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NUMERICAL SOLUTION OF THE CONTROL PROBLEM ON THE DEPLETION OF GAS RESERVOIRS WITH WEAKLY PERMEABLE TOP

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The current stage in the development of mathematical and software support for the processes of designing the development of hydrocarbon fields is characterized not only by the improvement of the means of geological and hydrodynamic modeling of reservoir fluid filtration but also by the use of algorithms for optimizing the development of gas deposits. The paper considers the problem of optimal control of the depletion of a gas reservoir with a low-permeability top. Using the so-called Myatiev-Girinsky hydraulic scheme, a two-dimensional equation describing the unsteady gas flow in a reservoir with a jumper is averaged over the capacity of the productive reservoir. This comes down to a one-dimensional equation with an additional term, taking into account gas-dynamic relationships between the reservoir and the jumper. For the numerical solution of process control problems, a formula for the gradient of the functional characterizing the reservoir depletion is found, and the method of successive approximations based on Pontryagin's maximum principle is applied. In this case, the direct and conjugate boundary value problems are solved by the method of straight lines, and the required flow rate, without taking it beyond the maximum and minimum possible, is found by the gradient projection method with a special choice of step. A brief block diagram of the algorithm for solving the problem is shown; on its basis, a computer program was compiled. The results of calculations are presented to identify the influence of the values of the complex communication parameter not only on the state of the object but also on the operating mode of the well. The expediency of using the presented optimization tool is dictated by an increase in the share of deposits

Keywords: Myatiev-Girinsky scheme, gas reservoir, optimal control, gradient method, maximum principle

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1. Introduction

When studying control problems associated with the depletion of a gas reservoir with a low-permeable top (or bottom), we have to consider possible flows of filtering gas from one horizon to another, which greatly complicates the mathematical solution to the problem. Bearing in mind that the reservoir in the vertical section consists of several layers, then when solving the problem of optimal control associated with the choice of well flow rate, it is necessary to fix a system of several two-dimensional equations under certain boundary conditions set on wells and reservoir boundaries. At the same time, the more layers in the section, the more equations and, therefore, the more difficult to obtain a solution to the optimization problem described by these equations. An analysis of gas-bearing reservoir recovery, i.e. a decline curve, shows that future productivity is well described by fitting an exponential equation to the rate of decline in gas-bearing layer productivity over time [1]. This kind of approach gives good results in the case of high reservoir productivity but does not adequately reflect the behavior of some production wells in depleted reservoirs. In [2], differential equations describing the productivity of gas-bearing layers are approximated by difference equations that are implicit for pressure and saturation and explicit for relative permeability. The joint solution of difference equations is obtained using either alternating directions or strongly implicit iterative procedures. Many authors agree that the solution of two-dimensional boundary value problems of parabolic type, describing the process of gas filtration, turns into a complex problem [3]. As seen, it is impossible to obtain exact solutions to these equations, even in the simplest case of filtration. Therefore, when compiling the basic differential equations, an approximate hydraulic method is often used, which is called the Myatiev-Girinsky scheme [4]. According to this scheme, the reservoir in the vertical section consists of several pore layers, where well-permeable layers alternate with low-permeable ones. In well-permeable formations, the vertical velocity component is neglected, assuming the flow to be horizontal, and in low-permeability formations, the horizontal filtration component is neglected, and the flow is assumed to be vertical. Based on this assumption, which does not introduce significant distortions into the flow pattern, according to the Myatiev-Girinsky scheme the vertical pressures in the reservoirs are averaged, taking into account the presence of low-permeability interlayers, and thereby the basic differential equations for gas flow only in the reservoir are obtained. These equations involve additional terms that take into account the weak permeability of the overlying and underlying layers. The degree of accuracy of the described scheme depends on the ratio of the permeability of productive and low-permeable formations.

Thus, in the presence of gas-dynamic connections between the reservoirs and a weakly permeable jumper, the distribution of gas pressure in a productive reservoir is reduced to solving nonlinear boundary value problems for partial differential equations under appropriate initial and boundary conditions. As boundary conditions, the impermeability of the outer boundaries of the productive formation and the conditions for production wells are set. The exchange processes occurring during the development of a multilayer field are taken into account by additional terms in the right parts of the equations.

Known exact and approximate solutions to non-stationary problems of liquid and gas filtration in reservoirs with a low-permeability jumper are obtained under the assumptions of several simplifications, the need for which is explained by the complexity of the corresponding boundary value problems. The complexity increases immeasurably when considering problems with a moving gas-water interface.

Examples of using the above scheme for solving a number of problems of stationary and non-stationary filtration of liquid and gas in reservoirs separated by a low-permeability jumper can be found in [5, 6]. Studies devoted to problems describing the flow of fluid in isotropic and anisotropic reservoirs, as well as in a multilayer system under elastic conditions, are of scientific relevance. At the same time, a mathematical model based on calculation formulas for well flow rates subsequently has high practical applicability. Such models define the required differential equations with partial derivatives, describing the filtration processes in reservoirs separated with a low-permeability jumper between the layers.

2. Literature review and problem statement

In [7], using the Gauss-Seidel iterative block-type method, numerical methods are developed for jointly solving the problems of geo-filtration and geo-migration in multilayer systems in the study of the transport of impurities. A multilayer system consists of several aquifers separated by low-permeability layers. Mathematical models according to the Myatiev-Girinsky scheme are built under the following assumption: longitudinal currents prevail in the aquifer, and transverse ones predominate in the separating layers. Implicit finite-volume difference schemes are used.

In [8], for problems of optimal control and forecasting of production processes in gas fields, the processes in which are described by two-dimensional equations of parabolic type, mathematical and computer models are studied, and computational algorithms are created. Nevertheless, the convergence issues related to the solution of the grid analog of the optimal control problem under consideration have not been studied, the structure of the optimal control software is given, but the calculation results are not presented.

In [9], the actual problem associated with the development of oil and gas fields in order to increase the gas recovery of reservoir systems and determine the main indicators of the object of study is considered. An analysis of scientific papers related to the problem of mathematical modeling of the process of gas filtration in a reservoir is given. To conduct a comprehensive study of the process under consideration, a mathematical model was developed based on the basic laws of hydromechanics. The proposed numerical method for solving the problem can be easily generalized for a system of three or more equations. The developed mathematical tools can be used to analyze and develop multi-layer gas fields in the presence of a gas-dynamic connection between the layers and make management decisions. On the basis of the proposed mathematical tool, computational experiments are carried out, the results are presented in the form of graphic objects, and analysis is given.

In [10], based on Pontryagin's maximum principle, the SMAC (Sequential Model-based Algorithm Configuration) algorithm is developed, which automatically determines the intelligent control that maximizes the net present value of the production process. The idea is to build an auto-adaptive parameterized decision tree that replaces arbitrarily chosen limit values for selected decision tree attributes with parameters. A new tool has been developed linking the parameterized decision tree to the reservoir simulator and optimization tool. The created tool allowed to increase revenue by 49 %.

In [11], second-order optimality conditions are studied for optimal fuel control problems with both ends on manifolds. Each locally optimal candidate extremal from Pontryagin's maximum principle is embedded in a parametrized family of extremals with classical second-order conditions for the absence of conjugate points or focal points. Two non-multiple conditions are developed for the projection of a parametrized family onto the state space to be a diffeomorphism. Thus, non-multiplicity conditions are sufficient (but not necessary) to ensure that a candidate extremal is locally optimal if the initial state is fixed.

The paper [12] considers a control problem in which the equation of state is described by a nonlinear partial differential equation. The authors examine the situation of control at the frontiers with the assignment of the corresponding boundary conditions. This paper presents the necessary and sufficient optimality conditions in the form of Pontryagin's maximum principle. Optimal control in the case when the control acts linearly is proved. However, the case of dynamic control is not studied in the work.

The paper [13] describes the optimal control method for maximizing oil revenues in oil reservoir systems. The fluid flow in an oil reservoir is represented by a system of non-linear partial differential equations of the second order. The model describes the interaction between the well and the reservoir; the paper pre-sets the boundary conditions for the fluid flow equations. At the same time, the flow rate is regulated by changing the bottom hole pressure. The presented method of increasing the recovery from the well takes into account the content of undesirable fluids, such as water or gas. The debit gradient is calculated using the adjoint method, and the optimal control setting is obtained using the linear search method, which slightly increases the time for finding the optimal solution.

The manuscript [14] presents a necessary and sufficient condition for an optimal control problem with distributed parameters based on Pontryagin's maximum principle, submitted by a second-order hyperbolic equation. The results obtained can be used in the theory of optimal processes for various controlled processes described by second-order hyperbolic equations. However, the paper mainly considers the case of boundary control, since in many cases it is possible to influence the course of the process only from the boundary of the examined area.

The purpose of the study [15] is to search for new indicators characterizing the mechanism for increasing reservoir productivity. The paper presents a method for early determination of reservoir drive. The research results diagnose a strict relationship between the reservoir operation mode and the ratio of porosity volume to reservoir pressure. New diagnostic indicators have been discovered to help identify the drive mechanism, called Reservoir Drive Performance Indexes (RDPIs). Despite the great promise of this approach, it is not entirely applicable to depleted wells.

The paper [16] presents a new empirical model for estimating Pd pressure for gas condensate reservoirs. Statistical error analysis was used to determine the accuracy of the model. The results of the proposed model were compared with the Soave-Redlich-Kwong equation of state (SRK-EOS) and the Peng-Robinson equation of state (PR-EOS). Gas condensate samples were used to verify the validity of the proposed model in relation to the equation of state. The proposed method requires large computing power.

The paper [17] presents the results of a research project aimed at developing heuristically driven development plans based on simple static simulation parameters rather than complex dynamic simulation parameters for traditional gas reservoirs. The combined use of heuristics based on the Maximum Efficient Rate (MER) criteria and the Maximum Depletion Rate Model (MDRM) was explored to model a systems approach to these reservoirs. The integration of these principles and their comparative evaluation led to conclusions about their relationship, in particular, that MER should be considered a special case of MDRM. The rules are based on the combined use of an exponential version of the Fundamental Equation of Mineral Production (EFE) with the exponential decline (ED) curve analysis. However, the presented production model is based on fairly accurate data and rules, which does not allow going beyond the prescribed volumes of reserves and production yields.

As can be seen from the review of the literature, the problems of unsteady gas flow are considered from different points of view, however, the experimental study of such flows is often associated with significant difficulties and costs. In most published works, the issue of simplifying multidimensional equations describing the control problem, as well as the issue of the applicability of the models used for depleted wells, is left without consideration. Also, attention is often not paid to the quality of the obtained numerical solution, its independence from circuit factors, algorithmic computability and software implementation.

To accurately take into account the behavior of gas flow in a multilayer medium, it is necessary to provide a numerical solution that simulates not only the state of the object, but also the well operation mode. This means the need to use very dense computational grids over the entire computational domain. These requirements under the conditions of non-stationary multidimensional calculations lead to large expenditures of computational resources, which until recently made such calculations practically impossible. The opportunities that have opened up today, due to the growth of computer power, are changing the situation, and carrying out accurate calculations of unsteady fluid flows, with the accumulation of experience in overcoming possible difficulties of a methodological nature, becomes a feasible task.

3. The aim and objectives of the study

The aim of the work is to demonstrate the methodology for applying the numerical method of successive approximations to solve problems of optimal control of depleted multilayer wells. This will reduce the significant cost of computational resources in the calculation of complex flows and achieve the required accuracy of the solution with relatively low requirements for the density of the computational grid and time steps. Also, an important part of the proposed solution is the choice of the Myatiev-Girinsky scheme to simplify the equations describing the filtration process.

To achieve this aim, the following objectives were set:

– to apply the Myatiev-Girinsky hydraulic scheme for averaging a two-dimensional equation describing the unsteady gas flow in a reservoir with a weakly permeable jumper;

– to find a formula for the gradient of the functional characterizing reservoir depletion and apply the method of successive approximations based on the Pontryagin maximum principle;

– to develop an algorithm and carry out software implementation to present the results of calculations to identify the influence of the values of the complex communication parameter not only on the state of the object but also on the wells' operation mode.

4. Materials and methods of research

Due to the significant difficulty in obtaining a solution to boundary value problems for two-dimensional differential equations of parabolic type, and consequently, to the associated optimal control problems, various numerical methods for solving are proposed. It is important to note that a particular difficulty in the numerical solution of control problems for processes described by two-dimensional equations of parabolic or hyperbolic type is directly related to the calculation of the gradient of the functional. So, to apply gradient methods at each step of the iterative process, it is necessary to integrate two two-dimensional parabolic boundary value problems – direct and adjoint. At present, Douglas' method and Samarsky's method in [6] have received wide distribution for the integration of these boundary value problems.

In order to avoid the above difficulties, this paper uses a different approach based on the reduction of a two-dimensional equation describing the process of non-stationary filtration in a reservoir with a low-permeability roof to a one-dimensional equation. In this case, the problems of the filtration theory can be considered two-dimensional, and the conditions of indivisibility of flow and continuity of pressure must be observed at the boundary of the reservoir and the jumper. The solution to two-dimensional boundary value problems and related optimal control problems is currently difficult.

To avoid this difficulty, using the Myatiev-Girinsky hydraulic scheme, a one-dimensional differential equation is derived that describes the filtration process in a productive reservoir with an additional term that takes into account possible gas flows through the top of the reservoir. In the presented paper, the solution to optimal control problems, the processes of which are described by one-dimensional equations, is found by the gradient method based on Pontryagin's maximum principle.

Computer numerical calculations and quantitative estimates of the research results are given. The method indicated in the paper can be used to solve more general problems of optimal control associated with the flow of gas, as well as an elastic fluid in reservoirs that are heterogeneous in terms of reservoir properties and are separated by low-permeability jumpers.

5. Results of the study on the control problem of the gas reservoirs' depletion

5. 1. Application of the Myatiev-Girinsky scheme describing the unsteady gas flow

For the case of unsteady plane-parallel filtration, the distribution of gas pressure in the reservoir is described by Leibenson's differential equation:

$$\frac{2m\mu}{k}\cdot\frac{\partial p}{\partial t}=\frac{\partial^2 p^2}{\partial x^2}+\frac{\partial^2 p^2}{\partial y^2}, \tag{1}$$

with the following boundary conditions:

$$t=0,\quad p=p_0=\text{const}, \tag{2}$$

$$x=0,\quad Q(t)=-vF\frac{p}{p_0}=\frac{kF}{2\mu p_0}\cdot\frac{\partial p^2}{\partial x},\quad x=l,\quad \frac{\partial p^2}{\partial x}=0, \tag{3}$$

$$y=0,\quad \frac{\partial p^2}{\partial y}=0,\quad y=b,\quad \frac{k}{\mu}\cdot\frac{\partial p^2}{\partial y}=-\frac{k_q}{\mu}\cdot\frac{p^2-p_0^2}{b_q}. \tag{4}$$

In (1)–(4), v is a gas filtration rate; F is a filtration area; k, k_q and b, b_q are the coefficients of permeability and capacity of the reservoir and the jumper, respectively; μ is a gas dynamic viscosity coefficient; m is a formation porosity; $Q(t)$ is the given positive values; l is a formation length; t is the time.

The initial condition (2) means that at the initial time, the reservoir was in an undisturbed state, i.e. the pressure at each point of the reservoir was equal to the initial pressure p_{H}. The first boundary condition in (3) shows that the well is being operated with a flow rate $Q(t)$. The second condition in (3) and the first condition in (4) indicate the impermeability of the outer boundary and bottom of the formation, respectively. The second condition in (4) shows that flow continuity is observed at the boundary between the reservoir and the jumper.

Introducing dimensionless variables and parameters:

$$p^*=\frac{p}{p_{\text{н}}},\quad x^*=\frac{x}{l},\quad y^*=\frac{y}{l},$$

$$t^*=\frac{kp_{\text{н}}}{2m\mu l^2}\cdot t,\quad Q^*(t^*)=\frac{2\mu l p_{\text{ат}}}{kFp_{\text{н}}}\cdot Q(t)$$

and discarding the asterisks for simplicity, we represent (1)–(4) in the form:

$$\frac{\partial p}{\partial t}=\frac{\partial^2 p^2}{\partial x^2}+\frac{\partial^2 p^2}{\partial y^2}, \tag{5}$$

$$t=0,\quad p=\text{const}=1, \tag{6}$$

$$x=0,\quad \frac{\partial p^2}{\partial x}=Q(t),\quad x=1,\quad \frac{\partial p^2}{\partial x}=0, \tag{7}$$

$$y=0,\quad \frac{\partial p^2}{\partial y}=0,\quad y=b,\quad \frac{k}{\mu}\cdot\frac{\partial p^2}{\partial y}=-\frac{lk}{\mu}\cdot\frac{p^2-1}{b}. \tag{8}$$

We assume l=1 and average equation (5) according to the above scheme. To do this, we multiply all the terms of equation (5) by $1/b$ and integrate (1) over y from 0 to b. Denoting:

$$P=\frac{1}{b}\int_0^b p\,\mathrm{d}y,$$

we get:

$$\frac{\partial P}{\partial t}=\frac{\partial^2 p^2}{\partial x^2}+\frac{1}{b}\int_0^b\frac{\partial^2 p^2}{\partial y^2}\mathrm{d}y.$$

Taking into account the second condition in (8), we have:

$$\frac{\partial p}{\partial t}=\frac{\partial^2 p^2}{\partial x^2}-a\cdot\left(p^2-1\right), \tag{9}$$

where $a=k_q/kbb_q$ – connection coefficient [5].

Note that due to the poor permeability of the jumper, p and P little differ from each other. Therefore, in (9) and in what follows, we will write P instead of p.

The boundary conditions for equations (9) have the form:

$$t=0,\quad p=\text{const}=1, \tag{10}$$

$$x=0,\quad \frac{\partial p^2}{\partial x}=Q(t),\quad x=1,\quad \frac{\partial p^2}{\partial x}=0. \tag{11}$$

The optimal control problem is formulated as follows. It is required to choose a piecewise-continuous control $Q(t)$ with a constraint $Q_1\le Q(t)\le Q_2$ so that at the end of the formation depletion process with a low-permeability top, the pressure $p(x,T)$ in the productive formation approaches the predetermined and based on technological considerations distribution $p^0(x)$.

As a measure of such a deviation, we take the quadratic functional:

$$I=\int_0^1\left[p(x,T)-p^0(x)\right]^2\mathrm{d}x. \tag{12}$$

The boundary value problem (9)–(11) with fixed values will be called the direct problem. To calculate the first variation of functional (9), we compose the Lagrange function of problem (5)–(9):

$$L=I+\int_0^1\int_0^T\psi(x,t)\left[\frac{\partial p}{\partial t}-\frac{\partial^2 p^2}{\partial x^2}+a\cdot\left(p^2-1\right)\right]\mathrm{d}x\mathrm{d}t, \tag{13}$$

where $\psi(x, t)$ is a Lagrange multiplier. Obviously, the extrema of the functionals and I coincide if the connection equations are satisfied.

We calculate the first variation L. The variation of the Lagrange function (13), which is the main linear part of the increment of this function, has the form:

$$\delta L=\int_0^1 2\left[p(x,T)-p^0(x)\right]\delta p(x,T)\mathrm{d}x\mathrm{d}t+$$
$$+\int_0^1\int_0^T\psi(x,t)\left[\frac{\partial\delta p}{\partial t}-\frac{\partial^2(2p\delta p)}{\partial x^2}+2a\cdot p\delta p\right]\mathrm{d}x\mathrm{d}t.$$

We transform the double integral using integration by parts, taking into account the initial and boundary conditions:

$$\delta p(x,0)=0,\quad \left.\frac{\partial(2p\delta p)}{\partial x}\right|_{x=0}=\delta Q(t),$$

$$\left.\frac{\partial(2p\delta p)}{\partial x}\right|_{x=1}=0, \qquad (14)$$

so that the expressions under the signs of the integrals do not contain partial derivatives of the variations of the phase variables. As a result, we get:

$$\delta L=\int_0^1\left\{\begin{array}{l}2\left[p(x,T)-p^0(x)\right]+\\+\psi(x,T)\end{array}\right\}\delta p(x,T)\mathrm{d}x\mathrm{d}t-$$
$$-\int_0^1\int_0^T\left[\frac{\partial\psi}{\partial t}+2p\frac{\partial^2\psi}{\partial x^2}-2ap\psi\right]\delta p(x,t)\mathrm{d}x\mathrm{d}t+$$
$$+\int_0^T\left(2p\frac{\partial\psi}{\partial x}\delta p\right)\Bigg|_{x=0}^{x=1}+\int_0^T\psi(x,0)\delta Q(t)\mathrm{d}t, \qquad (15)$$

where $\delta p(x, t)$, $\delta p(x, T)$, $\delta p(0, t)$, $\delta p(1, t)$ are arbitrary variations, $\delta Q(t)$ is allowable variation.

Assuming that the stationarity condition $\delta L=0$ is satisfied at the optimal point, and using a sufficiently large arbitrariness in the choice of variations of the phase variables, we equate the coefficients for the corresponding variations p to zero, that is, setting:

$$\frac{\partial\psi}{\partial t}=-2p\cdot\frac{\partial^2\psi}{\partial x^2}+2ap\psi,\quad 0<x<1,\quad 0\le t<T, \qquad (16)$$

$$\psi(x,T)=-2\left[p(x,T)-p^0(x)\right],\quad 0\le x\le 1, \qquad (17)$$

$$\frac{\partial\psi(0,t)}{\partial x}=\frac{\partial\psi(1,t)}{\partial x}=0,\quad 0\le t<T, \qquad (18)$$

from equality (15) we have:

$$\delta L=\int_0^T\frac{\partial H}{\partial Q}\delta Q(t)\mathrm{d}t. \qquad (19)$$

Here $H=\psi(0, t)Q(t)$ is the Hamilton function for the problem (9)–(12). Expressions (13)–(15) define the adjoint boundary value problem for the direct problem (9)–(11).

From formula (19) for the first variation of functional (12), it follows that the gradient of functional (12) is equal to $\psi(0, t)$. As can be seen, to obtain the gradient of functional (12) for given $Q(t)$ and α, we have to solve two boundary value problems. First, from (9)–(11) it is necessary to determine the function $p(x, t)$ then put the resulting $p(x, t)$ into (16)–(18), and from (16)–(18) find $\psi(x, t)$ and, finally, $\psi(0, t)$.

5. 2. Finding a formula for the gradient of the functional characterizing reservoir depletion

It is not possible to obtain exact solutions to these boundary value problems due to the nonlinearity of equations (9). For the numerical solution of these boundary value problems, implicit finite-difference schemes combined with a run or the method of straight lines are usually used, although various linearization methods and special approximate methods have been proposed for equations (9), often encountered in the theory of non-stationary gas filtration in a porous medium [6]. Functional (12) and its gradient are replaced by their approximating counterparts.

So, having formulas for the gradient for problem (9)–(12), we can state the gradient methods for solving it, particularly the gradient projection method.

Method for numerical solution of problem (9)–(12).

For the numerical solution of the problem by the gradient projection method based on the calculation of the gradient of the functional (12), as a rule, at each step of the iterative process, it is necessary to solve two boundary problems – problem (9)–(11) and problem (16)–(18), and then use the formula to calculate the gradient.

An approximate solution of the boundary value problems (9)–(11) and (16)–(18) will be sought by the method of straight lines from the theory of approximate solutions of boundary value problems. Let $x_i=ih$, $i=1, 2, \ldots, n$ is a grid with step $h=1/n$ on the segment [0,1]. By introducing the notation $p_i=p(x_i(t),t)$, $\psi_i(t)=\psi(x_i(t),t)$, $i=1, 2, \ldots, n$ subject to conditions $p_0^2(t)=p_0^2(t)-hQ(t)$, $p_n^2(t)=p_{n-1}^2(t)$, boundary value problem (9)–(11) can be approximated by a system of ordinary differential equations:

$$\frac{dp_1}{dt}=\frac{1}{h^2}\left[-p_1^2+p_2^2\right]-a\cdot\left(p_1^2-1\right)-\frac{Q(t)}{h},$$

$$\frac{dp_i}{dt}=\frac{1}{h^2}\left[p_{i-1}^2-2p_1^2+p_{i+1}^2\right]-a\left(p_i^2-1\right),\quad i=2,3,..,n-1, \qquad (20)$$

$$\frac{dp_n}{dt}=\frac{1}{h^2}\left[-p_{n-1}^2+p_n^2\right]-a\left(p_n^2-1\right),$$

with initial conditions:

$$p_i(0)=1,\quad i=1,2,\ldots,n, \qquad (21)$$

and conjugate system (16)–(18) subject to conditions $\psi_0(t)=\psi_1(t)$, $\psi_n(t)=\psi_{n-1}(t)$ can be approximated by the system of homogeneous equations:

$$\frac{d\psi_1}{dt}=-\frac{2p_1}{h_2}\left[-\psi_1+\psi_2\right]+2ap_1\psi_1,$$

$$\frac{d\psi_i}{dt}=-\frac{2p_i}{h^2}\left[\psi_{i-1}-2\psi_i+\psi_{i+1}\right]+2ap_i\psi_i,$$

$$i=2,3,\ldots,n-1, \qquad (22)$$

$$\frac{d\psi_n}{dt}=-\frac{2p_n}{h^2}\left[\psi_{n-1}-\psi_n\right]+2ap_n\psi_n,$$

with conditions at the right end:

$$\psi_i(T)=-2\left[p_i(T)-p^0(x_i)\right],\quad i=1,2,\ldots,n, \qquad (23)$$

where $p_i(t)$, $\psi_i(t)$ are the approximate values of functions $p(x, t)$, $\psi(x, t)$ at the grid nodes $x_i=ih$, $i=0, 1, \ldots, n$, $(n+1)h=1$, respectively.

5. 3. Developing an algorithm and carrying out software implementation of the calculation results

It is important to note that the approach, which is based on calculating the first variation of the minimized functional, seems to be promising in relation not only to the optimization of systems similar to (9)–(12), but also to more general optimization problems. In this case, the original distributed system can be solved by any numerical method without passing to systems of ordinary differential equations, as is done in [13].

The algorithm for solving the problem consists of the following steps:

1. Some admissible control $Q^k(t)$ is chosen.

2. According to the initial $Q^k(t)$, the direct system (20), (21) is integrated by the Runge – Kutta method (possibly also by the Euler method when certain relations between sampling steps in time and space coordinates are met) in the "forward direction" and the values of the functions $p_i(t)$, $i=1,..., n$ are found in the time interval $0\le t\le T$. Note that in the case of linearity of boundary value problems, it is important to preserve its solution only at the ends of the time interval.

3. The values of the approximating functional are calculated:

$$I_n=h\sum_{i=0}^{n}\left[p_i(T)-p^0(x_i)\right]^2, \qquad (24)$$

subject to conditions $p_0^2(T)=p_0^2(T)-hQ(T)$.

4. Formula (23) calculates the boundary values $\psi_i(t)$, $i=1, ..., n$ for the system of adjoint equations (22).

5. In the "reverse direction" of time, the adjoint system (22), (23) is integrated, the coefficients of which are calculated along the trajectories $p_i(t)$, $i=1, ..., n$.

6. At each integration step, the maximum or minimum value $\psi_0(t)=\psi_1(t)$ is found.

7. The new control $Q^{k+1}(t)$ is calculated by the formula:

$$Q^{k+1}(t)=\begin{cases}Q_1, \text{ if } Q^k(t)+\delta Q^k(t)<Q_1,\\ Q_2, \text{ if } Q^k(t)+\delta Q^k(t)>Q_2,\\ Q^k(t)+\delta Q^k(t), \text{ if } Q_1\le Q^k(t)+\\ +\delta Q^k(t)\le Q_2.\end{cases} \qquad (25)$$

Here:

$$\delta Q^k(t)=\lambda\cdot\frac{\psi_1^k(t)}{\left|\max\psi_1^k\right|}, k=0,1,2,... \qquad (26)$$

where k is an iteration number, and the parameter $\lambda>0$ is chosen depending on the change in the sign of the functions $y_1^k(t)$ during iterations [18]. That is, if $\psi_1(t)$ does not change sign during iterations, then can be increased to speed up convergence. If $\psi_1(t)$ changes sign at the previous iteration step, then for the next iteration we set the corresponding equal to $\lambda/2$ and so on. Note that the main work during the transition to the next iteration, as a rule, is associated with the calculation of the gradient.

8. A step is taken with a new control $Q^{k+1}(t)$ returning to step 2.

To implement the algorithms described above for solving problems (20), (21), and (24), a computer program was compiled. A brief block diagram of the program is shown in Fig. 1.

The iterative process (25), (26) continues until one of the end-of-count criteria described in [19] is met, in particular, the number of iterations is specified. Integration of systems (20), (22) is carried out according to the Runge-Kutta method with a constant step $\Delta t=0.01$, and the output of the results is carried out with a step $t=0.02$. The segment $0\le x\le 1$ is divided into five equal parts with lengths $h=0.2$.

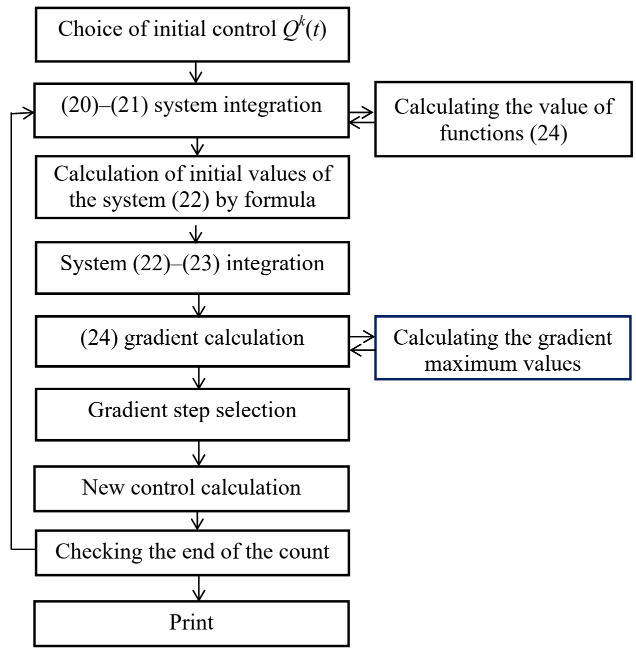

Fig. 1. Brief block diagram of the solution of problem (20), (21), (24)

It is easy to see that the chosen values of the parameters Δt and h satisfy the inequality:

$$\frac{\Delta t}{h^2}\le\frac{1}{2}. \qquad (27)$$

Therefore, when solving the direct and adjoint boundary value problems numerically, even the explicit difference scheme can be used. However, it should be noted that this limitation is very strict. For the stability of the difference method for solving boundary value problems, the step q has to be taken very small, which increases the total number in time and, consequently, the total amount of computational work. In this regard, despite the great simplicity of the explicit grid method, its use in practice is very limited.

To identify the effect of the coupling coefficient, first of all, Fig. 2 shows the dependence of the gas pressure over the reservoir at $t=0.2$ and $Q=0.5$ for different values of the connection coefficient. It follows from the figure that under the conditions of the considered problem for $\alpha\le 1$, the permeability of the reservoir top has almost no effect on the gas pressure in the productive reservoir, however, for $\alpha>1$ the effect of the permeability of the reservoir top becomes very noticeable and should be taken into account.

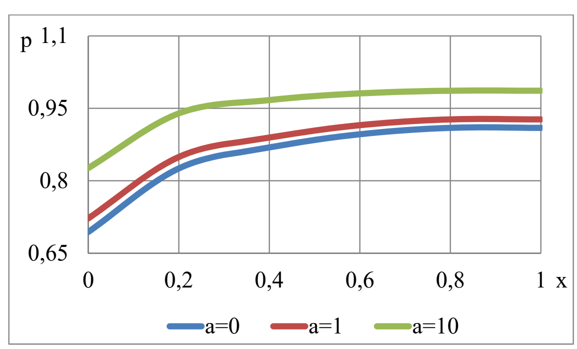

Fig. 2. Change in gas pressure in the reservoir for $t=0.2$ and $Q=0.5$ for different values of the connection coefficient

Fig. 3 shows graphs of the change in the debit functions in time. From the analysis of the graphs built on the basis of the obtained numerical data, it follows that the greater the complex connection parameter a, the sooner the stationary operation of the well occurs, and it is important to take into account the effect of gas-dynamic connection on $Q(t)$.

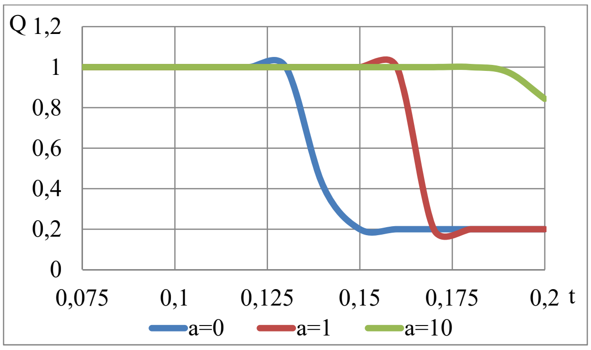

Fig. 3. Variation in time of the flow rate functions for different values of the connection coefficient

Table 1 shows the results of calculations for different values of the connection coefficient. The calculations were carried out for the following values of dimensionless parameters: Q_1=0.2, Q_2=1, T=0.2, $p^0(x)$=0.8. For the initial iteration, $Q^0(t)$ was assumed equal to 0.5.

Table 1

Calculation results for various values of the connection coefficient

Connection coefficient	Approximately optimal control	Functional value	Number of iterations
a=0	$Q(t)=\begin{cases}1, \text{if } 0\le t\le 0.13,\\ 0.4164, \text{if } 0.13<t\le 0.14,\\ 0.8437, \text{if } 0.14<t\le 0.2\end{cases}$	$1.1458\cdot10^{-3}$	54
a=1	$Q(t)=\begin{cases}1, \text{if } 0\le t\le 0.16,\\ 0.2, \text{if } 0.16<t\le 0.2\end{cases}$	$1.7911\cdot10^{-3}$	42
a=10	$Q(t)=\begin{cases}1, \text{if } 0\le t\le 0.18,\\ 0.9758, \text{if } 0.18<t\le 0.19,\\ 0.8437, \text{if } 0.19\le 0.2\end{cases}$	$1.1458\cdot10^{-2}$	160

Note that with a further increase in the number of reiterations for all values of the coupling parameter a, the well operation mode will be a piecewise constant function, receiving alternately the upper and lower boundaries, i.e. has structures $Q(t)=\text{sign}\psi_1(t)$. To check the optimality of one of the controls found, for example, for a=1 as $p^0(x)$ we set the solution $p_i(t)$, i=1, ..., n of problem (20), (21) with the control:

$$Q(t)=\begin{cases}1, \text{if } 0\le t\le 0.16,\\ 0.2, \text{if } 0.16<t\le 0.2,\end{cases} \quad (28)$$

having one switching point in segment $0\le t\le T$ (Fig. 4).

Obviously, in this case the value of the minimum of the functional is equal to zero. The approximately optimal controls found in this case, as can be seen from the graphs shown in Fig. 4, do not have a character different from the optimal one, and approach it with an increase in the number of iterations.

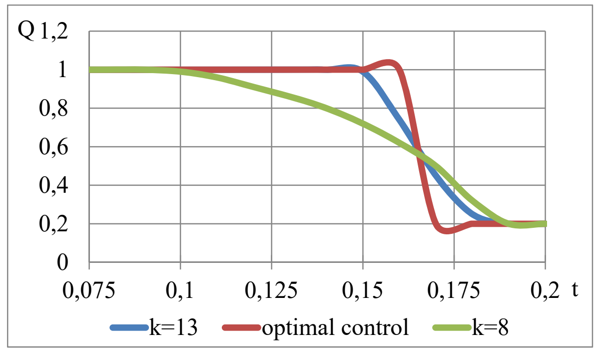

Fig. 4. Approximately optimal controls in intermediate iterations

6. Discussion of the results of the study on the control problem of the gas reservoirs' depletion

The solution to the most important problem of increasing the efficiency of developing new and especially long-term exploited gas and gas condensate fields is possible only with the widespread industrial use of artificial methods for controlling well productivity. In this case, marginal wells deserve special attention, the number of which, unfortunately, is steadily increasing, and both the total gas and condensate production in the country and the cost of gas and condensate production depend on the efficiency of working with such a fund. Compared to the existing literature in this area, the feature of the presented work is the reduction of a two-dimensional problem to a one-dimensional one, which greatly simplifies the calculations:

1. To solve the considered optimization problem, the Myatiev-Girinsky scheme was used in the work, which allows the two-dimensional equation to be reduced to one-dimensional equations with an additional term that takes into account the influence of overlying or underlying low-permeable pores, as evidenced by equation (9). To determine the desired flow rate, without taking it beyond the maximum and minimum possible, the gradient projection method with a special step selection was effectively used sequentially according to equations (20)–(26).

2. A numerical solution of the problem has been obtained based on the developed mathematical model of the medium with a low-permeability roof, which sufficiently fully takes into account the physical properties of the reservoir (a reference to equations (19), (25), (26)). Such solutions make it possible to more deeply and fully study the features of non-stationary gas filtration in a reservoir with a jumper, to find calculation formulas for non-stationary filtration of an elastic fluid applicable to engineering calculations, which is generally relevant and of significant scientific and practical interest.

3. Based on the calculations and plots, it was found that both in solving the boundary value problem and in solving the optimization problem associated with this boundary value problem, it is important to take into account the gas-dynamic relationships between the reservoir and the jumper. When running the model, despite the ill-posedness of optimal control problems with a quadratic functional, the gradient projection method did not show a tendency to "scatter" and gave a convergent sequence of controls ((26), Fig. 3, 4).

Solutions to many problems of the theory of filtration in case of violation of the rectilinear-parallelism or radiality of the flow are obtained in the form of complex series (often slowly converging) and, therefore, calculations on them become difficult. Due to the complexity of the resulting formulas, it is often impossible to draw any practical conclusions. The value of the presented approximate method lies not only in the simplification of calculations but also in the possibility, on the basis of simple formulas, to notice important qualitative patterns. The presented work is based on the Myatiev-Girinsky scheme on the vertical nature of filtration in separate layers. Unfortunately, the paper does not present a case of overflow with a strong deposit permeability. A significant limitation of the work is the difficulty in determining how and where the main supply and main discharge of interlayer gas-bearing horizons occur. The non-linear nature of the processes under consideration with increasing dimensions leads to a strong complication of the problem from a mathematical point of view. For future researchers, the extension of this approach can be the development of an algorithm for approximating differential equations of geofiltration by finite differences.

Although the approximate method presented in the paper does not claim to be more accurate, it greatly facilitates calculations when solving many problems.

7. Conclusions

1. On the basis of the Myatiev-Girinsky hydraulic scheme, the question of the interaction between the reservoir and the bridge was considered. The problem is solved under general conditions with respect to the reservoir boundaries. Based on the calculations, conclusions were drawn about the effect of the formation top permeability not only on the state of the object under study but also on the wells' operation mode. Based on the calculations, conclusions were drawn about the effect of the formation top permeability on the interaction of wells.

2. A formula for the gradient of the functional characterizing the reservoir depletion is found, and the method of successive approximations based on the Pontryagin maximum principle is applied. In this case, to calculate the value of the gradient of the functional according to the formula $I'(Q)=\partial H/\partial Q=\psi(0,\ t)$ in consecutive approximations, it is necessary to integrate two nonlinear boundary value problems for partial differential equations.

3. An algorithm was developed and a software implementation was carried out to present the results of calculations to identify the influence of the values of the complex communication parameter not only on the state of the object, but also on the well operation mode. Compared to existing methods, when using the proposed approach, convergence is accelerated, and what is more, this iterative method demonstrates enhanced stability. The main purpose of this work is to study the effect of the flow coefficient on the wells' operation mode under the condition of reservoir depletion by a given point in time.

Conflict of interest

The authors declare that they have no conflict of interest in relation to this research, whether financial, personal, authorship or otherwise, that could affect the research and its results presented in this paper.

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The study was performed without financial support.

Data availability

Data will be made available on reasonable request.

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