}\right)\times\\ \times(\eta+1)+q_k^2\end{array}\right]+\\ +\alpha_{0k}q_k\left(b_{11}^{(0)}-p_0\right)f_{2s}^{(1)}(\eta)\end{array}\right\}d\eta\times$$
$$\times\exp\left(\frac{\alpha_{0k}}{\sqrt{\varepsilon}}\ln\rho_s\right);$$

$$\tau_{1j}=\alpha_{0j}^2\left(\begin{array}{l}p_2+2p_1+p_0-\\ -b_{11}^{(2)}-2b_{11}^{(1)}-b_{11}^{(0)}\end{array}\right)-\left(t_0+t_1\right)\mathrm{ctg}\theta_0;$$

$$\tau_{2j}=\alpha_{0j}^2\left(b_{11}^{(0)}-b_{11}^{(2)}-p_0+p_2\right)+\left(t_0-t_1\right)\mathrm{ctg}\theta_0;$$

$$\gamma_{1s}^{(0)}=f_{1s}(\eta)-\frac{A}{\rho_s^2}\left[\frac{m_0}{g_0}+t_0\frac{b_{12}\left(b_{23}-b_{22}\right)}{b_{22}}\right]\mathrm{ctg}\theta_0;$$

$$g_{kn}=\int_{-1}^{1}\left\{\begin{array}{l}-\beta_{0n}^{-1}f_n^{''}(\eta)\left[\beta_{0k}^{-2}g_0f_k^{''}(\eta)-g_1f_k(\eta)\right]+\\ +f_n^{'}(\eta)\left[\begin{array}{l}\beta_{0k}^{-1}\left(g_1f_k(\eta)\right)'+\beta_{0k}^{-1}g_2f_k^{'}(\eta)-\\ -\beta_{0k}^{-3}\left(g_0f_k^{''}(\eta)\right)'\end{array}\right]\end{array}\right\}d\eta\times$$
$$\times\sum_{s=1}^{2}\exp\left(\frac{\left(\beta_{0n}+\beta_{0k}\right)}{\varepsilon}\ln\rho_s\right);$$

$$h_k^{''}=\sum_{s=1}^{2}\rho_s^{3/2}\exp\left(\frac{\beta_{0k}}{\varepsilon}\ln\rho_s\right)\times$$
$$\times\int_{-1}^{1}\left\{\begin{array}{l}\gamma_{1s}^{(0)}(\eta)\left[\beta_{0k}^{-2}g_0f_k^{''}(\eta)-g_1f_k(\eta)\right]+\\ +f_{2s}(\eta)\left[\begin{array}{l}-\beta_{0k}^{-3}\left(g_0f_k^{''}(\eta)\right)'+\\ +\beta_{0k}^{-1}g_2f_k^{'}(\eta)+\\ +\beta_{0k}^{-1}\left(g_1f_k(\eta)\right)'\end{array}\right]\end{array}\right\}d\eta;$$

$$B_j=B_{j0}+\varepsilon^{\frac{1}{2}}B_{j1}+....;\; D_n=D_{n0}+\varepsilon D_{n1}+... .$$

System (59) is always solvable under physically meaningful conditions superimposed on the right-hand side [28].

Determining B_{jp}, D_{np} (p=1, 2, ...) is invariably reduced to systems whose matrices coincide with the matrices of systems (58), (59).

5. 3. Homogeneous solutions for a heterogeneous transversal-isotropic cone in the degeneration of its median surface into a plane

A special case was noted when the opening angle of the median surface is $\theta_0=\pi/2$, which corresponds to a transversal-isotropic inhomogeneous plate of variable thickness. Here, not any plate is meant but that particular type of the cone considered in [18–22], which it takes when its median surface degenerates into a plane. This case of degeneration is special and requires a separate study.

In the case $\theta_0=\pi/2$ for solution (11), (12) based on the asymptotic method at $\varepsilon\to 0$ three groups of solutions are derived:

$$1)\left\{\begin{array}{l}u_\rho^{(1)}(\rho,\eta)=\varepsilon\rho^{-1}\tilde{D}\left[\begin{array}{l}-2\left(m_0+t_0\right)\eta+\\ +2t_1+m_1+O(\varepsilon)\end{array}\right],\\ u_\theta^{(1)}(\rho,\eta)=\tilde{D}\rho^{-1}\left[-\left(m_0+t_0\right)+O(\varepsilon)\right].\end{array}\right. \quad (60)$$

These solutions correspond to the eigenvalue $z_0^{(1)}=-\frac{1}{2}$;

$$2)\left\{\begin{array}{l}u_\rho^{(2)}(\rho,\eta)=\rho^{-\frac{1}{2}}\sum_{k=1}^{4}C_{2k}u_{\rho k}^{(2)}(\rho,\eta),\\ u_\theta^{(2)}(\rho,\eta)=\rho^{-\frac{1}{2}}\varepsilon^{-1}\sum_{k=1}^{4}C_{2k}u_{\theta k}^{(2)}(\rho,\eta),\end{array}\right. \quad (61)$$

where

$$u_{\rho k}^{(2)}(\rho,\eta)=$$
$$=\left\{\begin{array}{l}\left(\frac{3}{2}-z_{0k}\right)\left[\begin{array}{l}\left(z_{0k}^2-\frac{1}{4}\right)\left(p_0-b_{11}^{(0)}\right)+\\ +t_0+m_0\end{array}\right]\times\\ \times(\eta+1)+\\ +\left(z_{0k}-\frac{3}{2}\right)\left(m_0+m_1+t_1+t_0\right)-\\ -\left(z_{0k}-\frac{3}{2}\right)\left(z_{0k}^2-\frac{1}{4}\right)\times\\ \times\left(b_{11}^{(1)}-p_1+b_{11}^{(0)}-p_0\right)+\\ +\left(z_{0k}-\frac{1}{2}\right)t_1+m_1+O(\varepsilon)\end{array}\right\}\times$$
$$\times\exp\left(z_{0k}\ln\rho\right),$$

$$u_{\theta k}^{(2)}(\rho,\eta)=\left[\begin{array}{l}\left(z_{0k}^2-\frac{1}{4}\right)\left(p_0-b_{11}^{(0)}\right)+\\ +t_0+m_0+O(\varepsilon)\end{array}\right]\exp\left(z_{0k}\ln\rho\right).$$

Solutions to (61) correspond to the following eigenvalues

$$z_k=z_{0k}+\varepsilon^2z_{2k}+.... \quad (62)$$

Here, z_{0k} is the solution to the biquadrate equation

$$l_0z_{0k}^4+l_1z_{0k}^2+l_2=0, \quad (63)$$

where

$$l_0=\left(b_{11}^{(1)}-p_1+b_{11}^{(0)}-p_0\right)\times$$
$$\times\left(p_0-p_1-b_{11}^{(0)}+b_{11}^{(1)}\right)+\left(b_{11}^{(0)}-p_0\right)\times$$
$$\times\left(p_2-p_0-b_{11}^{(2)}+b_{11}^{(0)}\right);$$

$$l_1=\left(b_{11}^{(0)}-p_0\right)\left(3t_2+m_2-m_0-t_0\right)+$$
$$+\left(b_{11}^{(1)}+b_{11}^{(0)}-p_0-p_1\right)\times$$
$$\times\left(t_0+m_0-m_1-2t_1\right)-$$
$$-\left(b_{11}^{(1)}-b_{11}^{(0)}-p_1+p_0\right)\times$$
$$\times\left(2t_1+m_0+t_0+m_1\right)-$$
$$-\left(t_0+m_0\right)\left(p_2-b_{11}^{(2)}+b_{11}^{(0)}-p_0\right)-$$
$$-\frac{5}{2}\left(b_{11}^{(0)}-p_0\right)\left(p_2-b_{11}^{(2)}+b_{11}^{(0)}-p_0\right)-$$
$$-\frac{5}{2}\left(p_0-p_1-b_{11}^{(0)}+b_{11}^{(1)}\right)\times$$
$$\times\left(b_{11}^{(1)}+b_{11}^{(0)}-p_0-p_1\right);$$

$$\begin{aligned}
l_2 = & \left(p_0 - p_1 - b_{11}^{(0)} + b_{11}^{(1)}\right)\left(\frac{3}{2}t_1 + \frac{9}{4}m_0 + \frac{9}{4}t_0 + \frac{3}{4}m_1\right) + \\
& + \left(b_{11}^{(0)} + b_{11}^{(1)} - p_0 - p_1\right) \times \\
& \times \left(\frac{3}{4}m_1 + \frac{3}{2}t_1 - \frac{9}{4}t_0 - \frac{9}{4}m_0\right) + \left(p_2 - p_0 - b_{11}^{(2)} + b_{11}^{(0)}\right) \times \\
& \times \left(\frac{9}{16}b_{11}^{(0)} - \frac{9}{16}p_0 + \frac{9}{4}m_0 + \frac{9}{4}t_0\right) + \\
& + \left(b_{11}^{(0)} - p_0\right)\left(\frac{9}{4}m_0 + \frac{9}{4}t_0 - \frac{1}{4}m_2 - \frac{15}{4}t_1 - \frac{3}{4}t_2\right) + \\
& + \frac{9}{16}\left(b_{11}^{(1)} + b_{11}^{(0)} - p_0 - p_1\right) \times \\
& \times \left(p_0 - p_1 - b_{11}^{(0)} + b_{11}^{(1)}\right) + 4t_1^2 + 4m_1 t_1 + m_1^2 - \\
& - m_0 m_2 - 3m_0 t_2 - t_0 m_2 - 3t_0 t_2.
\end{aligned}$$

The second iterative process is absent here, i.e., there is no solution that has the character of an edge effect.

The displacements and stresses corresponding to the third iterative process take the form (38) to (43).

After a certain transformation from (48) for a transversal-isotropic inhomogeneous plate of variable thickness, the following is derived:

$$P = -2\pi\varepsilon^3 \tilde{D}\tilde{\omega}_0, \qquad (64)$$

where

$$\begin{aligned}
\tilde{\omega}_0 = & \left(2m_0 + 2t_0 + 2t_1 + m_1\right) \times \\
& \times \left(p_1 - 2t_1 + t_0 + m_0 - m_1 - b_{11}^{(1)}\right) - \\
& - \left(m_0 + t_0\right)\begin{pmatrix} 2p_1 - 2t_1 + 2p_2 - 3t_2 + \\ +2m_0 + 2t_0 - m_1 - \\ -m_2 - 2b_{11}^{(1)} - 2b_{11}^{(2)} \end{pmatrix} + O(\varepsilon).
\end{aligned}$$

The stressed state corresponding to the eigenvalue $z_0^{(1)} = -\dfrac{1}{2}$, is equivalent to the main vector of forces directed along the axis of symmetry. Solutions (61) define the main parts of the bending moment and the cutting force. The third asymptotic process defines solutions having the character of a boundary layer and the first terms of the asymptotic expansions of these solutions coincide with solutions of the boundary layer type for an inhomogeneous transversal-isotropic plate of constant thickness.

At $\theta_0=\pi/2$, the overall solution to (11), (12) will be a superposition of solutions (60), (61), (38), (39).

6. Discussion of results on the construction of a homogeneous solution to the problem of elasticity theory for a heterogeneous transversal-isotropic cone

The axisymmetric problem of the theory of elasticity for a heterogeneous transversal-isotropic truncated hollow cone of small thickness is investigated. It is assumed that the moduli of elasticity are arbitrary continuous functions of the cone opening angle whose values vary within the same order. A complete system of equations is given and the boundary problem (8), (9) is stated. By the method of asymptotic integration of the equations of the theory of elasticity, asymptotic decompositions (44), (45) for homogeneous solutions are constructed, which make it possible to calculate the stressed-strained state under various boundary conditions at the ends of the cone. Homogeneous solutions consist of three types: penetrating solutions, solutions such as simple edge effect, and boundary layer solutions.

An analysis of the stressed-strained states corresponding to various types of homogeneous solutions is carried out. It is shown that the first asymptotic process corresponds to some penetrating solution (13), (14). The stressed state determined by this solution is equivalent to the main vector of the forces applied in the arbitrary cross-section ρ=const. The second asymptotic process determines the solution to (22) of the type of edge effect, similar to the edge effect in applied shell theory. The first terms of the asymptotic stressed state decompositions determined by solution (22) are equivalent to the bending moment and cutting forces. In the first terms of the derived solution based on the first and second asymptotic process, decomposition by parameter ε can be considered as solutions to the applied theory of Kirchhoff-Love. The third asymptotic process corresponds to the solution of the type of boundary layer, which in the first term of asymptotic decomposition coincides with the boundary effect of Saint-Venant in the theory of inhomogeneous plates. Solutions to (38), (39), corresponding to the third asymptotic process, are localized at the ends, and, as they move away from the ends of the cone, they decrease. Solutions of the type of boundary layer corresponding to the imaginary β_{0k} can penetrate deeply enough and significantly change the picture of the stressed-strained state away from the ends of the cone. In this case, the stressed-strained state of the transversal-isotropic inhomogeneous cone is qualitatively different from the state of the isotropic inhomogeneous cone. Boundary layer type solutions, appropriate, real, or complex β_{0k} quickly fade away from the ends of the cone. When the β_{0k} are real or complex, the overall pattern of the stressed-strained state is qualitatively similar to the corresponding picture for an isotropic inhomogeneous cone, and they differ in the rate of attenuation of the border layers of Saint-Venant. Solutions to (38), (39), having the character of a boundary layer, are absent in applied theories of the shell. The stressed state corresponding to the second and third groups is self-balanced in each section ρ=const. Using the variational Lagrange principle, the boundary conditions at the ends of the cone are satisfied.

A particular type of cone has been studied during the degeneration of its median surface into a plane, that is, a heterogeneous plate of variable thickness. Given that this case of degeneration is special, based on the asymptotic method, homogeneous solutions have been constructed and an analysis of the stressed-strained state of a heterogeneous plate of variable thickness has been carried out. It is obtained that the stressed state corresponding to the first iterative process is equivalent to the main vector of forces directed along the axis of symmetry, the cutting force and bending moments. Solutions corresponding to the next iterative process quickly fade away from the edge of the plate and have the character of a boundary layer. The stressed-strained state in an inhomogeneous plate of variable thickness consists of a penetrating solution (60), (61) and a solution (38), (39), which has the character of a boundary layer.

It should be noted that the division of the stressed-strained state into internal and border layer solutions is characteristic only of a cone of small thickness. The resulting asymptotic formulas hold at $\varepsilon\to 0$.

The method of asymptotic integration is successfully used and has no drawbacks in solving the problem of elasticity theory for elastic bodies of small thickness. However, this method does not make it possible to correctly solve the problem of elasticity theory for a thick elastic body. For the first time, the problem of elasticity theory for a transversally isotropic inhomogeneous cone of small thickness was investigated and the method of asymptotic integration was used.

The results reported here are a generalization of certain results from [9, 10, 14, 15, 17–19]. In the case $b_{12}=b_{23}=\lambda$, $b_{44}=G$, $b_{11}=b_{22}=2G+\lambda$, all the solutions derived fully coincide with the solutions for the radial-inhomogeneous isotropic cone [14, 19]. When G and λ are constant, the solutions derived coincide with the solutions for the isotropic cone [9, 10, 18]. In particular, when b_{12}, b_{23}, b_{44}, b_{11}, b_{22} are constant, all the results derived for the transversal-isotropic cone follow from the constructed solutions [17]. Constructed asymptotic formulas for the displacements and stresses of a heterogeneous transversal-isotropic cone are valid, in particular, for a transversal-isotropic inhomogeneous cylinder.

The current work is theoretical in nature. Asymptotic formulas (44) to (47) make it possible to calculate the stressed-strained state of a transversal-isotropic cone of small thickness with any predetermined accuracy.

In the future, based on the derived asymptotic decompositions for displacements and stresses, the determination of the applicability of existing applied theories for a transversal-isotropic conical shell of small thickness will be considered.

7. Conclusions

1. Based on the asymptotic analysis, three groups of homogeneous solutions were derived: a penetrating solution; a solution that has the character of a marginal effect; a solution that has the character of a boundary layer. New classes of solutions (solutions having the character of a boundary layer) that are absent in applied theories are obtained. Asymptotic formulas for movements and stresses have been constructed, which make it possible to calculate the three-dimensional stressed-strained state of a non-uniform transversal-isotropic cone of small thickness. The constructed solutions could be a reference for assessing the accuracy of various applied theories for heterogeneous transversal-isotropic conic shells. Based on the constructed solutions and the analyses carried out, it is possible to propose a new refined applied theory for the transversal-isotropic inhomogeneous cone.

2. The features of the stressed-strained state of a heterogeneous transversal-isotropic cone of small thickness are revealed. It is shown that the stressed state determined by the penetrating solution is equivalent to the main vector of the forces applied in the arbitrary section ρ=const. The stressed state determined by the solution having the character of a boundary effect and the boundary layer is self-balanced in each section ρ=const. It is shown that, unlike the isotropic inhomogeneous cone, weakly decaying border layer solutions appear, which are characteristic only of a transversal-isotropic inhomogeneous cone of small thickness. For a heterogeneous transversal-isotropic cone, these solutions attenuate very weakly and can penetrate deep away from the ends and change the pattern of the stressed-strained state.

3. The features of the stressed-strained state, a particular type of inhomogeneous transversal-isotropic cone of small thickness during the degeneration of its median surface into a plane, have been studied. It is established that this case of degeneration is a special and the tense-deformed state consists of a penetrating solution and a solution that has the character of a boundary layer.

Conflicts of interest

The author declares that he has no conflicts of interest in relation to the current study, including financial, personal, authorship, or any other, that could affect the study and the results reported in this paper.

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Data availability

All data are available in the main text of the manuscript.

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