The study of the patterns of change in the hydrodynamic parameters under the conditions of non-stationary flow at the entry of the cylindrical pipe and the initial arbitrary distribution of velocities in the entry section was conducted based on the boundary layer equations. A boundary problem was formed under the axisymmetric change conditions in the flow. The boundary conditions were chosen in accordance with the pattern of an arbitrary distribution of velocities in the entry section. A general solution of the approximating Navier-Stokes equations is presented depending on the initial conditions and the Reynolds number. In accordance with the type of flow, the boundary conditions of the problem are established, and the boundary-value problem is formulated. Regularities for the change in velocities lengthwise in the entrance region have been obtained for a constant and parabolic velocity distribution in the inlet cross-sections. Analytical solutions have been obtained, allowing to obtain patterns of changes in velocities and pressures toward flow at any section and at any time. For the mentioned cases, the composite graphs of velocity changes in different sections along the length of the entrance transition area were constructed by computer analysis, for different time conditions. With the obtained composite graphs, the patterns of change over the entire length of the transition area of the entrance region were constructed, enabling to obtain fluid flow velocity at any point of the section. The length of the transition zone can be estimated based on the condition of reaching a certain percentage (99 %) of the maximum velocity of the flow.

The proposed solutions create the conditions for correctly constructing separate units of hydromechanical equipment

Keywords: cylindrical pipe, inlet section, non-stationary flow, viscous fluid, velocity distribution

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PATTERN IDENTIFICATION OF THE NON-STATIONARY LAMINAR FLOW OF A VISCOUS FLUID IN THE ROUND PIPE INLET SECTION

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1. Introduction

The study of non-stationary flow in pressurized cylindrical pipes is one of the main ways to improve machine tool structures. The management and regulation systems precise and stable operation depends on the accuracy of the research results. Of the cylindrical pipes, the fluid-bearing pressure system of a round cross-section is most often applicable, in which the flows can be stationary and non-stationary. According to the results on non-stationary flow patterns in the liquid channels of machines, their construction is carried out. The issue being discussed is pertinent and holds practical importance.

The main issue in studying fluid flow is developing a mathematical model of the given physical phenomenon. The results define the applicability limits of the chosen calculation method. The constructed model needs to describe the ongoing hydromechanical phenomena more accurately and, meanwhile, provide the possibility of getting analytical solutions. The study of non-stationary hydromechanical phenomena in pressure pipes is one of the most complicated problems in hydromechanics, where the change of values, besides the time, depends on the point data. From this point of view, the proposed topic has important theoretical interest and practical significance, which is due to its relevance.

2. Literature review and problem statement

The study of the patterns of viscous fluid flow in the transition zones of liquid channels is one of the most complex tasks of hydromechanics, the results of which are used in their design. Because of their practical importance, researching these problems is very relevant. It should be noted that studies of hydrodynamic phenomena in transition zones were mainly conducted in conditions of stationary fluid flow, which is one of the particular cases of flow. The flows in the transition zones of the liquid channels are non-stationary, and their study is aimed at solving important practical problems.

The system of Nave-Stokes equations in a non-deformable medium is used as the starting equation for non-stationary flows [1]. For each problem, boundary conditions are defined, and the basic equations are simplified. It is often impossible to obtain analytical solutions to this problem. Modern computing techniques make it possible to obtain approximate solutions for such problems, providing practically any accuracy. For studying the hydrodynamic processes in the entrance section transition area, many theoretical and approximate calculation methods have been developed. Conclusions concerning the nature of flow are drawn using each calculating method. A theoretical study and a presentation of the findings are conducted. Often, these conclusions are related to specific ranges of motion. It limits the applicability of the results obtained.

In the works [2, 3] the authors studied the patterns of change in hydrodynamic flow parameters at the transitional section of the pipe entrance. By successive approximations, which simplified the Navier-Stokes equations they obtained a boundary problem. Then, by analytic solutions, they found regularities of velocity and pressure change [4] and the results compared with those of experimental investigations. The reliability of the obtained results is confirmed by the comparative analyses which have been conducted. However, the deformation of the velocity field occurs not only from changes in the coordinate of the point, but also from the time parameter. Therefore, studies of the patterns of change in the hydrodynamic parameters of the flow at the inlet section with unsteady laminar motion are important and are of considerable practical interest.

In work [5], the problem of the pulsating flow of a viscous liquid at the entry region of the round pipe was solved on the basis of approximate equations. In [6], a comparison of the results of theoretical and experimental studies of the pulsation motion of a viscous fluid is given. The problem of a round pipe entry region with a suddenly applied velocity at the pipe inlet was solved in [7] using the hypothesis of self-similarity of the velocity profiles in the boundary layer and the momentum equation. Numerical integration of the Navier-Stokes equation solved a similar problem for a suddenly applied velocity at small Reynolds numbers in [8]. Under conditions of periodic perturbation and with the support of linear approximations, a thin boundary layer [9] was studied at the entrance region of a round pipe. However, the boundary layer is examined on a flat plate, which reduces the accuracy of the results. In [10], a non-stationary laminar flow of a viscous incompressible fluid at the entrance region of a round cylindrical pipe was considered for a constant velocity distribution at the entry. However, this condition limits the scope of the obtained results. These restrictions are partially overcome in [11]. A mathematical model has been created based on studies of changes in the hydrodynamic parameter pattern of a viscous incompressible fluid in the transitional sections of flat pipes, which allowed obtaining results with acceptable accuracy indicating motion dynamics patterns [12].

In [13], an attempt was made to find analytical solutions to simplify the hydrodynamic equation that describes the fluid flow under pressure in pipelines using new approximations of the Bessel function and its roots. Recommendations for the use of these solutions have been developed. The problem of a viscous fluid's laminar non-stationary flow in axisymmetric pipes is considered based on the changes in viscosity and pressure gradient. The proposed method and the obtained solutions reveal the regularities of the flow's hydrodynamic parameters while accounting for viscosity variability, and it can be applied to Newtonian fluid in particular [14]. An analytical solution of the momentum equation for non-stationary fluid flow in round pipes is presented in [15], where an arbitrary change in kinematic viscosity with the time change is permissible. Velocity and discharge are expressed as Bessel and Kelvin functions of the radial variable, while time dependence is expressed as a Fourier series. The analytical solution for velocity is compared to the direct momentum equation's numerical solution. Let's analyze the linear modal stability of the flow in a pipe with a stepwise increase in discharge from a stationary initial flow to a final flow. A stepwise increase in flow rate causes a non-periodic non-stationary flow. The analysis of the flow stability depending on the ratio of the current flow discharge to the steady-state laminar flow discharge was carried out in [16]. As a result, the conditions for the stability of an non-stationary flow in the pipe are obtained.

In [17], the author is studying the evolution of the main single-mode stationary flow of the viscous incompressible fluid in the flat diffuser. It is established that starting from some critical value of the Reynolds number, the existence of a stationary single-mode flow is impossible. The results of examining several laminar flow regimes in a flat diffuser/confuser with a small opening angle were presented by the authors in [18]. Consequently, patterns of changes in the hydrodynamic parameters of a viscous incompressible fluid had been obtained through numerical modeling based on the solution of Navier-Stokes equations. The areas of existence and transitions of flow regimes from stationary-symmetric to stationary-asymmetric and non-stationary ones in the diffuser and confuser, depending on the Reynolds number are found. The values of the Reynolds number, which determine the ranges of the existence of these fluid flow regimes for Newtonian and non-Newtonian fluids are given.

After conducting a literature review for the proposed problem, it is concluded that most of the research conducted in transitional areas of fluid channels has been on the conditions of stationary flow of viscous fluid. However, the nature of the fluid flow in these areas is non-stationary, making it an urgent task to investigate the patterns of changes in hydrodynamic parameters in these transition areas, under non-stationary flow conditions.

3. The aim and objectives of the study

The aim of the study is the identification reveals patterns of changes in the hydrodynamic parameters of a viscous fluid in a round cylindrical pipe inlet section during non-stationary laminar flow of a viscous fluid depending on the Reynolds number.

To achieve this aim, the following tasks must be solved:

– to formulate a boundary problem and determine the initial and boundary conditions;

– to develop a method for solving the boundary problem and identify the pattern of changes in the hydrodynamic parameters of viscous flow in the entrance region of a round cylindrical pipe;

– to draw graphs of changes in axial velocity depending on time and the Reynolds number;

– to identify the conditions for determining the length of the round pipe entrance section during non-stationary laminar flow.

4. Materials and methods

4. 1. Object and hypothesis of the study

The object of research is stationary laminar flow in axisymmetric channels.

The main hypothesis of the study is ignoring the influence of body forces on the change in the haydrodynamic parameters of the flow.

Assumptions made in the work are that the viscous fluid flow is considered axisymmetric and that the pressure in each fixed section does not depend on the radius.

The study was carried out on the basis of the boundary layer equations, which are simplified forms of the Navier-Stokes equations.

4. 2. Choosing a calculation scheme

Main symbols:

1. $V_r(r, z, t)+V_z(r, z, t)$ – fluid flow velocity components by coordinates r, z, t.
2. r, z, t – position of a point.
3. $p(z, t)$ – pressure.
4. R – pipe radius.
5. ν – kinetic coefficient of viscosity.
6. U_0 – average velocity of flow section.
7. $\varphi(r, t)$ – velocity function of the inlet cross-section.
8. y, x – dimensionless coordinates.
9. $\overline{t}$ –non-dimensional time.
10. $V_z(r, z, t)$ – dimensionless function of velocity.
11. $\overline{p}$ – dimensionless pressure.
12. Re – Reynolds number.
13. $\alpha = \dfrac{p_0}{\rho U_0^2}$ – dimensionless coefficient.
14. $-\alpha\dfrac{\partial \overline{p}}{\nu} = f\left(\overline{t}\right)$ – function for the pressure.
15. $\psi(y)$ – dimensionless function of the initial velocity distribution.

The transition area at the entrance of the cylindrical pipe with a round cross-section is considered for laminar viscous fluid flow. The origin of the Z axis is the center of the inlet section (Fig. 1) and the direction of flow is infinitely long.

The flow in a round pipe will be considered in cylindrical coordinates, starting from the zero point (Fig. 1).

It is assumed that in the entrance sections of the pipe at z=1, the velocity changes according to an arbitrary law. It is necessary to find patterns of change in the hydrodynamic parameters of a viscous fluid in the transition area, considering it to be axisymmetric and non-stationary.

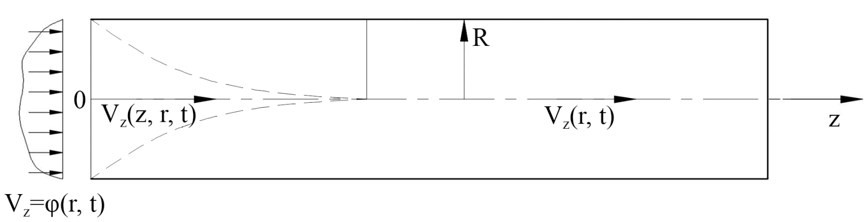

Fig. 1. On the study of a viscous incompressible fluid flow in the entrance region of a round cylindrical pipe

4. 3. Statement of the problem and formulation of the system of differential equations for the study

Lets suppose there is a viscous fluid laminar non-stationary flow in an infinitely long cylindrical pipe with R radius. In the initial section of the tube, where the starting point of the oz axis is located, the distribution of velocities is given by an arbitrary law, that is $u=\varphi(r, t)$, when z=0. Under these conditions, an axisymmetric isothermal viscous flow occurs. At the pipe entrance section with $u=\varphi(r, t)$, fluid velocity diagram on the pipe wall becomes zero. A deformation of the velocity diagram occurs, which extends over a certain distance along the pipe's length. A boundary layer occurs near the pipe walls, where the velocity gradient, du/dn, becomes very large, due to which the friction forces assume very large values regardless of the μ viscosity coefficient. The boundary layer gradually extends from the pipe walls to cover the entire pipe. Therefore, it is necessary to perform the studies in the transition area with the boundary layer equations. For the boundary layer, Prandtl [1] suggests using the Nave-Stokes equations, which simplify the equations for the boundary layer. Since the main forces in the boundary layer are the viscous forces, Prandtl ignored the parameters that were small compared to the viscous forces and got simplified equations for the boundary layer during the simplification of the Nave-Stokes equations.

To study the transition area of a cylindrical pipe of circular section, let's use the equations of the boundary layer, which in the cylindrical coordinate system have the following form [1]:

$$\frac{\partial V_z}{\partial t}+V_r\frac{\partial V_z}{\partial r}+V_z\frac{\partial V_z}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+\nu\left(\frac{\partial^2 V_z}{\partial r^2}+\frac{1}{r}\frac{\partial V_z}{\partial r}\right), \quad (1)$$

$$\frac{\partial V_z}{\partial z}+\frac{1}{r}\frac{\partial\left(V_r\cdot r\right)}{\partial r}=0. \quad (2)$$

To simplify equation (1), let's accept the conclusion made in the work [2, 3], according to which, therefore, there is:

$$\frac{\partial V_z}{\partial t}+V_z\frac{\partial V_z}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+\nu\left(\frac{\partial^2 V_z}{\partial r^2}+\frac{1}{r}\frac{\partial V_z}{\partial r}\right). \quad (3)$$

To integrate the resulting non-linear inhomogeneous differential equation [3], let's make an assumption, according to which let's replace the coefficient of the member $\partial V_z/\partial z$ with the average rate v_z of flow section:

$$V_z=U_0=\frac{2}{R^2}\int_0^R\int_0^T \phi\left(r,t\right)\cdot r\cdot \mathrm{d}r\cdot \mathrm{d}t. \quad (4)$$

After this assumption , the study of the transition area is brought to by integrating the equation [5, 6]:

$$\frac{\partial V_z}{\partial t}+U_0\frac{\partial V_z}{\partial z}= =-\frac{1}{\rho}\frac{\partial P}{\partial z}+\nu\left(\frac{\partial^2 V_z}{\partial r^2}+\frac{1}{r}\frac{\partial V_z}{\partial r}\right), \quad (5)$$

$$\frac{\partial V_z}{\partial z}+\frac{1}{r}\frac{\partial\left(V_r\cdot r\right)}{\partial r}=0. \quad (6)$$

for the following initial and boundary conditions [7–9]:

$$V_z=0,\ \ V_r=0,\ \text{when } r=R,\ z>0,\ t>0, \quad (7)$$

$$V_z=\phi\left(r,t\right),\ \text{when } z=0,\ 0\le r\le R, \quad (8)$$

$$V_z\to V',\ \text{when } z\to\infty,\ t>0,\ 0\le r\le R. \quad (9)$$

Here V' is the fluid flow velocity in the pipe stabilized section, which is determined by the following equation [10]:

$$\frac{\partial V_z}{\partial t}+\frac{1}{\rho}\frac{\partial P}{\partial z}=\nu\left(\frac{\partial^2 V'}{\partial r^2}+\frac{1}{r}\frac{\partial V'}{\partial r}\right). \tag{10}$$

Let's insert dimensionless variables:

$$y=\frac{r}{R},\ x=\frac{z}{R},\ \bar{t}=\frac{U_0\cdot t}{R},$$

$$\bar{V}_z\left(y,x,\bar{t}\right)=\frac{V_z\left(r,z,t\right)}{U_0},\ \bar{P}=\frac{P}{P_0}. \tag{11}$$

The equation system (5), (6) will take the following form:

$$\frac{\partial\bar{V}_z\left(y,x,\bar{t}\right)}{\partial\bar{t}}+\frac{\partial\bar{V}_z\left(y,x,\bar{t}\right)}{\partial x}=$$
$$=-\alpha\frac{\partial\bar{P}}{\partial x}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2\bar{V}_z\left(y,x,\bar{t}\right)}{\partial y^2}+\frac{1}{y}\frac{\partial\bar{V}_z\left(y,x,\bar{t}\right)}{\partial y}\right), \tag{12}$$

$$\frac{\partial\bar{V}_z}{\partial x}+\frac{1}{y}\frac{\partial\left(\bar{V}_r\cdot y\right)}{\partial y}=0, \tag{13}$$

where $\alpha=\frac{P_0}{\rho U_0^2},\ \mathrm{Re}=\frac{RU_0}{\upsilon},\ -\alpha\frac{\partial\bar{P}}{\partial x}=f\left(\bar{t}\right).$

To solve systems of equations (12), (13) it is necessary to formulate the boundary conditions of the problem.

4. 4. Choice of boundary conditions

The boundary conditions for the integration of (12), (13) will be:

$$\bar{V}_z\left(1,x,\bar{t}\right)=0,\ \bar{V}_z\left(y,0,0\right)=\psi\left(y\right),\ \frac{\partial\bar{V}_z}{\partial x}\to 0,$$

when

$$x\to\infty,\ \bar{V}_z=0,\ \bar{V}_z\left(y,\infty,\bar{t}\right)\to\bar{V}'\left(y,\bar{t}\right), \tag{14}$$

where $\bar{V}'\left(y,\bar{t}\right)$ is:

$$\frac{\partial\bar{V}_z'}{\partial\bar{t}}=f\left(\bar{t}\right)+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 V_z'\left(y,x,\bar{t}\right)}{\partial y^2}+\frac{1}{y}\frac{\partial V_z'\left(y,x,\bar{t}\right)}{\partial y}\right), \tag{15}$$

the general solution of the inhomogeneous equation:

$$\bar{V}_z'\left(1,x,\bar{t}\right)=0,\ \ \bar{V}_z'\left(y,0,0\right)=0, \tag{16}$$

in the case of homogeneous boundary conditions.

5. Results of research to identify patterns of changes in hydrodynamic parameters

5. 1. Integration of the boundary value problem to identify patterns of change in axial velocities and pressure

Let's find the solution to (12):

$$\bar{V}_z\left(y,x,\bar{t}\right)=U\left(y,x,\bar{t}\right)+\bar{\varphi}\left(y,\bar{t}\right), \tag{17}$$

in the form of a sum [4], where $U\left(y,x,\bar{t}\right)$:

$$\frac{\partial U\left(y,x,\bar{t}\right)}{\partial\bar{t}}+\frac{\partial U\left(y,x,\bar{t}\right)}{\partial x}=$$
$$=\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 U\left(y,x,\bar{t}\right)}{\partial y^2}+\frac{1}{y}\frac{\partial U\left(y,x,\bar{t}\right)}{\partial y}\right). \tag{18}$$

The solution of the homogeneous equation, in the case of inhomogeneous boundary conditions:

$$U\left(1,x,\bar{t}\right)=0,$$

$$U\left(y,0,0\right)=\phi\left(yR\right)=\psi\left(y\right), \tag{19}$$

$\bar{\varphi}\left(y,\bar{t}\right)$ is the partial solution of the inhomogeneous (15). (18) let's search for the solution to the equation in the form of a sum [4]:

$$U\left(y,x,\bar{t}\right)=\sum_{k=1}^{\infty}C_k\left(x,\bar{t}\right)J_0\left(\lambda_k y\right), \tag{20}$$

with this substitution in equation (18), let's obtain:

$$\sum_{k=1}^{\infty}\left[\frac{\partial C_k\left(x,\bar{t}\right)}{\partial\bar{t}}+\frac{\partial C_k\left(x,\bar{t}\right)}{\partial x}\right]J_0\left(\lambda_k y\right)=$$
$$=-\sum_{k=1}^{\infty}\frac{\lambda_k^2}{\mathrm{Re}}C_k\left(x,\bar{t}\right)J_0\left(\lambda_k y\right),$$

from where

$$\frac{\partial C_k\left(x,\bar{t}\right)}{\partial\bar{t}}+\frac{\partial C_k\left(x,\bar{t}\right)}{\partial x}+\frac{\lambda_k^2}{\mathrm{Re}}C_k\left(x,\bar{t}\right)=0. \tag{21}$$

The solution to equation (21) will be:

$$C_k\left(x,\bar{t}\right)=C_k\exp\left(-\frac{\lambda_k^2 x}{\mathrm{Re}}\right). \tag{22}$$

Considering the values of the $C_k\left(x,\bar{t}\right)$ coefficient, let's obtain the function $\bar{V}_z\left(y,x,\bar{t}\right)$:

$$\bar{V}_z\left(y,x,\bar{t}\right)=\sum_{k=1}^{\infty}C_k\exp\left(-\frac{\lambda_k^2 x}{\mathrm{Re}}\right)J_0\left(\lambda_k y\right)+\bar{\varphi}\left(x,\bar{t}\right). \tag{23}$$

To determine the function $\bar{\varphi}\left(y,\bar{t}\right)$, using

$$\int_0^1 y\frac{\partial\bar{V}_z\left(y,x,\bar{t}\right)}{\partial x}\mathrm{d}x=0$$

there is:

$$-\sum_{k=1}^{\infty}\int_0^1\frac{\lambda_k^2}{\mathrm{Re}}C_k\exp\left(-\frac{\lambda_k^2}{\mathrm{Re}}\right)J_0\left(\lambda_k y\right)y\mathrm{d}y+$$
$$+\int_0^1\frac{\partial\bar{\varphi}\left(y,\bar{t}\right)}{\partial x}y\mathrm{d}y=0,$$

from where:

$$\bar{\varphi}\left(x,\bar{t}\right)=-2\sum_{k=1}^{\infty}\frac{C_k}{\lambda_k}J_1\left(\lambda_k\right)\exp\left(-\frac{\lambda_k^2 x}{\mathrm{Re}}\right)+C_0\left(\bar{t}\right). \tag{24}$$

Considering equation (24) for the function $\bar{V}_z\left(y,x,\bar{t}\right)$ there is:

$$\bar{V}_z(y,x,\bar{t}) = \\ = \sum_{k=1}^{\infty} C_k \exp\left(-\frac{\lambda_k^2 x}{\mathrm{Re}}\right)\left[J_0(\lambda_k y) - \frac{2J_1(\lambda_k)}{\lambda_k}\right] + C_0(\bar{t}). \tag{25}$$

The value of the function $C_0(\bar{t})$ is determined from the boundary condition (14) when $x\to\infty$, $C_0(\bar{t})\to\bar{V}_z'(y,\bar{t})$.

The $\bar{V}_z'(y,\bar{t})$ function is determined from (15), the solution of which is found as a sum:

$$\bar{V}_z'(y,\bar{t}) = U_1(y,\bar{t}) + U_2(y,\bar{t}), \tag{26}$$

where $U_1(y,\bar{t})$ function is the solution of the inhomogeneous differential equation (27):

$$\frac{\partial U_1(y,\bar{t})}{\partial \bar{t}} = \\ = -\alpha\frac{\partial \bar{P}}{\partial x} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 U_1(y,\bar{t})}{\partial y^2} + \frac{1}{y}\frac{\partial U_1(y,\bar{t})}{\partial y}\right), \tag{27}$$

with homogeneous boundary conditions:

$$\begin{cases} U_1(y,\bar{t}) = 0, & \text{when } y = 1, \bar{t} > 0, \\ U_1(y,\bar{t}) = 0, & \text{when } 0 \le y \le 1, \bar{t} = 0, \end{cases} \tag{28}$$

but the function $U_2(y,\bar{t})$ is the solution of the homogeneous differential equation (29):

$$\frac{\partial U_2(y,\bar{t})}{\partial \bar{t}} = \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 U_2(y,\bar{t})}{\partial y^2} + \frac{1}{y}\frac{\partial U_2(y,\bar{t})}{\partial y}\right), \tag{29}$$

with inhomogeneous boundary conditions:

$$\begin{cases} U_2(y,\bar{t}) = 0, & \text{when } y = 1, \bar{t} > 0, \\ U_2(y,\bar{t}) = \psi(y), & \text{when } 0 \le y \le 1, \bar{t} = 0. \end{cases} \tag{30}$$

Let's find the solution of equation (29) in the following form:

$$U_2(y,\bar{t}) = \sum_{k=1}^{\infty} b_k \exp\left(-\frac{\lambda_k^2 \bar{t}}{\mathrm{Re}}\right) J_0(q_k y). \tag{31}$$

It follows from the first boundary condition (30) that q_k is the positive roots of the $J_0(q_k=0)$ equation, and from the second condition:

$$\psi(y) = \sum_{k=1}^{\infty} b_k J_0(q_k y). \tag{32}$$

From equation (32) let's obtain the value of the coefficient b_k:

$$b_k = \frac{I_k}{J_1^2(q_k)},$$

where

$$I_k = \int_0^1 \psi(y) J_0(q_k y) y \,\mathrm{d}y. \tag{33}$$

Inserting the value of the coefficient b_k, let's obtain:

$$U_2(y,\bar{t}) = \sum_{k=1}^{\infty} \frac{I_k}{J_1^2(q_k)} \exp\left(-\frac{\lambda_k^2 \bar{t}}{\mathrm{Re}}\right) J_0(q_k y), \tag{34}$$

let's search the solution to equation (27) in form:

$$U_1(y,\bar{t}) = \sum_{k=1}^{\infty} A_k(\bar{t}) J_0(q_k y), \tag{35}$$

let's represent the function $f(\bar{t})$ as a Fourier-Dalembert series:

$$f(\bar{t}) = \sum_{k=1}^{\infty} \frac{2f(\bar{t})}{q_k J_1(q_k)} J_0(q_k y). \tag{36}$$

Inserting expressions (35), (36) into equation (27), let's obtain:

$$\frac{\partial A_k(\bar{t})}{\partial \bar{t}} + \frac{q_k^2}{\mathrm{Re}} A_k(\bar{t}) = \frac{2f(\bar{t})}{q_k J_1(q_k)}. \tag{37}$$

The solution to equation (37) will be [4]:

$$A_k(\bar{t}) = \frac{2L_k(\bar{t}) \exp\left(-\frac{q_k^2 \bar{t}}{\mathrm{Re}}\right)}{q_k J_1(q_k)},$$

where

$$L_k^{(1)}(\bar{t}) = \int_0^{\bar{t}} f(\xi) \exp\left(\frac{q_k^2 \xi}{\mathrm{Re}}\right) \mathrm{d}\xi. \tag{38}$$

Considering equation (38):

$$U_1(y,\bar{t}) = \sum_{k=1}^{\infty} \frac{2L_k(\bar{t}) \exp\left(-\frac{q_k^2 \bar{t}}{\mathrm{Re}}\right)}{q_k J_1(q_k)} J_0(q_k y). \tag{39}$$

Substituting equations (34), (39) let's obtain equation (26):

$$\bar{V}_z'(y,\bar{t}) = \sum_{k=1}^{\infty} \exp\left(-\frac{q_k^2 \bar{t}}{\mathrm{Re}}\right) \left[\begin{array}{l} \frac{2L_k(\bar{t})}{q_k J_1(q_k)} + \\ + \frac{I_k}{J_1^2(q_k)} \end{array}\right] J_0(q_k y). \tag{40}$$

From equations (25), (40) let's obtain the general solution to the problem:

$$\bar{V}_z(y,x,\bar{t}) = \\ = \sum_{k=1}^{\infty} C_k \exp\left(-\frac{\lambda_k^2 x}{\mathrm{Re}}\right)\left[J_0(\lambda_k y) - \frac{2J_1(\lambda_k)}{\lambda_k}\right] + \\ + \sum_{k=1}^{\infty} \exp\left(-\frac{q_k^2 \bar{t}}{\mathrm{Re}}\right)\left[\frac{2L_k^{(1)}(\bar{t})}{q_k J_1(q_k)} + \frac{I_k}{J_1^2(q_k)}\right] J_0(q_k y). \tag{41}$$

The obtained equation must satisfy the (14) initial condition $\bar{V}_z(y,0,0) = \psi(y)$, :

$$\psi(y) = \sum_{k=1}^{\infty} b_k J_0(q_k y) + \sum_{k=1}^{\infty} C_k \left[J_0(\lambda_k y) - J_0(\lambda_k)\right]. \tag{42}$$

It is possible to multiply both parts of the equation (42) with an expression $[J_0(\lambda_n y)-J_0(\lambda_n)]y\mathrm{d}y$ and integrate over the interval (0;1).

Considering that λ_n is the positive roots of the equation $J_2(\lambda_n)=0$:

$$\int_0^1\left[J_0(\lambda_k y)-J_0(\lambda_k)\right]\left[J_0(\lambda_n y)-J_0(\lambda_n)\right]y\mathrm{d}y= =\begin{cases}0,\lambda_k\neq\lambda_n\\ \dfrac{1}{2}J_1^2(\lambda_n),\lambda_k=\lambda_n\end{cases}, \tag{43}$$

let's obtain:

$$C_n=\frac{2}{J_1^2(\lambda_n)}\int_0^1\left[J_0(\lambda_n y)-J_0(\lambda_n)\right]\psi(y)y\mathrm{d}y- -\frac{2}{J_1^2(\lambda_n)}\sum_{k=1}^{\infty}b_k\int_0^1 J_0(q_k y)\left[J_0(\lambda_n y)-J_0(\lambda_n)\right]y\mathrm{d}y. \tag{44}$$

Considering that:

$$\int_0^1 J_0(q_k y)\left[J_0(\lambda_n y)-J_0(\lambda_n)\right]y\mathrm{d}y= =\frac{2\lambda_n J_1(\lambda_n)J_1(q_k)}{q_k\left(q_k^2-\lambda_n^2\right)},$$

let's obtain the values of coefficient C_n:

$$C_n=\frac{2L_n^{(2)}}{J_1^2(\lambda_n)}-\frac{4\lambda_n}{J_1(\lambda_n)}\sum_{k=1}^{\infty}b_k\frac{J_1(q_k)}{q_k\left(q_k^2-\lambda_n^2\right)}, \tag{45}$$

where:

$$L_n^{(2)}=\int_0^1\left[J_0(\lambda_n y)-J_0(\lambda_n)\right]\psi(y)y\mathrm{d}y. \tag{46}$$

The resulting solutions refer to the general boundary and initial conditions of the problem. Using general solutions, it is possible to obtain solutions that are adequate to the given conditions for each private case. Let's consider two private cases:

5. 2. Integrating a boundary value problem to identify patterns of changes in axial velocities and pressure at a constant distribution of initial velocities

Let's assume that the velocity of the entering fluid is constant at all points, therefore: $\varphi(r)=u_0^*=\text{const},\ 0\leq r<R$ from where $\psi(y)=A_0$. In the case of constant values of velocity distribution and pressures in the inlet section, in order to obtain the patterns of velocity distribution in the transition area, let's determine the values of the functions in formula (41):

$$I_k=\int_0^1\psi(y)J_0(q_k y)y\mathrm{d}y=\frac{A_0 J_1(q_k)}{q_k}.$$

$$b_k=\frac{I_k}{J_1^2(q_k)}=\frac{A_0}{q_k J_1(q_k)},$$

$$L_k^{(1)}(\bar{t})=\frac{B_0\,\mathrm{Re}}{q_k^2}\exp\left(\frac{q_k^2\xi}{\mathrm{Re}}-1\right),$$

$$L_n^{(2)}=0,\ C_n=\frac{4A_0G_{kn}}{J_1(\lambda_n)},$$

where $G_{kn}=\sum_{k=1}^{\infty}\dfrac{\lambda_n}{q_k^2\left(\lambda_n^2-q_k^2\right)}$.

Inserting these values of the functions into equation (41):

$$\bar{V}_z(y,x,\bar{t})= =\sum_{k=1}^{\infty}\frac{4A_0G_{kn}}{J_1(\lambda_n)}\exp\left(-\frac{\lambda_k^2 x}{\mathrm{Re}}\right)\left[J_0(\lambda_k y)-\frac{2J_1(\lambda_k)}{\lambda_k}\right]+ +\sum_{k=1}^{\infty}\left[\frac{A_0}{q_k}-\frac{2B_0\,\mathrm{Re}}{q_k^3}\right]\frac{J_0(q_k y)}{J_1(q_k)}\exp\left(-\frac{q_k^2\bar{t}}{\mathrm{Re}}\right)+ +\sum_{k=1}^{\infty}\frac{2B_0\,\mathrm{Re}}{q_k^3}\frac{J_0(q_k y)}{J_1(q_k)}. \tag{47}$$

Taking into account, that $\sum_{k=1}^{\infty}\dfrac{J_0(q_k y)}{q_k^3 J_1(q_k)}=\dfrac{1}{8}\left(1-y^2\right)$:

$$\bar{V}_z(y,x,\bar{t})= =\sum_{k=1}^{\infty}\frac{4A_0G_{kn}}{J_1(\lambda_n)}\exp\left(-\frac{\lambda_k^2 x}{\mathrm{Re}}\right)\left[J_0(\lambda_k y)-\frac{2J_1(\lambda_k)}{\lambda_k}\right]+ +\sum_{k=1}^{\infty}\left[\frac{A_0}{q_k}-\frac{2B_0\,\mathrm{Re}}{q_k^3}\right]\frac{J_0(q_k y)}{J_1(q_k)}\exp\left(-\frac{q_k^2\bar{t}}{\mathrm{Re}}\right)+ +\frac{B_0\,\mathrm{Re}}{4}\left(1-y^2\right). \tag{48}$$

Based on the velosity change equation (48), let's obtain the pressure change pattern from equation (12). Accepting that the pressures at any point of the fixed cross-section are equal, from equation (12) let's get the pressure change function in the pipe axis, where $x=0$. It will depend on the variables of $(x,\bar{t})$, so there is:

$$\frac{\partial\bar{V}_z(y,0,\bar{t})}{\partial\bar{t}}+\frac{\partial\bar{V}_z(y,0,\bar{t})}{\partial x}=-\alpha\frac{\partial\bar{P}(x,\bar{t})}{\partial x}. \tag{49}$$

By inserting the value of the function from equation (48) into equation (49), determining the function:

$$\bar{P}(x,\bar{t})=\bar{P}_0+ +\frac{1}{\alpha}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{4A_0\lambda_n\left[J_0(\lambda_n)-1\right]}{q_k^2\left(\lambda_n^2-q_k^2\right)J_1(\lambda_n)}\left(1-\exp\left(-\frac{\lambda_n^2}{\mathrm{Re}}x\right)\right)- -\frac{1}{\alpha}\sum_{k=1}^{\infty}\left(\frac{A_0q_k}{\mathrm{Re}}-\frac{2B_0}{q_k}\right)\frac{x\exp\left(-\dfrac{q_k^2\bar{t}}{\mathrm{Re}}\right)}{J_1(q_k)}. \tag{50}$$

The obtained regularities enable one to determine the patterns of changes in axial velocities and pressures along the entire length of the transition area of the cylindrical pipe inlet section at any time.

5. 3. Integrating of a boundary value problem to identify patterns of changes in axial velocities and pressure with parabolic distributions of initial velocities

let's assume the distribution of the velocity of the fluid entering the cylindrical pipe is parabolic, therefore $\psi(y)=A_0(1-y^2)$, $0\leq r<R$, corresponding to which there

is $\psi(y)=A_0(1-y^2)$. In the case of constant values of parabolic velocity and pressure distribution in the inlet section, in order to obtain the patterns of velocity distribution in the transition area, let's determine the values of the functions in formula (41):

$$I_k=\int_0^1\psi(y)J_0(q_ky)y\,dy=\frac{8A_0J_1(q_k)}{q_k^3},$$

$$b_k=\frac{I_k}{J_1^2(q_k)}=\frac{A_0}{q_k^3J_1(q_k)},$$

$$L_k^{(1)}(\bar t)=\frac{B_0\,\mathrm{Re}}{q_k^2}\exp\left(\frac{q_k^2\xi}{\mathrm{Re}}-1\right),$$

$$L_n^{(2)}=\frac{A_0}{2}\left(\frac{16}{q_k^3}-\frac{1}{q_k}\right)J_1(q_k),$$

$$C_n=A_0\left(\frac{16}{\lambda_n^3}-\frac{1}{\lambda_n}\right)+\frac{32A_0\lambda_n}{J_1(\lambda_n)}\sum_{k=1}^{\infty}G_{kn}^{(2)},$$

where $G_{kn}^{(2)}=\sum_{k=1}^{\infty}\frac{\lambda_n}{q_k^4\left(\lambda_n^2-q_k^2\right)}$.

Inserting these function values into equation (41), let's obtain:

$$\bar V_z(y,x,\bar t)=\sum_{k=1}^{\infty}\left\{A_0\left(\frac{16}{\lambda_n^3}-\frac{1}{\lambda_n}\right)+\frac{32A_0\lambda_n}{J_1(\lambda_n)}\sum_{k=1}^{\infty}G_{kn}\right\}\times$$
$$\times\exp\left(-\frac{\lambda_n^2x}{\mathrm{Re}}\right)\left[J_0(\lambda_ny)-\frac{2J_1(\lambda_n)}{\lambda_n}\right]+$$
$$+\sum_{k=1}^{\infty}\frac{2B_0\,\mathrm{Re}}{q_k^3}\frac{J_0(q_ky)}{J_1(q_k)}\exp\left(-\frac{q_k^2\bar t}{\mathrm{Re}}\right)+$$
$$+\sum_{k=1}^{\infty}\left[\frac{8A_0}{q_k^3J_1(q_k)}-\frac{2B_0\,\mathrm{Re}}{q_k^3J_1(q_k)}\right]\exp\left(-\frac{q_k^2\bar t}{\mathrm{Re}}\right)J_0(q_ky). \qquad (51)$$

Taking into account that $\sum_{k=1}^{\infty}\frac{J_0(q_ky)}{q_k^3J_1(q_k)}=\frac{1}{8}\left(1-y^2\right)$:

$$\bar V_z(y,x,\bar t)=$$
$$=A_0\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left\{\frac{16}{\lambda_n^3}-\frac{1}{\lambda_n}+\frac{32\lambda_n}{J_1(\lambda_n)}\frac{1}{q_k^4\left(\lambda_n^2-q_k^2\right)}\right\}\times$$
$$\times\exp\left(-\frac{\lambda_n^2x}{\mathrm{Re}}\right)\left[J_0(\lambda_ny)-\frac{2J_1(\lambda_n)}{\lambda_n}\right]+$$
$$+\sum_{k=1}^{\infty}\frac{2B_0\,\mathrm{Re}}{q_k^3}\frac{J_0(q_ky)}{J_1(q_k)}\exp\left(-\frac{q_k^2\bar t}{\mathrm{Re}}\right)+$$
$$+\sum_{k=1}^{\infty}\left[\frac{8A_0-2B_0\,\mathrm{Re}}{q_k^3J_1(q_k)}\right]\exp\left(-\frac{q_k^2\bar t}{\mathrm{Re}}\right)J_0(q_ky)+$$
$$+\frac{B_0\,\mathrm{Re}}{4}\left(1-y^2\right). \qquad (52)$$

Having determined the regularity of the change in the axial velocity according to the formula (52), let's similarly calculate the regularity of the change in pressure and get:

$$\bar P(x,\bar t)=\bar P_0+\frac{1}{\alpha}\times$$
$$\times\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}A_0\left(\frac{16}{\lambda_n^3}-\frac{1}{\lambda_n}+\frac{32\lambda_n}{J_1(\lambda_n)}\frac{1}{q_k^4\left(\lambda_n^2-q_k^2\right)}\right)\times$$
$$\times\left(J_0(\lambda_n)-1\right)\left(1-\exp\left(\frac{-\lambda_n^2}{\mathrm{Re}}x\right)\right)+$$
$$+\frac{1}{\alpha}\sum_{k=1}^{\infty}\left(\frac{8A_0-2B_0\,\mathrm{Re}}{\mathrm{Re}\,q_kJ_1(q_k)}\right)x\exp\left(-\frac{q_k^2}{\mathrm{Re}}\bar t\right). \qquad (53)$$

The obtained solutions enable the measurement of axial velocities and pressure changes along the entire length of the transition area at any point in the transition section at any moment.

5. 4. Graphs of changes in the hydrodynamic parameters of a viscous fluid at the inlet region of a round pipe

Based on the solutions obtained, let's investigate the nature of the flow features in the transition area of the round pipe inlet section. Based on the integration results of differential equations for viscous fluid flow, regularities of change in the distribution of axial velocities $\bar V_z(y,x,\bar t)$ were obtained.

To visualize the patterns of changes in the axial velocity $\bar V_z(y,x,\bar t)$ along the transverse section and along the length of the transitional section depending on the initial distribution of velocities $\bar V_z(y,0,0)=\psi(y)$ and the Reynolds number Re=20, 40, 60, 80, 100 their graphs of change were constructed. Fig. 2–7 show the indicated graphs for cases $\bar V_z(y,0,0)=A_0=1$ at Re=40 and $\bar V_z(y,0,0)=A_0\left(1-y^2\right)=\left(1-y^2\right)$ at Re=100.

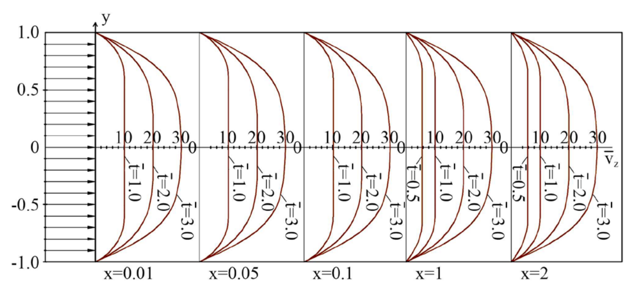

Fig. 2. Graphs of changes in axial velocities $\bar V_z(y,x,\bar t)$ along the transition point of the round pipe inlet section at A_0=1, B_0=10, Reynolds numbers of Re=40 and x=0.01, x=0.05, x=0.1, x=1.0, x=2.0

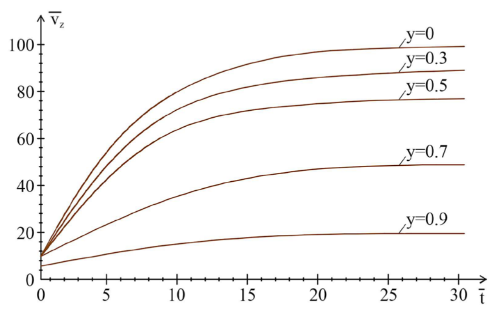

Fig. 3. Graphs of changes in axial velocities $\bar V_z(y,x,\bar t)$, at A_0=1, B_0=10, x=1.0, Reynolds numbers Re=40 and y=0.2, y=0.3, y=0.5, y=0.7, y=0.9

The deviation of the axial velocities in the transition section at y=0 should not exceed 1 % of the non-stationary velocity of the stabilized section. Based on this condition, a calculation formula was obtained to determine the length of the transition section and the graph is shown in (Fig. 8), which has an important practical application in the design of various hydraulic automation systems [2, 3, 10].

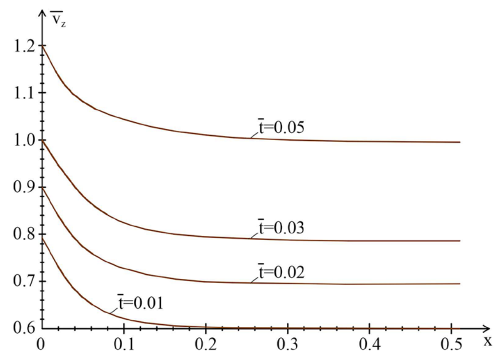

Fig. 4. Graphs of changes in axial velocities $\overline{V}_z\left(y,x,\overline{t}\right)$ along the transition point of the round pipe inlet section at y=0, A_0=1, B_0=10, Reynolds numbers of Re=40 and $\overline{t}=0.01$, $\overline{t}=0.02$, $\overline{t}=0.03$ $\overline{t}=0.05$

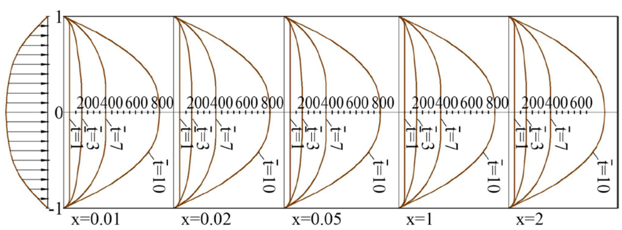

Fig. 5. Graphs of changes in axial velocities $\overline{V}_z\left(y,x,\overline{t}\right)$, along the cross-section at the transition point of the round pipe inlet section at $A_0(1+y^2)=1+y^2$, B_0=50, Reynolds numbers of Re=100 and x=0.01, x=0.02, x=0.05, x=1.0, x=2.0

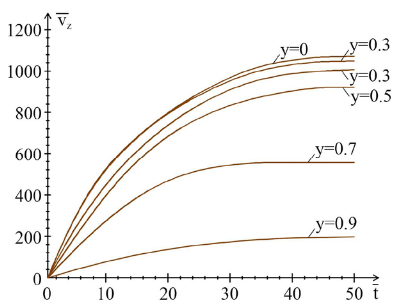

Fig. 6. Graphs of changes in the axial velocities $\overline{V}_z\left(y,x,\overline{t}\right)$ of the inlet section of a round pipe at x=0.02, B_0=50, Reynolds numbers of Re=100

An analysis of the numerical calculations results and the resulting graphs determined the dynamics of changes in axial velocities and the length of the transition section depending on the Reynolds number. It can be seen that at the start of the process, the length of the transition point is equal to Z=0.00345R·Re (Fig. 8). In the process of non-stationary development, the length of the transition point practically does not change, at $\overline{t}=5$, Z=0.0035R·Re. Therefore, with practical accuracy, the length of the transition point can be taken to be equal to Z=0.0035R·Re.

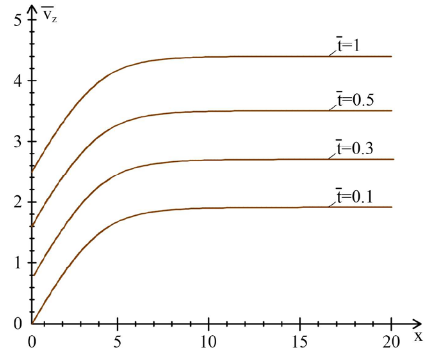

Fig. 7. Graphs of changes in axial velocities $\overline{V}_z\left(y,x,\overline{t}\right)$ along the transition point of the inlet section of a round pipe at y=0; $A_0(1+y^2)=1+y^2$, B_0=50, Reynolds numbers of Re=100, and $\overline{t}=0.1$, $\overline{t}=0.3$ $\overline{t}=0.5$ $\overline{t}=1.0$

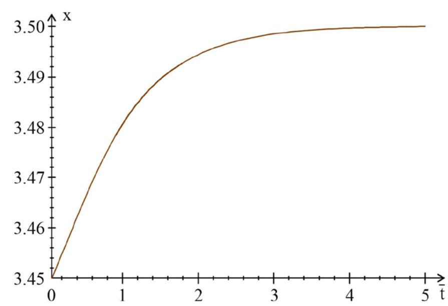

Fig. 8. Graph of the change in the length of the transition point at the Reynolds number of Re=1000

6. Discussion of the results on the development of a viscous fluid non-stationary flow at the round pipe inlet section

Formulas for the axial velocities and pressure distribution along the round cylindrical pipe inlet section length with non-stationary laminar flow of a viscous, incompressible fluid are obtained from solving a boundary value problem. The studies were carried out with a uniform and parabolic distribution of velocities in the entrance section of a round pipe, which corresponds to the processes taking place in reality. Therefore, the calculated results correspond to the natural data with practical accuracy.

The graphs constructed by computer calculations using the formulas (48) and (52) demonstrate the development of the process in the round pipe entrance section. An analysis of the results of the numerical calculation and the obtained graphs (Fig. 2–7) showed that the process degree of development depends on the pressure gradient, the initial distribution of velocities in the entrance cross-section, and the Reynolds number. The viscous fluid flow at the entrance transition section during non-stationary flow is unstable. The shapes of the velocity distribution diagrams in each fixed section and in the transition section change over time (Fig. 2–7) due to the deformation of the distribution diagrams and the impact of the pressure gradient. The velocity distribution diagrams change outside the transition section because of the pressure gradient. With an increase in the Reynolds number, the length of the transition section decreases, which is explained by the intense dissipation of the flow energy. The study of the viscous fluid flow development process in the entrance region of a round pipe was performed using simplified Navier-Stokes equations. Based on the integration results, approximate results were obtained. However, the integration accuracy results in engineering calculations are quite appropriate. The results of this study can contribute to the improvement of constructive changes in the transition sections of the hydraulic systems of various mechanisms and machines, which will lead to an increase in their reliable operation.

Based on the problem of relevance, further, development is associated with the specification of the initial section length and the design changes in the entrance region of the cylindrical channel. Analysis of the numerical calculation results and the resulting graphs determined the length of the entrance region depending on the Reynolds number. The condition of the entrance region is the coincidence of the numerical values of the velocities at each fixed point of the cross-section.

7. Conclusions

1. On the basis of the approximating Navier-Stokes equations, a boundary value problem is formulated to study non-stationary laminar motion at the inlet section of a round pipe. The boundary conditions of the problem were formulated as closely as possible to the processes occurring in the transition sections of hydraulic systems. This ensures that the calculation results can be effectively used in the construction of these areas.

2. A universal method for integrating a boundary value problem has been developed, which makes it possible to obtain patterns of change in hydrodynamic parameters along the length of the inlet section for an arbitrary initial velocity distribution. The results obtained suggest that hydraulic automatic devices can be designed based on the condition of reliable operation.

3. Graphs of the change in the dimensionless hydrodynamic flow parameters for uniform and parabolic distributions of the initial velocities at the pipe inlet, depending on the Reynolds number for different values of the dimensionless time, are plotted. The results obtained make it possible to reveal the influence of the pipe and liquid parameters on the change in the parameters of the initial section. Due to the universality of the obtained graphs, it is possible to conclude the nature of the non-stationary flow and its impact on the length.

4. Conditions have been established for determining the length of the initial section depending on the dimensionless time, with uniform and parabolic distributions of initial velocities, which is important information in the design of various mechanisms and machines of hydropneumoautomatics.The correct design of the pressure hydraulic system transition sections depends on the nature of the non-stationary flow. Depending on the geometric dimensions, a controlled mode can be obtained.

Conflict of interest

The authors declare that they have no conflict of interest in relation to this research, whether financial, personal, authorship or otherwise, that could affect the research and its results presented in this paper.

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Data availability

Manuscript has no associated data.

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