The object of research is the processes of radiation transfer in the «Sun – paraboloid concentrator – heat receiver» system. There are many factors that affect the value of the density of the concentrated heat flux that reaches the surface of the heat sink. The study of the influence of these factors on the overall energy indicators of the system is an important scientific problem that was solved in this work. To solve this problem, a generalized mathematical model of the radiation transfer process in the «Sun – concentrator – heat receiver» system was built, which was adapted for a paraboloid concentrator. The constructed mathematical model was solved by an approximate analytical method, which took into account the integral and discrete parameters of the system, as well as the probability distribution of aberrations of the concentrator surface, its defocusing, and other random influences. The dimensionless density of the heat flux on the surface of the heat sink for a mathematically ideal and real paraboloid concentrator of a fixed geometry was determined. Using the found analytical solution, the results obtained on the basis of the Monte Carlo method were verified. Analytical and numerical results for a mathematically ideal and a real concentrator with minor aberrations and a clear orientation to the Sun agree within the permissible error. For a real concentrator with defocusing, a deviation of numerical data from analytical data was observed. The presence of deviations is associated with a simplification in the interpretation of the analytical probability distribution, in which it is impossible to take into account each influence separately. The obtained analytical results will be useful in the development of real power plants and can be used practically at the stage of checking the adequacy of the system model

Keywords: paraboloid concentrator, mathematical model, analytical solution, error function, verification of results

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CONSTRUCTION OF A MATHEMATICAL MODEL AND APPROXIMATE ANALYTICAL SOLUTION TO THE PROBLEM OF ENERGY EXCHANGE IN THE «SUN – PARABOLOID CONCENTRATOR – HEAT RECEIVER» SYSTEM

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1. Introduction

Against the background of the global energy crisis, research into alternative energy sources is gaining more and more popularity. One of the promising directions is the study of thermodynamic solar energy systems with concentrators of various types. These studies are based on the modeling of interrelated energy transfer processes in the relevant systems with further verification of the obtained results. The basis of such modeling is the development of a generalized mathematical model of the process of transferring solar radiation in the «Sun – concentrator – heat receiver» system. Such a model is proposed to be adapted for a paraboloid concentrator and solved analytically. The obtained analytical solution will be used to verify the results of the numerical algorithms obtained in works [1, 2].

The results of such studies are needed in practice because the problem of verification of data obtained with the help of numerical solutions is still relevant. At the stage of modeling of real power plants with paraboloid concentrators, an important aspect is checking the model for adequacy by comparing the obtained numerical results with analytical data or real results of existing power plants. That is why an analytical solution that allows for a quick check of the fact of the validity of numerical data will be useful in the development of real energy systems.

2. Literature review and problem statement

Most of the scientific research related to the modeling of paraboloid concentrators is performed using the Monte Carlo method based on proprietary and commercial software products. A general analysis of the state of research on solar paraboloid concentrators, prospects for their use, and scientific problems that arise during the design of appropriate solar energy conversion systems are given in works [3, 4].

Paper [5] presents the results of numerical studies of a two-mirror paraboloid concentrator of a new design, which were carried out by the Monte Carlo method. It is shown that the proposed concentrator has higher energy indicators in the radial direction compared to the concentrator of traditional design. The authors compared their own numerical MATLAB results with the results obtained using the commercial Tracepro® package. But issues related to taking into account the effect of defocus remained unresolved. In addition, the basic mathematical model was not highlighted, no attempts were made to obtain an analytical solution. Optical surface errors were taken into account using a probability distribution, which takes into account all types of possible surface errors. This approach does not make it possible to determine the contribution of each of these types separately. An option to overcome the relevant difficulties can be the approach proposed in [6], where a separate type of optical errors related to environmental conditions (wind load, etc.) was studied in detail. A combination of the finite element method and the Monte Carlo method with a new type of discretization of the calculation area was used for the calculation. This approach was used for a system with a payload in the Stirling cycle. Approaches to the design of paraboloid concentrators, which involve the use of the Stirling cycle, are summarized in [7]. It is shown that the efficiency of such systems is 22–23 % and can be increased by choosing the optimal geometry of the concentrator and cavity heat receiver, the type of material for their manufacture, etc. The authors emphasize the significant difficulties that arise during the optimization of systems with paraboloid concentrators. One of the options for overcoming these difficulties is proposed in [8], where the method of optimization of the «paraboloid concentrator – conical heat receiver» system is given based on multi-criteria experimental studies and numerical calculations in the COMSOL Multiphysics® system. It was determined that the efficiency of such a power system is 66 % at a paraboloid concentration level of 31.15. The work did not consider in detail the methods of taking into account the errors of the surface of the concentrators, as well as the possibility of scaling the obtained data to another geometry of the heat sink. An option to overcome the relevant difficulties can be the study of cavity heat receivers of different geometries. This is the approach used in work [9], where several types of cavity heat receivers were compared from the point of view of their energy efficiency with a fixed paraboloid geometry. The experimental data obtained on the basis of the photometric approach agreed with the numerical results of the Monte Carlo method. However, analytical approaches were not used. In [10], a Monte Carlo model was developed to determine the relationship between defocusing errors and surface irregularities in a paraboloid concentrator. The main attention was paid to the peculiarities of the functioning of the tracking system and the possibility of optimizing its operation by taking into account surface errors. To take into account these errors, an approach was used, which involves taking into account the unevenness of the distribution of rays in the solar beam that falls on the concentrator and is reflected from it. This distribution corresponded to the classical astronomical distribution of energy in the solar beam. This approach is impractical when constructing an analytical solution, which is described in [11]. The analytical solution constructed for the ideal concentrator is based on the infinitesimal pyramid method, and surface errors are taken into account by introducing an appropriate measure of accuracy. This is one of the few analytical solutions used to verify most numerical studies. One of these studies is described in [12], where the heat flux density distributions for ideal concentrators are presented, as well as the influence of concentrator surface aberrations on these distributions. The numerical algorithm built on the basis of the Monte Carlo method was verified using the numerical model based on COMPREC proposed by the authors of [13], as well as the analytical solution presented in [11].

Work [1] also reports the results of computer modeling of paraboloid concentrator parameters, which was carried out on the basis of the Monte Carlo method. The developed model takes into account the influence of concentrator surface aberrations on the heat flow distribution on the surface of the heat sink. Aberrations were modeled numerically using uniform and normal distribution laws. Study [1] was continued in [2] by taking into account the defocusing factor of the paraboloid concentrator relative to the direction to the Sun. The results obtained in [1, 2] require further verification, which is planned to be carried out using an analytical approach. In contrast to the classical analytical solution [11], this work involves the construction of an analytical solution in which the probability distribution of surface errors is introduced. Such a solution is built on the basis of a mathematical model that is typical for conducting numerical studies using the Monte Carlo method.

All this gives reason to argue that it is advisable to conduct an appropriate study and obtain an approximate analytical solution that will be relevant to most numerical models that use the probability distribution of surface errors.

3. The aim and objectives of the study

The purpose of this study is to find an analytical solution to the problem of solar radiation transfer in the system «Sun – paraboloid concentrator – heat sink». This will make it possible to obtain a tool for verifying the fact of the reliability of the data obtained with the help of a numerical solution.

To achieve the goal, the following tasks were set:

– to build a generalized mathematical model of solar radiation concentration;

– to analyze the main assumptions regarding the consideration of the indicatrix in the mathematical model of the paraboloid concentrator. Based on the chosen assumption, create an authentic calculation model for a paraboloid concentrator of arbitrary geometry;

– based on the calculation model, find an analytical solution to the problem of solar radiation concentration;

– using the obtained analytical solution, prove the fact of the reliability of the numerical data obtained by the Monte Carlo method in [1, 2].

4. The study materials and methods

The object of our research is the process of transferring solar radiation in systems with concentrators. The construction of a mathematical model of the concentration process «Sun – concentrator – heat receiver» was carried out in two stages. At the first stage, the process of reflection of solar radiation from the mirror surface of the concentrator was analyzed, and at the second stage, the characteristics of the heat receiver were determined.

It is assumed that the mirror surface of the concentrator has an arbitrary geometric shape and is described by an equation of the form:

$$F_C(x,y,z)=0, \quad (1)$$

where $x, y, z \in S_C$.

A heat sink is located at a suitable distance from the concentrator, the surface of which is given by the equation:

$$F_A(x,y,z)=0, \quad (2)$$

where $x, y, z \in S_A$.

The main hypothesis of the study assumes that the analytical results of the distribution of heat flux on the surface of the heat sink, obtained using the developed solution, will coincide within the permissible error with the numerical results obtained in [1, 2], as well as with the classical analytical solution presented in [11]. This will make it possible to confirm the adequacy of the obtained analytical solution and allow it to be used as a test when creating models of real concentrators.

A photometric approach was used to build a mathematical model [14].

The main assumptions, which were considered, classified, and accepted in the study, related to the indicator of a beam of solar rays that falls on the mirror surface of the concentrator. Taking into account these assumptions and applying the developed generalized model to the paraboloid equation, a model of a paraboloid concentrator of arbitrary geometry was obtained.

To find an analytical solution to the concentration problem, the normal law of the distribution of angular errors was used, which was chosen to characterize the deviation of a beam of sunlight reflected from a real concentrator relative to its ideal direction. The probability parameter that characterizes the deviation of the solar beam very accurately simulates the real negative factors of the concentration process, such as: irregularities of the mirror surface of the concentrator, defocusing of the concentrator relative to the direction to the Sun.

The analytical data obtained as a result of the study were compared with the numerical results obtained using the Monte Carlo algorithm in [1, 2].

5. Results of research into the process of solar radiation concentration

5. 1. Construction of a generalized mathematical model of the solar radiation transfer process

Fig. 1 schematically depicts the process of reflection of the sun's beam from the surface of the concentrator. On the mirror surface of the concentrator, we select the elementary area dS_C, which is the neighborhood of the point $Q_i(x_C, y_C, z_C)$. The location of the section dS_C is uniquely determined by the direction of the normal vector $\overline{n}_C$. A beam of solar rays $\overline{l}_S^{\,0}$, falls on this point and forms a solid angle of $2\psi_C$, where $\psi_S=16'$. The beam of rays $\overline{l}_C^{\,0}$, reflected from the surface element of the concentrator dS_C is characterized by the angle φ. The normal of the concentrator $\overline{n}_C$ forms with vectors $-\overline{l}_S^{\,0}$, and $\overline{l}_C^{\,0}$, a system of angles ϑ_C, ϑ_A, which determine the location of the concentrator relative to the Sun and the heat sink. The vector $\overline{l}_{CA}^{\,0}$, that forms the solid angle φ_0 characterizes the direction from the concentrator to the receiver for the beam of rays falling on the heat receiver.

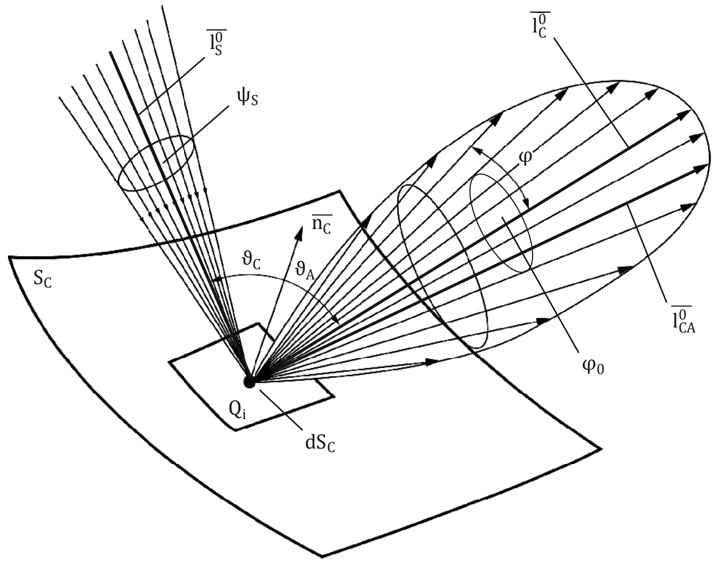


Fig. 1. Scheme of reflection of solar radiation from the surface of the concentrator

The total flux of radiation reflected by the element dS_C is determined by the ratio:

$$d\Phi_0 = \beta_C d\Phi_F = \beta_C E_S dS_{C\perp}, \quad (3)$$

where E_S is the heat flux density coming from the Sun, W/m^2; β_C is the reflection coefficient of the concentrator surface.

We define the projection of an elementary section as:

$$dS_{C\perp} = dS_C \cos\vartheta_C, \quad (4)$$

where $\cos\vartheta_C = \overline{l}_S^{\,0}\overline{n}_C$.

The reflected radiation propagates in the solar beam with the indicatrix function $f_0(\varphi)$:

$$f_0(\varphi) = dI(\varphi)/dI(\varphi), \quad (5)$$

where $dI(\varphi)$, $dI(0)$ is the radiation intensity within the corresponding solid angle and in the main direction of the solid angle, respectively.

Normalization is carried out according to the ratio:

$$d\Phi_0 = \int_0^{2\pi}\int_0^{\varphi} dI(\varphi)\,\mathrm{d}\omega, \quad (6)$$

where $\mathrm{d}\omega=\sin\varphi d\varphi d\vartheta$ is the elementary area of a spherical surface of radius 1.

We substitute (5) in (6), and we get:

$$d\Phi_0 = dI(0)2\pi\int_0^{\varphi_0} f_0(\varphi)\sin\varphi\,\mathrm{d}\varphi. \quad (7)$$

Equating the right-hand sides of equations (3) and (7), we have:

$$dI(0) = \frac{\beta_C E_S dS_{C\perp}}{2\pi\int_0^{\varphi_0} f_0(\varphi)\sin\varphi\,\mathrm{d}\varphi}. \quad (8)$$

Based on equations (8) and (5), we obtain the ratio for the radiation force in any direction inside the reflected beam:

$$dI(\varphi)=\frac{f_0(\varphi)\beta_C E_S dS_{C\perp}}{2\pi\int_0^{\varphi_0} f_0(\varphi)\sin\varphi d\varphi}. \quad (9)$$

Let's move on to defining the characteristics of the heat receiver. The transfer of radiation between the concentrator and the receiver is a heat exchange between two surfaces that are arbitrarily oriented in space.

Let's select any point $P(x_A, y_A, z_A)$ on the surface of the heat receiver, given by equation (2), and pass the $\bar{l}_{CA}$, ray to it from the point $Q_i(x_C, y_C, z_C)$ of the concentrator. The unit vector of this ray, which characterizes its direction in space, is determined by the equation:

$$\bar{l}_{CA}=\frac{(x_A-x_C)i+(y_A-y_C)j+(z_A-z_C)k}{l_{CA}}, \quad (10)$$

where $l_{CA}=|\bar{l}_{CA}|$.

Based on the known photometric relation [14], we have:

$$dE_A=\frac{dI(l_{CA})\cos\vartheta_A}{l_{CA}^2}, \quad (11)$$

where $\cos\vartheta_A=-\bar{l}_{CA}^0\bar{n}_A$, a $\bar{n}_A=\frac{\operatorname{grad}F_A(x,y,z)}{|\operatorname{grad}F_A(x,y,z)|}$.

To determine the radiation force in the direction of the vector, it is necessary to establish whether the flow from the concentrator to the given point of the receiver is transferred in this direction, and also to determine the magnitude of this flow.

In order to determine that the beam $\bar{l}_{CA}^0$ enters the angle φ_0 and that this beam enters the inlet of the heat receiver S_{inl}, we introduce two functions of the Heaviside type, respectively:

$$\xi(\bar{l}_{CA})=\begin{cases}1, \text{if } (\bar{l}_{CA}^0\bar{l}_C^0)\geq\cos\varphi_0,\\ 0, \text{if } (\bar{l}_{CA}^0\bar{l}_C^0)<\cos\varphi_0,\end{cases}$$

$$\chi(\bar{l}_{CA})=\begin{cases}1, \text{if } \bar{l}_{CA}\cap S_{inl}\neq 0,\\ 0, \text{if } \bar{l}_{CA}\cap S_{inl}=0.\end{cases} \quad (12)$$

Taking into account the functions $\xi(\bar{l}_{CA})$ and $\chi(\bar{l}_{CA})$, the expression for the radiation force in the direction $\bar{l}_{CA}$ can be written as:

$$dI(\bar{l}_{CA})=dI(\varphi)\xi(\bar{l}_{CA})\chi(\bar{l}_{CA}). \quad (13)$$

Substituting equality (13) in relation (11) and using expression (9), we have:

$$dE_A=\frac{E_S\beta_C f_0(\varphi)\xi(\bar{l}_{CA})\chi(\bar{l}_{CA})\cos\vartheta_C\cos\vartheta_A dS_C}{2\pi l_{CA}^2\int_0^{\varphi_0} f_0(\varphi)\sin\varphi d\varphi}. \quad (14)$$

Taking into account the properties of flux additivity, the full radiation density at point P is found by integrating expression (14) over the entire mirror surface of the concentrator:

$$E_A=\frac{E_S}{2\pi}\iint_{S_C}\frac{\beta_C f_0(\varphi)\xi(\bar{l}_{CA})\chi(\bar{l}_{CA})\cos\vartheta_C\cos\vartheta_A \mathrm{d}S_C}{l_{CA}^2\int_0^{\varphi_0} f_0(\varphi)\sin\varphi d\varphi}, \quad (15)$$

or its dimensionless analog:

$$\varepsilon_A=\frac{E_A}{E_S}=$$
$$=\frac{1}{2\pi}\iint_{S_C}\frac{\beta_C f_0(\varphi)\xi(\bar{l}_{CA})\chi(\bar{l}_{CA})\cos\vartheta_C\cos\vartheta_A \mathrm{d}S_C}{l_{CA}^2\int_0^{\varphi_0} f_0(\varphi)\sin\varphi d\varphi}. \quad (16)$$

Thus, expression (16) can be considered a generalized model of solar radiation transfer in the «Sun – concentrator – heat receiver» system. This generalized mathematical model does not depend on the geometry of the concentrator.

5. 2. Constructing a calculation model for a paraboloid concentrator

Various assumptions are used to solve the constructed generalized mathematical model (12) to (16), which can be divided into 4 main groups. These assumptions are classified relative to the value of the indicatrix function. This distribution is shown schematically in Fig. 2.

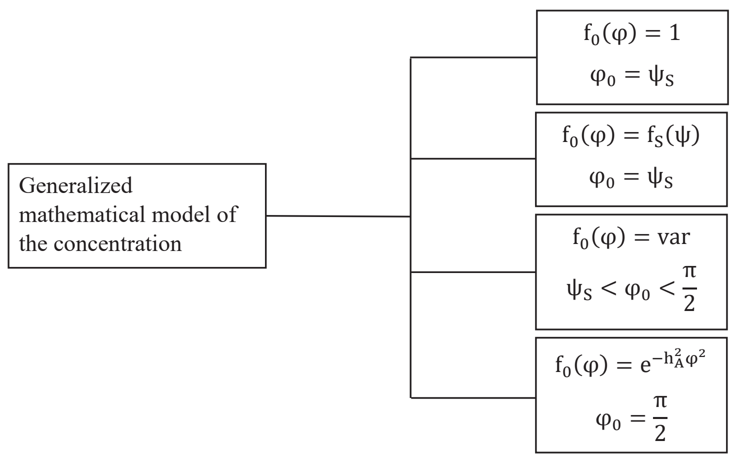


Fig. 2. Classification of concentration process models by type of indicator

The first group includes models built on the assumption that the size of the reflected beam is equal to the size of the incident beam and the distribution of radiation in the beam is uniform. Mathematically, these assumptions can be written as follows:

$$f_0(\varphi)=1,\ \varphi_0=\psi_S. \quad (17)$$

The second group consists of models built on the assumption that the incident and reflected beams of rays are completely identical from the point of view of the value of the radiation indicatrix. Mathematically, this condition looks like this:

$$f_0(\varphi)=f_S(\psi),\ \varphi_0=\psi_S. \quad (18)$$

The third group of models is built with a real view of the indicatrix function of the reflected beam, provided that the scattering effect is taken into account. Mathematically, it looks like this:

$$f_0(\varphi)=\text{var},\ \psi_S<\varphi_0<\frac{\pi}{2}. \quad (19)$$

The models belonging to the fourth group are built on the assumption proposed in [15]. This assumption is that the conventional beam reflected from the concentrator has an indicatrix of the form:

$$f_0(\varphi)=e^{-h_A^2\varphi^2};\ \varphi_0=\frac{\pi}{2}, \tag{20}$$

where h_A is the parameter of the energy distribution function in the conditional reflected beam (indicatrix parameter).

The peculiarity of this approach is that the real mirror surface is replaced by a geometrically accurate (ideal) one, and surface aberrations are implicitly taken into account in the indicatrix of the reflected beam using the h_A parameter.

After analyzing the possible approaches to modeling the concentration process, the assumption (17) about the uniform distribution of energy in the reflected beam, which is equal to the solar one, was chosen. Taking into account this assumption, expression (16) is simplified to the form [1]:

$$\varepsilon_A=\frac{E_A}{E_S}=\frac{\beta_C}{\pi\psi_S^2}=\iint_{S_C}\frac{\xi\left(\bar{l}_{CA}\right)\cos\vartheta_C\cos\vartheta_A\,\mathrm{d}S_C}{l_{CA}^2}. \tag{21}$$

Mathematical model (21) was applied for a paraboloid concentrator of the appropriate geometry, the equation of which takes the following form:

$$x^2+y^2=4f_C z, \tag{22}$$

where f_C is the focal length of the concentrator.

Then, passing from the surface integral to the multiple, we have:

$$\varepsilon_A=\frac{E_A}{E_S}=\frac{\beta_C}{2\pi f_C\psi_S^2}= \\ =\int_0^{2\pi}\int_0^{R_C}\frac{\xi\left(\bar{l}_{CA}\right)\cos\vartheta_C\cos\vartheta_A\sqrt{4f_C^2+r^2}\,\mathrm{d}r\mathrm{d}\theta}{l_{CA}^2}, \tag{23}$$

where R_C is the concentrator radius; r, ϑ – radial and angular coordinate, respectively.

5. 3. Approximate method for calculating radiation density

The analytical solution was built on the assumption that the deviation of rays reflected from the surface of a real and mathematically ideal concentrator corresponds to the normal distribution law:

$$P_{\bar{\varphi}}(\varphi)=\frac{1}{\sqrt{2\pi}\sigma_{\bar{\varphi}}}e^{-\frac{\varphi^2}{2\sigma_{\varphi}^2}}, \tag{24}$$

where φ is the random angular deviation of the axis of the reflected beam from the ideal direction; $\sigma_{\bar{\varphi}}$ is the root mean square deviation φ.

Fig. 3 shows the cross-section of path of the rays from the point Q of the concentrator to the point A on the surface of the heat sink.

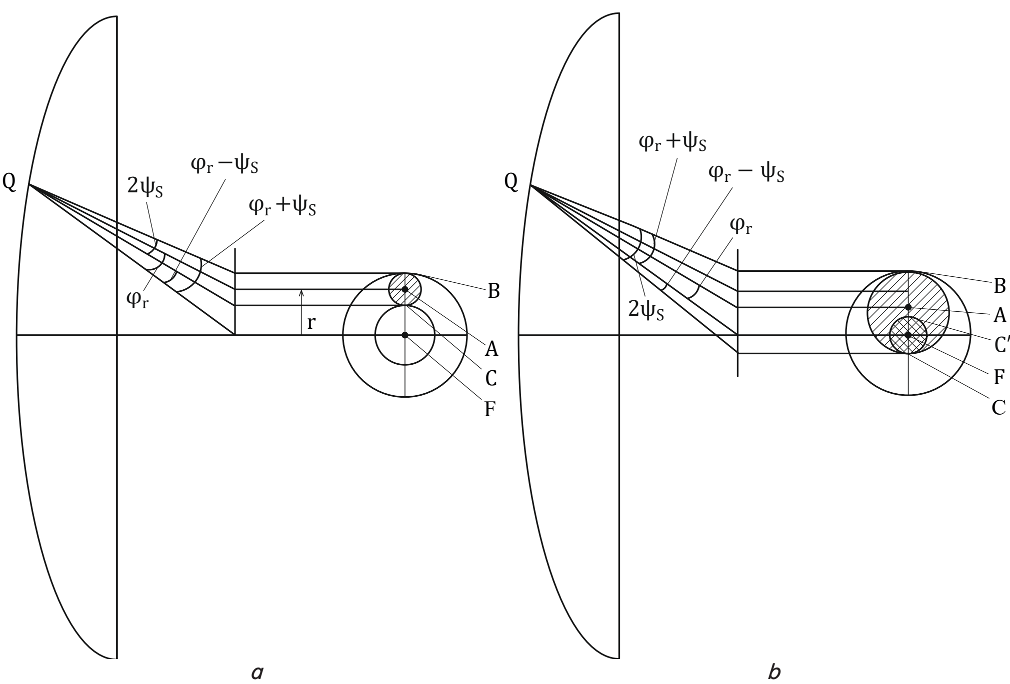
If the reflected beam emanating from point Q on the surface of the concentrator covers point A on the surface of the heat receiver, then the radiation density at this point, taking into account assumption (17) and expression (14), can be determined as follows:

$$dE_A=\frac{E_S\beta_C\cos\vartheta_{QA}dS_{C\perp}}{\pi\psi_S^2 l_{QA}^2}, \tag{25}$$

where ϑ_{QA} is the angle between the ray QA and the normal to the surface of the heat sink at point A; l_{QA} is the distance between points Q and A.

We shall assume that the concentrated heat flux has its maximum value at the focus, i.e., in this case point A coincides with point F, and:

$$l_{QA}\approx l_{QF}=\rho^*,\ \vartheta_{QA}\approx\vartheta_F. \tag{26}$$



a *b*

Fig. 3. Irradiation of a point in the focal plane by the annular zone of the concentrator: *a* – $\varphi_r \geq \psi_S$; *b* – $\varphi_r \leq \psi_S$

Then expression (25) can be written as:

$$dE_{A_{\max}} = \frac{E_S \beta_C \cos\vartheta_F dS_{C\perp}}{\pi\psi_S^2 \rho^{*2}}. \tag{27}$$

To determine the average value of the total radiation density at point A, it is necessary to establish the probability that the set of beams reflected from the surface of the concentrator will cover this point. As can be seen from Fig. 3, point A belongs to a circle of radius r with the center at focus F. The position of any point of this circle in the corresponding meridional plane of the paraboloid concentrator can be described by the following angle [16]:

$$\varphi_r = \frac{r}{p^*}(1+\cos u)\cos u, \tag{28}$$

where $p^*=2f_C$; r, u are the current radius and the current opening angle of the paraboloid, respectively.

On the mirror surface of the concentrator, we highlight an elementary ring area bounded by the angles u and $u+du$. The probability of the event D_{φ_r,ψ_S}, which consists in the fact that the beams reflected by this area will cover point A, is determined as follows [17–19]:

a) at $\varphi_r \geq \psi_S$:

$$P\left(D_{\varphi_r,\psi_S}\right) = 0.25\frac{\psi_S}{\varphi_r} 2\int_{\varphi_r-\psi_S}^{\varphi_r+\psi_S} p_{\tilde{\varphi}}(\varphi)\mathrm{d}\varphi, \tag{29}$$

where $2\int_{\varphi_r-\psi_S}^{\varphi_r+\psi_S} p_{\tilde{\varphi}}(\varphi)\mathrm{d}\varphi$ is the probability that the axis of the beam, reflected by any point of the annular section on the concentrator, will fall into the ring on the focal plane, which is obtained as a result of the rotation of the segment BC around the optical axis; $0.25\psi_S/\varphi_r$ is the ratio of the area of a circle with diameter BC to the area of a ring with width BC.

b) at $\varphi_r \leq \psi_S$:

$$P\left(D_{\varphi_r,\psi_S}\right) = 2\int_0^{\psi_S-\varphi_r} p_{\tilde{\varphi}}(\varphi)\mathrm{d}\varphi + \\ + 0.25\left(2-\frac{\varphi_r}{\psi_S}\right) 2\int_{\psi_S-\varphi_r}^{\psi_S+\varphi_r} p_{\tilde{\varphi}}(\varphi)\mathrm{d}\varphi, \tag{30}$$

where $2\int_0^{\psi_S-\varphi_r} p_{\tilde{\varphi}}(\varphi)\mathrm{d}\varphi$ and $2\int_{\psi_S-\varphi_r}^{\psi_S+\varphi_r} p_{\tilde{\varphi}}(\varphi)\mathrm{d}\varphi$ is the probability of the beam axis hitting a circle with a diameter CC' and a ring obtained as a result of rotating the segment $C'B$ around the optical axis, respectively; $0.25(2-\varphi_r/\psi_S)$ is the ratio of the area of the circle with diameter BC, from which the area of the central circle is subtracted, to the area of the ring with width $C'B$.

Since for the points of the focal plane $\vartheta_{QA} \approx \vartheta_F = u$, the average value of the irradiation, which is created by the annular zone on the concentrator at point A, can be determined as follows:

$$\delta E_r = P\left(D_{\varphi_r,\psi_S}\right)\int_{\theta=0}^{2\pi} \mathrm{d}E_{A_{\max}}. \tag{31}$$

Taking into account the dependence (27), we have:

$$\delta E_r = \frac{2E_S\beta_C}{\psi_S^2} P\left(D_{\varphi_r,\psi_S}\right)\sin u\cos u du. \tag{32}$$

The average value of the total flux density at point A is determined by integrating expression (32) over the opening angle of the concentrator u_C:

$$E_r = \frac{2E_S\beta_C}{\psi_S^2}\int_0^{u_C} P\left(D_{\varphi_r,\psi_S}\right)\sin u\cos u\mathrm{d}u. \tag{33}$$

It should be noted that the probability $P\left(D_{\varphi_r,\psi_S}\right)$ depends on the current angle φ_r, which, in turn, depends on the current value of the angle u. With this approach, it is analytically impossible to calculate the integral (33). To solve this problem, it is necessary to determine the average integral value of the angle φ_r in the interval $[0; u_C]$, which will simplify the integrand expression (33). Based on the mean value theorem, we have:

$$\bar{\varphi}_r = \frac{\int_0^{u_C}\varphi_r \mathrm{d}u}{u_C} \frac{r}{p^*}\frac{\sin u_C + 0.5u_C + 0.25\sin 2u_C}{u_C}. \tag{34}$$

We substitute the obtained value $\bar{\varphi}_r$ into equation (29) or (30) and find the average probability value $P\left(D_{\bar{\varphi}_r,\psi_S}\right)$, which does not depend on the angle u. After substituting this value into expression (33) and carrying out integration, we have:

$$E_r = \frac{2E_S\beta_C}{\psi_S^2} P\left(D_{\bar{\varphi}_r,\psi_S}\right)\int_0^{u_C}\sin u\cos u\mathrm{d}u = \\ = \frac{E_S\beta_C\sin^2 u_C}{\psi_S^2} P\left(D_{\bar{\varphi}_r,\psi_S}\right). \tag{35}$$

For a mathematically ideal concentrator $P\left(D_{\bar{\varphi}_r,\psi_S}\right)=1$. Then the maximum value of irradiation at the focus of such a concentrator is determined as follows:

$$E_{\max}^i = \frac{E_S\beta_C\sin^2 u_C}{\psi_S^2}. \tag{36}$$

For the focus of the paraboloid concentrator, the value $r=0$ and expression (30) will take the following form:

$$P\left(D_{\varphi_r,\psi_S}\right) = \int_{-\psi_S}^{\psi_S} p_{\tilde{\varphi}}(\varphi)\mathrm{d}\varphi = erf\frac{\psi_S}{\sqrt{2}\sigma_{\tilde{\varphi}}}. \tag{37}$$

Therefore, the maximum value of the flux density in the focal plane for a real concentrator can be determined from expression (33) as:

$$E_{\max} = \frac{E_S\beta_C\sin^2 u_C}{\psi_S^2} erf\frac{\psi_S}{\sqrt{2}\sigma_{\tilde{\varphi}}}. \tag{38}$$

Thus, equations (36) and (38) approximately analytically describe the maximum value of the heat flux density in the focal plane of the ideal and real concentrator, respectively. These equations were used to verify the numerical model obtained by the Monte Carlo method and described in detail in [1, 2].

5. 4. Verification of the results obtained using the Monte Carlo method

The comparison of analytical and numerical results was carried out for the concentrator model with the following geometry: radius of the concentrator R_C=2 m, opening angle u_C=60°, radius of the heat receiver r_A=0.04 m, reflection coefficient of the concentrator surface β_C=1, heat flux density $E_S = 600\ \mathrm{W/m^2}$.

The maximum value of the dimensionless density of the heat flux in the focus of such a concentrator $E^{i}_{\max}/E_S = 34621$, which sufficiently accurately corresponds to the results obtained using the Monte Carlo numerical model developed in [1].

To obtain the distribution, the radius of the heat receiver r_A was divided into N points $r_i \in [0; r_A]$. Substitute the value of r_i into expression (34) and obtain the average integral value of the angle $\bar{\varphi}_{r_i}$ for the corresponding point of the radius. If the value $\bar{\varphi}_{r_i} \geq \psi_S$, then the value $P(D_{\bar{\varphi}_{r_i},\psi_S})$ is calculated according to expression (29), and if it is $\bar{\varphi}_{r_i} \leq \psi_S$, according to expression (30). After that, we find the value of radiation E_{r_i} according to formula (35) at the selected point, and we determine the local value ε_{A_i} for each of the points as follows:

$$\varepsilon_{A_i} = \frac{E_{r_i}}{E_S}. \qquad (39)$$

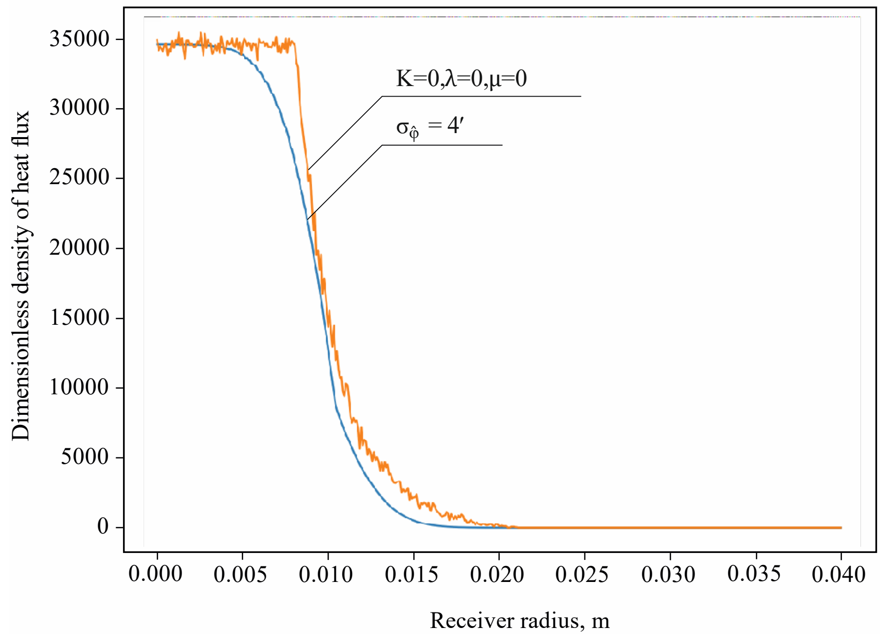


Fig. 4. Comparison of heat flux density distribution obtained by analytical ($\sigma_{\bar{\varphi}} = 4'$) and numerical methods (ideal concentrator)

Fig. 4 shows a comparison of the distributions of dimensionless heat flux density obtained by the proposed analytical method and the Monte Carlo numerical method developed in [1]. In this paper, the coefficient K was introduced, which shows the degree of aberrations of the concentrator surface, which cause the deviation of the reflected beam of rays from its ideal direction. The root-mean-square deviation of the angular errors σ_{γ_C} was calculated in fractions of the magnitude of the opening angle of the Sun $\sigma_{\gamma_C} = K\psi_S$ [20, 21]. Studies [1] were continued in work [2], where flat angles μ and λ were introduced, which characterize the deviation of the beam of rays reflected from the concentrator, which occurs in the process of defocusing the concentrator relative to the direction to the Sun.

As can be seen from Fig. 4, the numerical results for an ideal concentrator without defocusing almost coincide with the analytical results at $\sigma_{\bar{\varphi}} = 0.25\psi_S = 4'$, which proves the adequacy of the developed analytical models and numerical algorithms.

When the deviation value $\sigma_{\bar{\varphi}}$ increases, the effect of energy losses is observed, which fairly accurately simulates the negative effects of a real concentrator, such as aberrations of the concentrator surface and defocusing relative to the direction of the Sun. Fig. 5 shows a comparison of the distribution of the heat flux density according to the analytical method at $\sigma_{\bar{\varphi}} = \psi_S = 16'$, and concentrators with surface aberrations at $K=0.5$ without defocusing and with defocusing at $\lambda=0.002$, $\mu=0.002$.

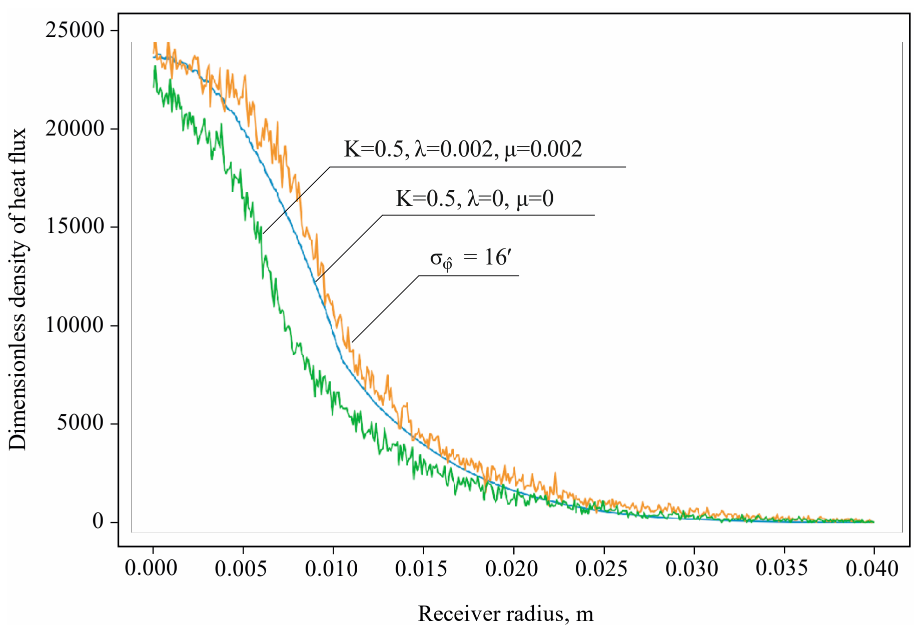


Fig. 5. Comparison of heat flux density distributions obtained by the analytical method ($\sigma_{\bar{\varphi}} = 16'$) and the numerical method (real concentrator)

As can be seen from Fig. 5, the numerical results for a real concentrator with exact orientation to the Sun and aberrations (K=0.5), which correspond to $\sigma_{\varphi}=16'$, almost completely coincide with the analytical data. When defocusing, there is a deviation of numerical data from analytical data, which is especially noticeable when approaching the focus. Despite this, the numerical results obtained in this case are sufficiently close to the analytical ones, which proves the adequacy of the developed mathematical model and the developed Monte Carlo numerical algorithm.

6. Discussion of results of investigating the process of finding an analytical solution to the problem of radiation concentration of a paraboloid concentrator

Our results obtained within the framework of this study are determined by the following aspects:

– the universal generalized mathematical model of the radiation transfer process (16) can be applied in studies of concentration systems of any geometry;

– the mathematical model of solar radiation transfer in a system with a paraboloid concentrator (23) takes into account all aspects of the process and fully meets the set goals;

– the analytical solution to the problem of determining the radiation distribution on the surface of the heat receiver for a real concentrator (expressions (29) to (35)) is easy to apply and takes into account the error factor that occurs in the concentration process;

– comparison of analytical data with numerical results of the Monte Carlo model [1, 2], which is shown in Fig. 4, 5, proves the adequacy of the obtained analytical solution and the reliability of the numerical results.

The main advantages of the proposed method are:

– the unambiguity of the obtained analytical estimate of the heat flow distribution on the surface of the heat sink in a system with a paraboloid concentrator of fixed geometry;

– a generalized mathematical model of the radiation transfer process makes it possible to build a system model with a concentrator of any geometry on its basis;

– the simplicity of the process of finding the distribution of the heat flow on the surface of the heat receiver makes it possible to obtain an analytical result quickly and without the use of additional computing equipment;

– the integral value, which contains all types of errors, including the defocusing factor, gives significant advantages in engineering and design calculations. In comparison with the classical analytical solution [11], which is used to verify most of the numerical studies. The proposed solution demonstrates a more detailed approach in taking into account the errors that occur in the process of radiation transfer and gives a closer approximation to the real distribution of the heat flow.

The obtained analytical solution to the problem of solar radiation transfer in a system with a paraboloid concentrator fully meets the set goals. The integral parameter, which takes into account all types of errors, provides a sufficiently accurate approximation of analytical data to real ones. This allows us to claim that the proposed solution is a worthy alternative to existing classical solutions. Analytical data obtained with the help of this solution can be used as a test for verification of numerical data.

The limitations of the study are that it is impossible to apply the theory of similarity to the mathematical model, which is obtained on the basis of the photometric approach. That is, the generalized mathematical model can be used for any geometric shape of the concentrator but given geometric dimensions.

The disadvantages of the proposed method include the fact that the integral value of the errors of the concentration system sufficiently accurately corresponds to the numerical results only with small error values. With an increased degree of error, there is a discrepancy between analytical and numerical data.

It is advisable to use the proposed approach in the field of development of real power plants with paraboloid concentrators for the verification of data obtained using numerical methods. The presented analytical solution makes it possible to obtain the distribution of the heat flow on the surface of the heat receiver for a concentrator of any radius and with any opening angle. Comparison of numerical data with analytical data obtained by the proposed method will prove the fact of the reliability of the obtained results. This study is a further development of research aimed at developing a mathematical model of a paraboloid concentrator, which takes into account all possible error factors, such as: aberrations of the mirror surface of the concentrator, defocusing relative to the direction to the Sun. Considering the shortcoming of this method, which consists in the divergence of analytical and numerical data with a high degree of error, the directions of further research should be aimed at finding new methods for constructing an analytical solution in order to increase its accuracy.

7. Conclusions

1. This work describes the stages of developing a generalized mathematical model of the process of solar radiation transfer in the «Sun – concentrator – heat receiver» system. The processes of reflection of the solar beam from the mirror surface of the concentrator, as well as the transfer of solar radiation from the concentrator to the heat receiver are described in detail. The generalized mathematical model does not depend on the geometry of the concentrator.

2. Possible approaches to modeling the concentration process were analyzed. The assumption of a uniform distribution of energy in the reflected beam, which is equal to the solar one, is chosen. Taking into account the selected assumption and applying the generalized model for the equation of the paraboloid, a mathematical model of the transfer of solar radiation in the system «Sun – paraboloid concentrator – heat sink» was built.

3. The distribution of the heat flux density on the surface of the heat sink for a paraboloid concentrator of fixed geometry was analytically found. In contrast to the classical analytical solution, which is used to verify most of the numerical studies, the probability distribution of surface errors was introduced in this work. This approach gives a closer to the real distribution of the heat flow.

4. On the basis of the obtained analytical solution, the numerical results obtained by the Monte Carlo method were verified. The maximum values of dimensionless radiation density at the focus of a mathematically ideal concentrator, obtained by analytical and numerical methods, are equal to $E_{\max}$=34621 and $E_{\max}$=34937, respectively. For a real concentrator with aberrations of the mirror surface, these values

are equal to E_{max}=23635 and E_{max}=23862, respectively, which is within the permissible error. It is proved that the results obtained using the numerical Monte Carlo model correspond to the analytical results to a sufficient extent, which proves the fact of the reliability of the numerical Monte Carlo results.

Conflicts of interest

The authors declare that they have no conflicts of interest in relation to the current study, including financial, personal, authorship, or any other, that could affect the study and the results reported in this paper.

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Data availability

All data are available in the main text of the manuscript.

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