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DETERMINING REGULARITIES IN THE CONSTRUCTION OF CURVES AND SURFACES USING THE DARBOUX TRIHEDRON

Ali Kadhim Ahmed
PhD, Lecturer
Department of Soil Science and Water Resources
College of Agriculture
University of Diyala
Bagdad str., 1, Baqubah, Iraq, 32001

Andrii Nesvidomin
PhD, Associate Professor*

Serhii Pylypaka
Doctor of Technical Sciences, Professor,
Head of Department*

Tatiana Volina
Corresponding author
PhD, Associate Professor*
E-mail: t.n.zaharova@ukr.net

Serhii Dieniezhnikov
PhD, Associate Professor
Department of Management of Education
and Pedagogy of the Higher School
Sumy State Pedagogical University
named after A. S. Makarenko
Romenska str., 87, Sumy, Ukraine, 40002

*Department of Descriptive Geometry,
Computer Graphics and Design
National University of Life
and Environmental Sciences of Ukraine
Heroyiv Oborony str., 15, Kyiv, Ukraine, 03041

The Frénet trihedron, known in differential geometry, is accompanying for a spatial and, as a special case, for a flat curve. Its three mutually perpendicular unit orts are defined uniquely for any point on the curve except for some special ones. Unlike the Frénet trihedron, the Darboux trihedron relates to the surface. Two of its unit orts are located in a plane tangent to the surface, and the third is directed normally to the surface. It can also be accompanying for the curve, which is located on the surface. To this end, one of the orts in the plane tangent to the surface must be tangent to the curve.

Trihedra are movable and, with respect to a fixed coordinate system, change their position due to movement and rotation. The object of research is the process of formation of curves and surfaces, as a result of the geometric sum of the bulk motion of the Darboux trihedron and the relative motion of the point in its system under given conditions. In the study of the geometric characteristics of curves and surfaces, it is necessary to have formulas for the transition from the position of the elements of these objects in the system of a moving trihedron to the position in a fixed Cartesian coordinate system. This is exactly what needs to be solved. The results obtained are parametric equations of curves and surfaces that are tied to the initial surface. Nine guide cosines were found, three for each ort.

A distinctive feature of this approach in comparison with the traditional one is the use of two systems: fixed and mobile, which is the Darboux trihedron. This approach allows us to consider in a new way the problem of the construction of curves and surfaces. The scope of practical application can be the construction of geometric shapes on a given surface. An example of such a construction is the laying of a pipeline along a given line on the surface. In addition, the sum of the relative motion of a point in a trihedron and the bulk motion of the trihedron itself over the surface gives an absolute trajectory of motion. Its sequential differentiation produces absolute speed and absolute acceleration without finding individual components, including the Coriolis acceleration. This could be used in point dynamics problems

Keywords: accompanying trihedron, Darboux trihedron, parametric equations of curves and surfaces, transition formulas

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1. Introduction

In differential geometry, the Frénet trihedron is widely known, which is accompanying for a spatial and, as a special case, for a flat curve. Its three mutually perpendicular unit orts are defined uniquely for any point on the curve except for some special ones. For example, for the inflection point of a flat curve or for the straightening point of a spatial curve, the direction of the principal normal becomes undefined. The Darboux trihedron relates to the surface. Its two single orts are located in a plane tangent to the surface, and the third is directed normally to the surface. It can also be accompanying for a curve that is located on the surface. To this end, one of the orts in the plane tangent to the surface must be tangent to the curve. Then these two trihedra have a common ort tangent to the curve, and there is a certain angle between the other

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two orts. However, the direction of the ort, which is directed normally to the surface, can be chosen in one or the opposite direction. Thus, unlike the Frénet trihedron, a Darboux trihedron at a point on a curve on a surface can have two positions.

The Frénet and Darboux trihedra are movable and, with respect to a fixed coordinate system, change their position due to movement and rotation. For a Frénet trihedron, the guide cosines of its orts are determined in terms of the differential characteristics of the curve in which the first and second derivatives of this curve are involved. At a point on a curve with a curvature equal to zero, the position of the orts of the main normal and binormal becomes undefined. For a Darboux trihedron, one of the orts points normally towards the surface, that is, its direction is determined in terms of differential surface characteristics and is definite for a regular surface. The ort of the tangent is also definite, hence the third ort, perpendicular to the first two, will also be defined. When studying the geometric characteristics of curves and surfaces using accompanying trihedra, it is necessary to have formulas for the transition from the position of the elements of these objects in the system of a moving trihedron to the position in a fixed Cartesian coordinate system. To this end, there are nine guide cosines, three for each ort. For a Frénet trihedron, they are completely defined in terms of the first and second derivatives of the parametric equations of the guide curve. For a Darboux trihedron – through parametric surface equations.

The use of two coordinate systems – fixed and moving – is used in mechanics to find the absolute motion of a point. The motion of a point with respect to a moving coordinate system is relative, and the motion of a moving system with respect to a fixed system is portable. The coordination of these movements in space occurs in a function of time. The geometric sum of these movements gives the absolute trajectory of a point with respect to a fixed coordinate system.

This principle can be transferred to the construction of curves and surfaces when the moving coordinate system is a Darboux trihedron that moves along a given line on the surface. If a curve is fixed in its system, then its trace in space forms a surface when the trihedron moves. If such a curve is a circle, then a tubular surface will form, for example, a pipeline that lies on the surface. The coordination of the motion of a trihedron on a surface with a fixed coordinate system occurs not in a function of time but in a function of an independent variable describing a curve on the surface along which the trihedron moves. Such studies are important because they allow expanding the shaping of curves and surfaces under given conditions. The practical application of such studies, for example, in the design of a pipeline on the surface, determines their relevance. In addition, the proposed description of the complex motion of a point makes it possible to simply find its absolute velocity and acceleration by differentiating the equations of absolute trajectory, which could be used in point dynamics problems.

2. Literature review and problem statement

First of all, it is necessary to dwell on a scientific task that does not lose its relevance for a long time: increasing the durability and wear resistance of parts. In work [1], technological support for the protection of contact surfaces of press joints against wear is substantiated. Often, it is proposed to solve such a problem by improving the coatings of parts. In [2], the evaluation of the quality parameters of the aluminized coating obtained by electro spark doping was given. In [3], the defective structure of nitride coatings under the action of ion irradiation was investigated. In [4], the technological parameters of the manufacture of combined electric spark coatings are given. However, with this approach, objective difficulties arise associated with the cost of the developed methods, the necessary tools, and materials. An option to overcome these difficulties may be the use of geometric modeling of objects and processes. This makes it possible, to a certain extent, to solve urgent issues by geometric methods, namely providing a geometric shape according to predetermined conditions at the design stage.

Geometric modeling can be performed in various ways. Thus, in [5], the solid-state modeling of geometric objects in point calculus is considered. This provides an expansion of the instrumental capabilities of computer graphics. The principles and advantages of the proposed methodology are described in detail in [6]. Geometric modeling of torso surfaces in the Baluba-Nadish calculus is described in [7]. Work [8] is more applied and highlights numerical modeling in heat and fluid dynamics. Simulation of aerodynamic flow is considered in [9].

Often, process modeling requires an analytical description of the movement of a material point on the surface. This situation arises in the study of the movement of mixtures consisting of a set of elements that can be mistaken for a material particle. For example, when dispersing seed [10]. For an analytical description of the motion of a material point on a surface under the action of applied forces, a Darboux trihedron can be used. Its movement in the vicinity of an infinitesimal area of the surface can be considered as movement along the tangent to the surface of the plane corresponding to this area. A special case of such motion in the plane, when the Frénet and Darboux trihedra coincide, is considered in [11].

The problem of finding the trajectory of motion of a particle under the action of forces applied to it is not easy and is generally reduced to solving systems of nonlinear differential equations of the second order [1, 2]. The use of the Darboux trihedron enables an alternative approach to solving such problems. If the curve along which the trihedron moves on the surface is described in the function of the length of its own arc, then Frénet formulas can be applied. Constructing such curves is considered in monograph [13], in which the issues of modeling plane and spatial curves and surfaces in natural parametrization are addressed. For this purpose, various laws of distribution of curvature and twist (for spatial lines) and distribution of curvature along the main directions of the surface are applied. In addition, the monograph provides examples of application of the proposed methods of geometric modeling in transport and energy industries. The curve is a guide for the accompanying trihedra. The construction of curves under given conditions is considered in [14]. It outlines the modeling of one-dimensional contours with the provision of a given accuracy of interpolation. All this suggests that it is advisable to conduct research on the expansion of the shaping of curves and surfaces under the predefined conditions.

3. The aim and objectives of the study

The aim of this study is to determine the patterns of shaping curves and surfaces using the guide cosines of the Darboux trihedron with respect to a fixed coordinate system. This will make it possible to construct curves and surfaces as a result of the interaction of two coordinate systems: fixed and moving, which is the Darboux trihedron.

To accomplish the aim, the following tasks have been set:

– to analytically substantiate the position of the Darboux trihedron in a fixed coordinate system through the parameters of the surface and the curve on it;

– to show the convergence of expressions for guide cosines of a Darboux trihedron with respect to a fixed coordinate system using a specific example;

– to consider the practical application of the obtained dependences for the construction of curves and surfaces.

4. The study materials and methods

Based on the object of our study, which is the process of shaping curves and surfaces using the Darboux trihedron, an algorithm for constructing these objects is formulated. It is assumed that the use of two coordinate systems – fixed and mobile – will make it possible to obtain curves and surfaces with predetermined properties.

At each point of the spatial curve, one can construct the accompanying Frénet trihedron. Figure 1*a* shows the arc *AB* as a flat curve belonging to a conical surface. At point *A*, the Frénet $\overline{\tau}$, $\overline{n}$, $\overline{b}$ and Darboux $\overline{T}$, $\overline{P}$, $\overline{N}$ trihedra are constructed. They have a common ort $\overline{\tau} \equiv \overline{T}$ and a certain angle φ between the other two orts. The difference between them is the fact that the Frénet trihedron is built without taking into account the surface on which the curve is located. Its ort $\overline{\tau}$ is tangent to the curve, the principal normal ort $\overline{n}$ is directed towards the center of curvature and the binormal ort $\overline{b}$ is directed so that it forms the right-hand system of three mutually orthogonal vectors. Their direction is completely determined by the differential characteristics of the curve. In this regard, the position of the Frénet trihedron at some special points of the curve becomes undefined. For example, at point *B* (Fig. 1, *a*), which is the inflection point and at which the Darboux trihedron is constructed, the position of the Frénet trihedron is undefined since at this point the curvature of the curve is zero and the center of its curvature is absent. In the Darboux trihedron, the ort $\overline{N}$ is normalized towards the surface, the ort *P* is defined from the vector product provided that the orts form the right-hand coordinate system. As a result of the fact that the orts $\overline{T}$ and $\overline{P}$ are perpendicular to the ort $\overline{N}$, they form a tangent plane to the surface at a given point. For an expanded surface, the direction of the ort $\overline{P}$ may coincide with the rectilinear generatrix (as, for example, for the cone in Fig. 1, *a*).

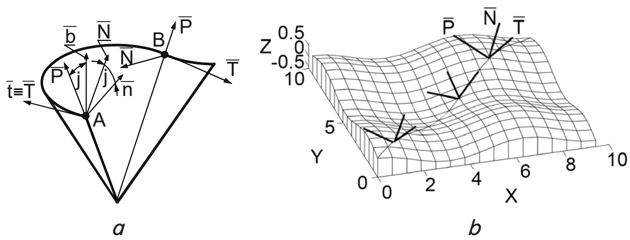

Fig. 1. Graphic illustrations to construct the accompanying Darboux trihedron of a curve on the surface: *a* — Frénet and Darboux trihedra of a flat curve belonging to a conic surface; *b* — separate positions of the Darboux trihedron of the curve on the transfer surface

For a Darboux trihedron, one must choose the direction of the ort $\overline{N}$ since the normal vector to the surface can be directed both in one direction and in the opposite direction. For example, at point *A* (Fig. 1, *a*), it is directed inside the cone. When moving to point *B*, its direction is also directed inward into the cone, that is, the Darboux trihedron at point *B* is defined, unlike the Frénet trihedron. If the direction of the ort $\overline{N}$ is changed to the opposite, then the direction of the ort $\overline{P}$ will also change to the opposite.

The calculations were carried out in the computer algebra system Wolfram Mathematica [15]. Figures were drawn in the environment of the commercial computer algebra system Maple [16].

5. Results of constructing curves and surfaces using the Darboux trihedron

5. 1. Analytical substantiation of the position of the Darboux trihedron through the parameters of the surface and the curve on it

Let the surface be given by the parametric equations $X=X(u, v)$, $Y=Y(u, v)$, $Z=Z(u, v)$, where u, v are independent variables of the surface. To set a line on it, you need to make variables u and v dependent on each other. This can be done in different ways: assign the dependence $v=v(u)$ or $u=u(v)$ or relate them through the third variable t: $v=v(t)$, $u=u(t)$. On the surface then there is a description a line in a function of one of these variables: u, v, or t. If the dependence $v=v(u)$ is assigned, then the parametric equations of the line are written as $x=x(u, v(u))$, $y=y(u, v(u))$, $z=z(u, v(u))$. To find the ort of the tangent $\overline{T}$, it is necessary to differentiate the obtained equations by the variable u and, via normalization, reduce to the unit vector. After such normalization, the numerical value of each of the three expressions at a specific value of u=const will be equal to the corresponding directional cosine of ort $\overline{T}$. The normal $\overline{N}$ to the surface is the vector product of vectors tangent to the coordinate lines that assign partial derivatives:

$$\overline{N} = \begin{vmatrix} X & Y & Z \\ X_u & Y_u & Z_u \\ X_v & Y_v & Z_v \end{vmatrix} = \\ = \left\{ Y_u Z_v - Y_v Z_u;\ -X_u Z_v + X_v Z_u;\ X_u Y_v - X_v Y_u \right\}. \tag{1}$$

A variable with an index at the bottom means a partial derivative of the corresponding variable, for example, $X_u=\partial X/\partial u$. If you swap the last two lines in determinant (1), the vector $\overline{N}$ will change its direction to the opposite one. For a vector $\overline{N}$ to become a unit vector, it needs to be normalized. The resulting three projections of the unit vector $\overline{N}$ will be its guide cosines. It should be borne in mind that the three expressions obtained as a result of the disclosure of determinant (1) can be functions of two variables u and v. Substituting specific numerical values of these curved coordinates will give the direction of the ort at the corresponding point on the surface. It is necessary to have an ort at a point of the curve on the surface, so you need to go to one variable, substituting the relationship between the variables u and v into the projection expressions.

The third ort $\overline{P}$ is found from the vector product of the orts $\overline{N}$ and $\overline{T}$. There is a condition that the ort $\overline{P}$ has the desired direction (the trihedron is right-hand). In the bottom two lines of determinant (1), the expressions of the ort $\overline{N}$ projections must be higher, and the expressions of the ort projections $\overline{T}$ must be lower. It is convenient to open such determinants with the help of symbolic mathematics software products. If the orts $\overline{N}$ and $\overline{T}$ were single, then the resulting ort $\overline{P}$ would also be single, that is, it does not need to be normalized.

5. 2. Demonstration of determining the guide cosines of a Darboux trihedron with respect to a fixed coordinate system

For example, take the transfer surface formed by moving a sine wave by a sine wave in two mutually perpendicular planes. This will not narrow the overall solution, but it will somewhat simplify the expressions. Parametric surface equations are:

$$X = u;$$

$$Y = v;$$

$$Z = a(\sin v + \cos u), \tag{2}$$

where a is a constant value.

Find the guide cosines of the ort $\overline{N}$. The partial derivatives of equation (2) are written:

$$X_u = 1; Y_u = 0; Z_u = -a\sin u;$$

$$X_v = 0; Y_v = 1; Z_v = a\cos v. \tag{3}$$

We open the determinant (1) when substituting partial derivatives (3) into it:

$$\{a\sin u \quad -a\cos v \quad 1\}. \tag{4}$$

Dividing the projections of vector (4) by its module, we obtain the guide cosines l, m, n of the ort $\overline{N}$:

$$l_N = \frac{a\sin u}{\sqrt{1+a^2\left(\cos^2 v+\sin^2 u\right)}};$$

$$m_N = -\frac{a\cos v}{\sqrt{1+a^2\left(\cos^2 v+\sin^2 u\right)}};$$

$$n_N = \frac{1}{\sqrt{1+a^2\left(\cos^2 v+\sin^2 u\right)}}. \tag{5}$$

If the dependence $v=v(u)$ is established, the vector of the tangent to the curve on the surface (1) is found by differentiating these equations (that is, already the equations of the curve) by the variable u:

$$x' = 1;$$

$$y' = v';$$

$$z' = a(v'\cos v - \sin u). \tag{6}$$

The guide cosines of the ort $\overline{T}$ are found by normalizing its projections (6):

$$l_T = \frac{1}{\sqrt{1+v'^2+a^2\left(v'\cos v-\sin u\right)^2}};$$

$$m_T = \frac{v'}{\sqrt{1+v'^2+a^2\left(v'\cos v-\sin u\right)^2}};$$

$$n_T = \frac{a\left(v'\cos v-\sin u\right)}{\sqrt{1+v'^2+a^2\left(v'\cos v-\sin u\right)^2}}. \tag{7}$$

The guide cosines of the ort $\overline{P}$ are found by vector multiplication of the orts $\overline{N}$ (5) and $\overline{T}$ (7):

$$l_P = -\frac{v'+a^2\cos v\left(v'\cos v-\sin u\right)}{\sqrt{\left[1+a^2\left(\cos^2 v+\sin^2 u\right)\right]\left[1+v'^2+a^2\left(v'\cos v-\sin u\right)^2\right]}};$$

$$m_P = \frac{1-a^2\sin u\left(v'\cos v-\sin u\right)}{\sqrt{\left[1+a^2\left(\cos^2 v+\sin^2 u\right)\right]\left[1+v'^2+a^2\left(v'\cos v-\sin u\right)^2\right]}};$$

$$n_P = \frac{a\left(\cos v+v'\sin u\right)}{\sqrt{\left[1+a^2\left(\cos^2 v+\sin^2 u\right)\right]\left[1+v'^2+a^2\left(v'\cos v-\sin u\right)^2\right]}}. \tag{8}$$

Guide cosines (5), (7), (8) of orts $\overline{N}, \overline{T}, \overline{P}$ are represented in general form with an unknown dependence $v=v(u)$, that is, for an unspecified curve on the surface (2). Let such a dependence be the simplest: $v=u$. So, $v'=1$. Substitution $v=u$ in equation of the surface (2) will give a line on its surface that passes diagonally through the cells of the coordinate grid (Fig. 1, *b*). The substitution $v=u$ and $v'=1$ in the expressions of guide cosines (5), (7), (8) makes them dependent on only one variable – u, the numerical value of which specifies a point on the curve. For example, the guide cosines of ort $\overline{N}$ after substitution will be recorded:

$$l_N = \frac{a\sin u}{\sqrt{1+a^2}};$$

$$m_N = -\frac{a\cos u}{\sqrt{1+a^2}};$$

$$n_N = \frac{1}{\sqrt{1+a^2}}. \tag{9}$$

The projection n_N of the ort onto the OZ axis is positive, indicating that the ort of the normal points upwards towards the surface. In Fig. 1, *b*, surface (2) is constructed at $a=0.3$ and changing independent variables within $u=0...3\pi$, $v=0...3\pi$. The orts of the Darboux trihedron are constructed at different values of the variable u.

5. 3. Practical application of the obtained dependences for the construction of curves and surfaces

Fig. 1, *b* shows three positions of the Darboux trihedron on a given line. Such positions can be constructed as much as you like at a given interval Δu. If the parameter u changes continuously, the Darboux trihedron will move along the curve. In the Darboux trihedron system, you can specify a point C with its coordinates: $C\{T_C, P_C, N_C\}$. When a Darboux trihedron moves along a curve, a fixed point C in its system will describe a certain line with respect to the fixed coordinate system $OXYZ$. There are formulas for the transition from the coordinates in the trihedron system to the coordinates of the main (fixed) system using guide cosines. They take the form:

$$x_C = T_C l_T + P_C l_P + N_C l_N;$$

$$y_C = T_C m_T + P_C m_P + N_C m_N;$$

$$z_C = T_C n_T + P_C n_P + N_C n_N. \tag{10}$$

Relations (10) establish the correspondence between a point in a Darboux trihedron and a fixed coordinate system with their vertices combined. To obtain the coordinates of point C at a given point of the curve in a fixed coordinate system, you need to carry out a parallel transfer by the value of the coordinates of the point of the curve in the fixed system $OXYZ$. Taking into account the above at $v=u$ the fixed point in the trihedron system with coordinates $C\{T_C, P_C, N_C\}$ describes the trajectory in a fixed coordinate system according to parametric equations:

$$x_C = T_C\frac{1}{\sqrt{2+a^2-a^2\sin 2u}}+$$
$$+P_C\frac{1+a^2\cos u(\cos v-\sin u)}{\sqrt{(1+a^2)(2+a^2-a^2\sin 2u)}}+N_C\frac{a\sin u}{\sqrt{1+a^2}}+u;$$
$$y_C = T_C\frac{1}{\sqrt{2+a^2-a^2\sin 2u}}+$$
$$+P_C\frac{1-a^2\sin u(\cos u-\sin u)}{\sqrt{(1+a^2)(2+a^2-a^2\sin 2u)}}-N_C\frac{a\cos u}{\sqrt{1+a^2}}+u;$$
$$z_C = T_C\frac{a(\cos u-\sin u)}{\sqrt{2+a^2-a^2\sin 2u}}+$$
$$+P_C\frac{a(\cos v+\sin u)}{\sqrt{(1+a^2)(2+a^2-a^2\sin 2u)}}+$$
$$+N_C\frac{1}{\sqrt{1+a^2}}+a(\sin u+\cos u). \quad (11)$$

Point C can be mobile in the Darboux trihedron system. Let it describe the circle of radius r in the plane of the trihedron formed by the orts $\overline{T}$ and $\overline{N}$ with the center shifted by the distance r along the ort $\overline{N}$. The rotation of the point of a circle in the plane and the motion of the Darboux trihedron are consistent with each other and depend on the variable u. The parametric equations of a circle in the trihedron system are written:

$$T_C = r\cos\omega u;$$
$$P_C = 0;$$
$$N_C = r - r\sin\omega u, \quad (12)$$

where ω is a constant that affects the speed of rotation of a point.

Substitution (12) in (11) will give parametric equations of the curve, which is the result of adding two movements: the bulk motion of the trihedron along the curve and the rotational motion of a point in the trihedron system. In Fig. 2, *a*, the curve is constructed at $r=0.5$ and $\omega=3$. The constant ω is chosen in such a way that the curve resembles a cycloid when a circle rolls in a straight line. Our rolling occurs along a curve and the curve itself is not flat, but spatial.

You can specify the rotational motion of point C in the plane of the trihedron formed by the orts $\overline{P}$ and $\overline{N}$. Then the parametric equations of the circle in the trihedron system are written:

$$T_C = 0;$$
$$P_C = r\cos\omega u;$$
$$N_C = r\sin\omega u. \quad (13)$$

In this case, the rotation of the point occurs in the normal plane of the trihedron around the origin. The result is a curve similar to a helix with a curved axis, which is the given curve on the surface. In Fig. 2, *b*, a curve is constructed at $r=1$ and $\omega=15$.

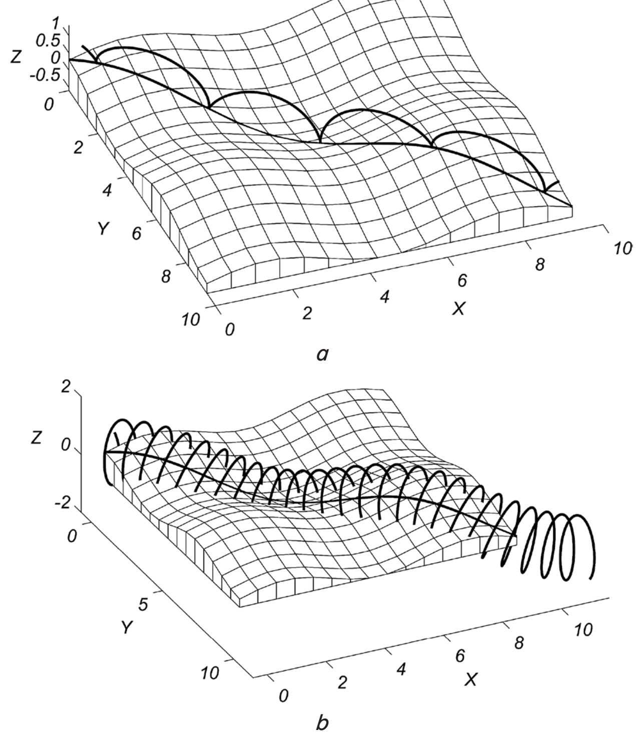

Fig. 2. Spatial curves as a result of adding the motion of a Darboux trihedron along the predefined curve on the surface and the rotational motion of a point in the trihedron system: *a* – the rotation of a point occurs in the plane formed by the orts $\overline{T}$ and $\overline{N}$; *b* – the rotation of the point occurs in the plane formed by the orts $\overline{P}$ and $\overline{N}$

The constructed curves are absolute trajectories of addition of two movements – the bulk motion of the Darboux trihedron on the surface and the relative motion of a point in the system of the trihedron itself. In this case, if necessary, one can easily find expressions of absolute velocity and acceleration by sequential differentiation of absolute trajectory equations. A more difficult way is to find the acceleration of the bulk motion, relative motion, and Coriolis acceleration followed by their vector addition.

Using the Darboux trihedron, surfaces can also be built. To this end, you need to place a certain curve in its system. When the Darboux trihedron moves, the set of curve positions will form a surface. In the parametric equations of the surface there are two independent variables. In our case, one of them is the variable u, which specifies the displacement and orientation of the Darboux trihedron. As the second variable, we take t, with which we describe the generatrix curve in the trihedron system. Let it be a circle of radius r located in the normal plane of a Darboux trihedron with center shifted by the value of radius r in the direction of the normal $\overline{N}$. Its parametric equations will be written as:

$$T_C = 0;$$
$$P_C = r\cos t;$$
$$N_C = r + r\sin t. \quad (14)$$

To close curve (14), the variable t must vary within $t=0...2\pi$. By substituting equations (14) into (11), parametric equations of the tubular surface can be obtained. In Fig. 3, *a*, it is built at $r=0.5$. The axis of the tubular surface is raised above the transfer surface by the value of radius r, so it touches the transfer surface along the line set on it by the internal equation $v=u$. You can specify a different line on the transfer surface. For example, assign a line using a trigonometric function and find its derivative:

$$v = b\sin u + c,$$

$$v' = b\cos u. \tag{15}$$

Substituting the expression v from (15) into (2) forms a new line on the transfer surface. When substituting expressions (15) in (5), (7), and (8), we obtain the guide cosines of the Darboux trihedron for the new curve. After that, you can build both lines and surfaces using a trihedron. In Fig. 3, *b*, we constructed a tubular surface that touches the line on the transfer surface given by equation (15). To this end, equation (14) was substituted into the updated guide cosines and the subsequent surface equation.

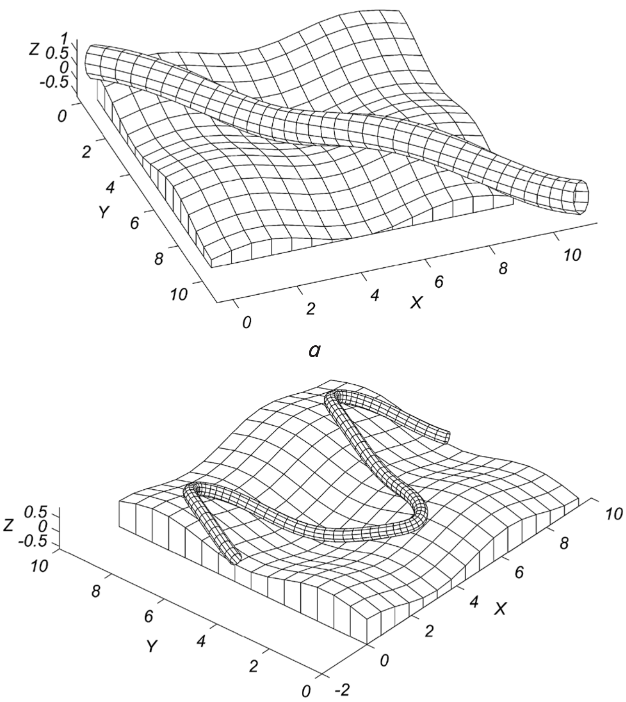

Fig. 3. Tubular surfaces that touch the transfer surface along the given lines on it: *a* — the line on the transfer surface is given by the equation $v=u$; *b* — the line on the transfer surface is given by the equation $v=b\sin u+c$

The approach devised allows us to build other surfaces with a generating curve of variable shape. For example, in equation (14) one can specify the generating circle of variable radius with dependence $r=r(u)$.

6. Discussion of results of the construction of curves and surfaces using the Darboux trihedron

Our results on the construction of curves and surfaces using the Darboux trihedron with the predefined trajectory of its motion can be explained by using two coordinate systems. One of them is a fixed system, and the second is a mobile one, the role of which belongs to the Darboux trihedron. The fixed system is known. A feature of the formation of a moving system, the role of which belongs to the Darboux trihedron, is the definition of orts so that this system is right-hand. This is an important condition since the transition from a moving system of a Darboux trihedron to a fixed one via guide cosines requires that both systems be right-hand. In this respect, the position of the Darboux trihedron can be determined in two ways: the ort of the normal to the surface, which is determined by the disclosure of determinant (1), can be directed in one or the opposite direction of the surface. The peculiarity of our study is that its original author makes a decision regarding the direction of the normal in each case, based on the conditions of the problem set. The direction of the ort is changed by changing the last two lines in the determinant (1). For example, when constructing curves (Fig. 2) and surfaces (Fig. 3), the direction of the normal to the surface was chosen positive. When changing the direction of the normal vector, the tubular surfaces in Fig. 3 would be built not above the surface, but below it.

The proposed approach allows obtaining curves and surfaces based on the physical essence of the problem. Typically, lines and surfaces are described by parametric equations in a fixed coordinate system, for example, in [13, 14]. The advantage of the proposed approach makes it possible not only to expand the possibilities of shaping lines and surfaces but also to carry out their analytical description for specific tasks, for example, for laying a pipeline on the predefined surface.

The possibilities of the proposed approach are greatly expanded if the guide line of the Darboux trihedron on the surface is described in the eigen arc length function. This fact is a limitation to the wider use of the Darboux trihedron since among the variety of curves on the surface, only a small part can be described as a function of the length of the natural arc. If the trajectory of the bulk motion of a trihedron is described in a function of the length of its eigen arc, then the derivatives of the trajectory equations are a unit vector that does not need to be normalized. In addition, you can apply formulas of differential geometry, similar to Frénet formulas for the Frénet trihedron. The disadvantages of the method include cumbersome expressions of the guide cosines of the Darboux trihedron for some surfaces. However, this is not an insurmountable obstacle to improving the proposed approach. In the future, further research can be directed to numerical methods of calculation.

7. Conclusions

1. At each point of the curve that lies on the surface, you can construct a Darboux trihedron. One of its orts is the unit vector, pointing normally towards the surface. However, the normal itself can be directed in one or the opposite direction from a point on the surface. It has been justified how to find three orts of a Darboux trihedron so that its system is right-hand.

2. Analytic expressions for guide cosines of each ort of the Darboux trihedron have been found. These expressions are represented in terms of the parameters of the surface and the curve on it. The curve on the surface is given by a certain relationship between independent surface variables. The type of this dependence is due to the shape of the curve on the surface. When establishing such a dependence, the position of the Darboux trihedron on the surface, which is accompanying the curve, depends only on the value of one independent variable.

3. Our theoretical results of the analytical description of the motion of a point or line in the system of a moving trihedron were used to construct curves and surfaces using the accompanying Darboux trihedron of a curve on the surface. The transfer surface and the line on it were given, which runs diagonally through the cells of the curved coordinate grid of the surface. When a Darboux trihedron moved along this line in its system in one of the coordinate planes, the point described a circle. As a result, the absolute trajectory of a point in projections onto a fixed coordinate system was obtained. In one case, it is analogous to a cycloid for a straight line on a plane, in the other – a helix with a curved axis. If a Darboux trihedron is moved with a fixed circle in its normal plane, a tubular surface is formed. In the work, we built such surfaces in the form of a pipeline with a curved axis, which lies on the surface.

Conflicts of interest

The authors declare that they have no conflicts of interest in relation to the current study, including financial, personal, authorship, or any other, that could affect the study and the results reported in this paper.

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Data availability

All data are available in the main text of the manuscript.

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