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CONSTRUCTING GEOMETRICAL MODELS OF SPHERICAL ANALOGS OF THE INVOLUTE OF A CIRCLE AND CYCLOID

Andrii Nesvidomin
PhD, Associate Professor*
Serhii Pylypaka
Doctor of Technical Sciences, Professor, Head of Department*
Tatiana Volina
Corresponding author
PhD, Associate Professor*
E-mail: t.n.zaharova@ukr.net
Mykhailo Kalenyk
PhD, Associate Professor
Department of Mathematics, Physics and Methods of their Education
Sumy State Pedagogical University named after A. S. Makarenko
Romenska str., 87, Sumy, Ukraine, 40002
Ivan Shuliak
PhD, Associate Professor
Department of System Design of Objects of Transport Infrastructure and Geodesy
National Transport University
Omelyanovicha-Pavlenko str., 1, Kyiv, Ukraine, 01010
Yuriy Semirnenko
PhD, Associate Professor, Head of Department**
Nataliia Tarelnyk
PhD, Associate Professor**
Iryna Hryshchenko
PhD, Associate Professor*
Yuliia Kholodniak
PhD, Associate Professor
Department of Computer Sciences
Dmytro Motornyi Tavria State Agrotechnological University
B. Khmelnytsky ave., 18, Melitopol, Ukraine, 72312
Larysa Sierykh
PhD, Associate Professor
Department of Theory and Methods of Educational Content
Sumy Regional Institute of Postgraduate Pedagogical Education
Rymskoho-Korsakova str., 5, Sumy, Ukraine, 40007
*Department of Descriptive Geometry, Computer Graphics and Design
National University of Life and Environmental Sciences of Ukraine
Heroyiv Oborony str., 15, Kyiv, Ukraine, 03041
**Department of Engineering Systems Design
Sumy National Agrarian University
Herasyma Kondratieva str., 160, Sumy, Ukraine, 40000

The common properties of images on a plane and a sphere are considered in the scientific works by scientists-designers of spherical mechanisms. This is due to the fact that the plane and the sphere share common geometric parameters. They include constancy at all points of the Gaussian curve, which has a zero value for a plane and a positive value for a sphere. Figures belonging to them can slide freely on both surfaces. With unlimited growth of the radius of the sphere, its limited section approaches the plane, and the spherical shape transforms into a plane. Thus, a loxodrome that crosses all meridians at a constant angle is transformed into a logarithmic spiral that intersects at a constant angle the radius vectors that come from the pole. The tooth profile of cylindrical gears is outlined by the involute of a circle. A spherical involute is used for the corresponding bevel gears. Other spherical curves are also known, which are analogs of flat ones.

The formation of a cycloid and an involute of a circle are associated with the mutual rolling of a line segment with each of these figures. If the segment is fixed and the circle rolls along it, then the point of the circle describes the cycloid. In the case of a stationary circle along which a segment is rolled, the point of the segment will execute the involute. To move to the spherical analogs of these curves, it is necessary to replace the circle with a cone, and the straight line with a plane. The spherical prototype of the cycloid will be the trajectory of the point of the base of the cone, which rolls along the plane, that is, along the sweep of the cone. The sweep of a cone is a sector, the radius of the limiting circle of which is equal to the generating cone. If this sweep, like a section of a plane, is rolled around a fixed cone, when its top coincides with the center of the sector, then the point of the limiting radius of the sector will execute a spherical involute. This paper analytically implements these two motions and reports the parametric equations of the spherical analogs of the circle involute and the cycloid

Keywords: involute, cycloid, spatial curves, parametric equations, geometric model, spherical analogs

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1. Introduction

The involute of a circle is widely known as a curve along which the contour of a gear tooth is outlined. In addition to the involute one, there is a cycloidal engagement, the teeth of which are outlined by the arcs of cycloids and epo- and hypocycloids. When designing bevel gears with an involute profile, a spherical involute is used – an analog of the involute of a circle on a sphere. Despite the fact that such curves are widely known, there is very little information in scientific sources on the techniques of constructing involute circles and cycloids, and even less data on their mathematical notation. The concept of a spherical cycloid also exists and applies to a spherical pendulum. It refers more to mechanics than to geometry. However, the properties of cycloids and involutes of a circle can be transferred to their spherical analogs, which predetermines the relevance of research in this area.

2. Literature review and problem statement

In the scientific literature, it is quite difficult to find information about the techniques of constructing spatial analogs of plane curves. Although quite a lot of attention is paid to the study of the properties of such curves and their application. Thus, the concept of spherical helicity and its application in magnetohydrodynamics is quite fully explained in paper [1]. However, nothing is said about the technique of obtaining such spatial curves.

Work [2] is quite interesting from the point of view of techniques of construction of curves. Its authors study spherical curves, the curvature of which is expressed by the distance to a great circle (or to a point). New characteristics of some known spatial curves, as well as several new families of spherical curves, were found. The internal equations of the latter are expressed through elementary functions or elliptic Jacobi functions. In addition, parametric equations can be obtained for them as a function of the arc length. However, with the help of the approach proposed by the authors, it is impossible to find spherical analogs spherical analogs of the plane curves that interest us. The same situation occurs in work [3], where spherical curves are considered in relation to a modified orthogonal system with twist in three-dimensional Euclidean space. The authors of article [4] propose the transition from spherical curves to Mannheim curves as a possible method of constructing the latter. But even this method is not considered possible to apply to the search for spherical analogs of plane curves.

In [5], spherical curves are studied using the Bishop system. In the role of initial data, the authors propose general differential equations of spherical curves. In this case, it is possible to switch to their flat analogs, but not vice versa.

It should be noted separately that work [6] investigates spherical mechanisms, and the concept of a spherical ellipse is used to design a bevel transmission, which is an analog of a transmission between parallel axes, in which ellipses act as non-circular wheels. In work [7], a special grid is created using spherical curves, which is an analog of a rectangular square grid on a plane. Approximation of the sphere by a continuous strip of the unfolding surface is considered in work [8].

All this allows us to argue that it is expedient to conduct a study aimed at building a geometric model of the formation of spherical analogs of the involute of a circle and a cycloid.

3. The aim and objectives of the study

The purpose of this study is to build a geometric model of the formation of spherical analogs of the involute circle and cycloid and their mathematical notation. This will make it possible to construct involute and cycloidal coupling using the obtained parametric equations.

To achieve the goal, the following tasks must be solved:

– to derive parametric equations of the spherical involute of a circle;

– to derive parametric equations of a spherical cycloid;

– to visualize the results using computer graphics in relation to spherical objects.

4. The study materials and methods

The object of our research is spherical curves, the technique of their formation is similar to the technique of formation of known plane curves, namely involutes of circles and cycloids. The subject of the study is the development of analytical techniques for constructing spherical curves – analogs of known plane curves.

The main hypothesis assumed that on the surface of the sphere one can get curves that are analogs of the involute of a circle and a cycloid. To this end, the mutual rolling of a plane and a cylinder in the formation of plane curves must be replaced by the mutual rolling of a plane and a cone in the formation of their spherical analogs.

As mentioned above, the involute of a circle and a cycloid on a plane can be formed as the trajectory of a point during the mutual rolling of a segment of a straight line and a circle. The trajectory of a point of a straight line when it rolls along a fixed circle is an involute, and vice versa, when a circle rolls along a fixed segment, the trajectory of a point of a circle executes a cycloid. From this scheme of formation of flat curves, it is possible to proceed to the scheme of formation of spherical analogs of these same curves. The mutual rolling of a circle and a segment is a simplified variant of the mutual rolling of surfaces – a plane and a cylinder. Simply, in this case, the rolling of the surfaces can be replaced by the rolling of curves – orthogonal sections of the cylinder and the plane. If the cylinder is replaced by a cone, the mutual rolling of the cone and the plane makes it possible to obtain spherical curves. And there is a complete analogy in this – when rolling a cone on a plane, its top will be a fixed point, and the trajectory of the point of the circle (the base of the cone) will execute the spherical analog of the cycloid. If you roll the plane (sweep) around a fixed cone, then the point of the plane executes the spherical analog of the involute of a circle. To implement the described techniques of mutual rolling of a cone and a plane, the apparatus of analytical and differential geometry is used.

Calculations were performed using the Wolfram Mathematica computer algebra system [9]. Drawings were created in the environment of the commercial Maple computer algebra system [10].

5. Results of the construction of spherical analogs of the involute of a circle and cycloid

5. 1. Parametric equations of a spherical involute

The parametric equations of a cone with a vertical axis, in which rectilinear generators inclined at an angle of β to the horizontal plane are:

$$X = u\cos\beta\cos\gamma;$$

$$Y = u\cos\beta\sin\gamma;$$

$$Z = u\sin\beta, \tag{1}$$

where γ and u are independent surface variables, and γ is numerically equal to the angle of rotation of the surface point around the Oz axis, and u is the second variable, which is numerically equal to the length of the rectilinear generating cone, the count of which starts from the origin of the coordinates (top of the cone). The element of the sweep of the cone will be a flat sector μ bounded by an arc of radius R, where R is the radius of the sphere on the surface of which the base of the cone lies – a circle of radius r (Fig. 1). There is a relationship between the radii r and R:

$$r = \cos\beta. \tag{2}$$

The flat sector μ must roll around the cone. Fig. 1 shows the position of the sector when it touches the base of the cone at point A. If it is rolled around a stationary cone, point B on the arc of the sector, executing a certain trajectory BB_0, will coincide with point A_0 on the base of the cone (Fig. 1). At the same time, the lengths of arcs AB and AA_0 will be equal since rolling occurs without sliding. The segment OB is equal to the radius of the sphere R, so it follows that the arc BB_0 is a spherical curve. But on the other hand, its formation corresponds to the technique of constructing an involute. Thus, the arc BB_0 is part of the sought-after curve – the spherical involute of a circle. It is necessary to give its mathematical notation.

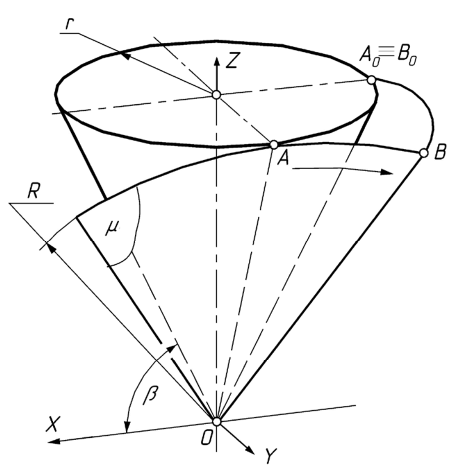


Fig. 1. Geometric model of the formation of a spherical involute of a circle by rolling a flat sector (sweep) on the surface of a cone

When the sector μ is rolled around the cone, a set of circles of radius R with the center at point O and inclined to the horizontal plane at an angle β is formed. Let's write down the parametric equations of one circle that passes through point O (origin of coordinates) and is inclined at an angle β to the horizontal plane:

$$x = R\sin\alpha;$$

$$y = -R\sin\beta\cos\alpha;$$

$$z = R\cos\beta\cos\alpha, \tag{3}$$

where α is the independent variable (the angle that varies from 0 to 2π for a closed circle). At α=0, the starting point of the circle of radius R will be at the top of the circle of radius r. Let's form a set of circles that will go around the cone and will have a common point with a circle of radius r. To this end, we turn the inclined circle (3) around the OZ axis by an angle γ. As a result of rotation, its equation will be written as follows:

$$x = R\sin\alpha\cos\psi + R\sin\beta\sin\psi\cos\alpha;$$

$$y = R\sin\psi\sin\alpha - R\sin\beta\cos\psi\cos\alpha;$$

$$z = R\cos\beta\cos\alpha. \tag{4}$$

With constant values of the angles β and ψ, one circle can be constructed according to equations (4). By setting a new value to the angle ψ, a new circle will be obtained, the plane of which will be tangent to the cone. The set of values of the angle ψ will correspond to the set of positions of circles of radius R, which will have a common point with the circle of radius r and whose planes will surround the cone.

We obtain a circle of radius r on the surface of the cone from its equations (1) at u=R. When the sector μ is rolled around the cone, the contact point A will move to the position A_0 (Fig. 1), which will correspond to the rotation of the radius r by the angle γ. The length s of the arc AA_0 will be equal to $s=r\cdot\gamma$. When rolling in the reverse order, the length of the arc AB of the circle of radius R must be equal to the length of the arc AA_0 of the circle of radius r, i.e., $r\cdot\gamma$. The length of the corresponding arc of a circle of radius R is determined similarly – by the product of its radius by the angle of rotation α: $R\cdot\alpha$. From the equality of the arcs, we find: $\alpha=-r\cdot\gamma/R$. The sign "–" means that when changing the angle γ, which is equal to the angle ψ, the angle α must change in the opposite direction, which corresponds to the direction of movement of point A towards point B (in Fig. 1, the direction of movement is shown by an arrow). Substitution of $\alpha=-r\cdot\gamma/R$ and $\psi=\gamma$ in (4) will make it possible to obtain the parametric equations of the spherical involute:

$$x = R\sin\left(-\frac{r}{R}\gamma\right)\cos\gamma + R\sin\beta\sin\gamma\cos\left(-\frac{r}{R}\gamma\right);$$

$$y = R\sin\gamma\sin\left(-\frac{r}{R}\gamma\right) - R\sin\beta\cos\gamma\cos\left(-\frac{r}{R}\gamma\right);$$

$$z = R\cos\beta\cos\left(-\frac{r}{R}\gamma\right). \tag{5}$$

Equation (5) can be simplified. The value of the angle β depends on the ratio of the radii r and R according to (2). Therefore, it can be excluded from equations (5). In addition, the ratio r/R is replaced by the constant a. Under such conditions, equation (5) will be finally written:

$$x = a\sin\gamma\cos a\gamma - \cos\gamma\sin a\gamma;$$

$$y = -a\cos\gamma\cos a\gamma - \sin\gamma\sin a\gamma;$$

$$z = \sqrt{1-a^2}\cos a\gamma. \tag{6}$$

Parametric equations (6) describe a spherical involute on a sphere of unit radius. The constant a is the radius of the circle on this sphere, so it must be less than unity. To go from a unit sphere to its desired size, three equations (6) need to be multiplied by a scale factor, which is the radius of the sphere R.

5. 2. Parametric equations of spherical cycloids

The geometric model of obtaining a spherical cycloid is based on the common property of the involute and the cycloid – the mutual rolling of the cone and the plane. Fig. 2, *a* shows the diagram of the formation of a spherical cycloid. A cone with a fixed vertex *O* rolls on a plane, and its base (a circle of radius *r*) rolls on a circle of radius *R*. The point of the circle, which is the base of the cone, executes a certain trajectory (arc *AB*), which is an element of a spherical cycloid. The base of the cone (circle of radius *r*) lies with all its points on the surface of the sphere. This follows from the fact that a flat section of a sphere is a circle. Fig. 2, *b* shows the base of the cone, projected in a straight line, which is a flat section of the sphere. When rolling a cone, its base (a circle of radius *r*) rolls along a circle of radius *R*. Such rolling creates a set of circles of radius *r*, inclined at an angle β to the horizontal plane (Fig. 2, *b*), which lie on the surface of a sphere of radius *R*. In this sense, there is a similarity between the model of mutual rolling of a cone and a plane with the previous problem. In both cases, the set of circles circumscribes the cone. But in the first problem, the set of circles of radius *R* of the sphere surrounds the cone of radius *r* of its section, and in the second problem, it is the other way around. However, the commonality of the approach allows solving the second problem in a similar way.

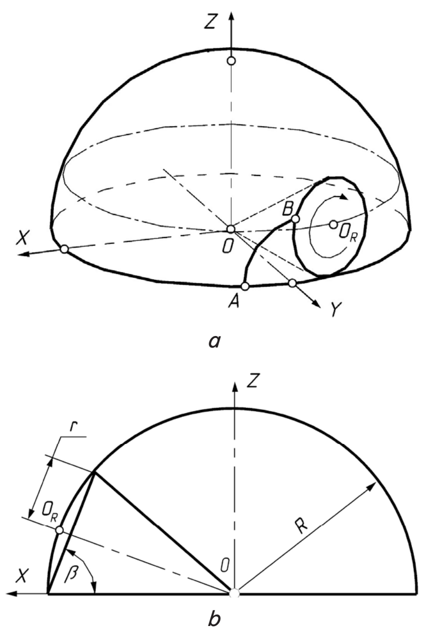


Fig. 2. Graphical illustrations of the formation of a spherical cycloid: *a* — a diagram of the rolling of a cone on a plane and the formation of an arc *AB*, which is an element of a spherical cycloid; *b* — designating the structural parameters of the cone and the sphere, which are interconnected

After matching the geometric parameters, it is possible to write the equation of a one-parameter set of circles, which are inclined at an angle β to the horizontal plane and touch the circle of the hemisphere (Fig. 2, *b*). Since there is a relationship (2) between the angle β and the radii *R* and *r*, it is possible to proceed to the ratio $a=r/R$ by analogy with the previous problem. At the same time, a transition to a sphere of unit radius (R=1) is made, and the value a<1 is the radius of a circle on a sphere of unit radius. The set of circles will be written by parametric equations:

$$x = a\sin\alpha\cos\gamma - \left[1 - a^2\left(1+\cos\alpha\right)\right]\sin\gamma;$$

$$y = a\sin\alpha\sin\gamma + \left[1 - a^2\left(1+\cos\alpha\right)\right]\cos\gamma;$$

$$z = a\sqrt{1-a^2}\left(1+\cos\alpha\right). \qquad (7)$$

When γ=const and the angle α changes within α=0...2π, equations (7) will describe a circle of radius *a*, which belongs to a sphere of unit radius. At different values of the angle γ, a set of such circles is formed. A certain circle will correspond to the angle γ. At α=0 and γ=0, the circle of radius *a* and the circle of unit radius of the sphere will have a common point of contact. When the unit radius vector is rotated by the angle γ, the radius vector *a* must turn at angle α with the condition that the corresponding arcs of the circles are equal. From here, we can write $\gamma=a\cdot\alpha$, from which we find: $\alpha=-\gamma/a$. The sign "–" corresponds to the physical essence of the formation of a spherical cycloid. By substituting the expression $\alpha=-\gamma/a$ into equation (7) and simplifications, the parametric equations of the spherical cycloid, which is located on a sphere of unit radius, will finally be constructed:

$$x = -\sin\gamma + a^2\left(1+\cos\frac{\gamma}{a}\right)\sin\gamma - a\cos\gamma\sin\frac{\gamma}{a};$$

$$y = \cos\gamma - a^2\left(1+\cos\frac{\gamma}{a}\right)\cos\gamma - a\sin\gamma\sin\frac{\gamma}{a};$$

$$z = a\sqrt{1-a^2}\left(1+\cos\frac{\gamma}{a}\right). \qquad (8)$$

With the integer value of *n* arcs, the constant *a* is defined as the inverse of *n*: a=1/*n*.

5. 3. The construction of a spherical involute and cycloid according to the parametric equations built

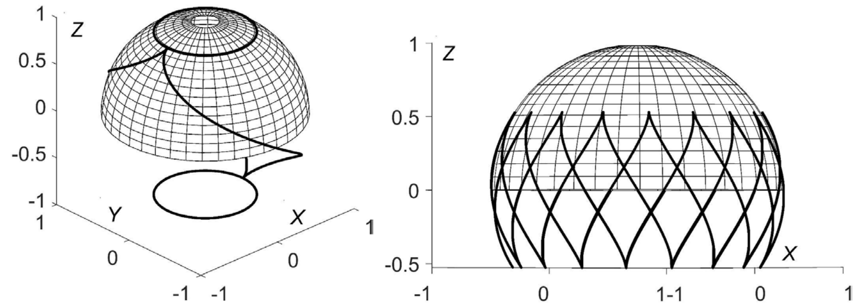
Before constructing the spherical involute, let's pay attention to the main properties of the involute of a circle in the plane. In Fig. 3, *a*, its symmetrical turns are constructed, where it is shown that the tangent lines to the circle cross them at a right angle. In Fig. 3, *b*, according to equations (6) at a=0.75, a spherical involute is constructed. An analog of tangent lines to a circle in a plane are tangent circles of radius R=1 on a sphere of unit radius. All of them, by analogy, cross the spherical involute at a right angle. The main difference between planar and spherical involutes is that the turns of a planar involute can continue to infinity while the turns of a spherical involute go to a congruent circle in the opposite hemisphere.

A spherical cycloid can be constructed according to equations (8). In Fig. 4, *a*, a spherical cycloid is constructed with the number of arcs n=5, that is, at a=0.2.

If we specify the number of arcs not as an integer but as an integer and a fractional part, then the number of arcs can also be an integer, but they will intersect and the point will return to its original position after passing all turns. This is demonstrated in Fig. 4, *b* with n=4.5.

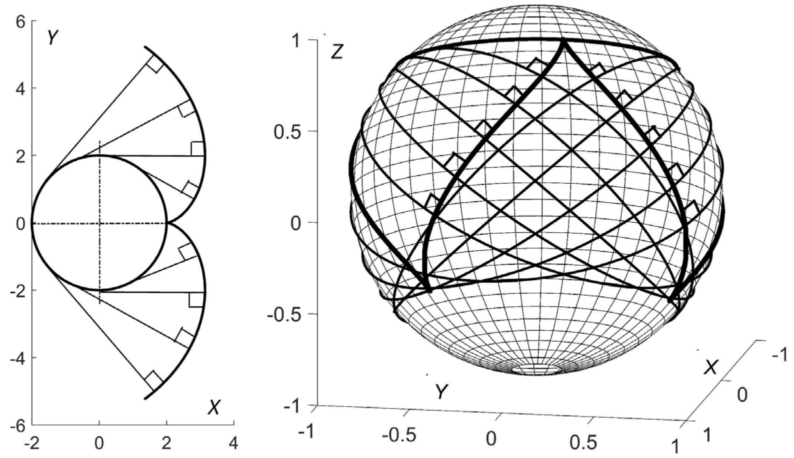
A similar situation, as with the construction of spherical cycloids, occurs when constructing spherical involutes. The turns of the involute may not intersect and be located between two symmetrical circles on the sphere (Fig. 5, *a*) or they may intersect (Fig. 5, *b*). In all cases, at the point of contact with the circle, they form a right angle with it in both hemispheres.

If a is an irrational fraction, then the point of the involute will not return to its initial position at any number of revolutions around the OZ axis. If a is a rational fraction, then after a certain number of revolutions the point will return to its original position. Such curves can be used to form patterns on the sphere. The number of shaping options can be increased by rotating the involute turn around the OZ axis at a given interval. For example, in Fig. 6, a, the involute shown in Fig. 5, a, was rotated around the OZ axis by 30°. In Fig. 6, b, two involutes are constructed at a=0.4, which are symmetrically rotated relative to each other.



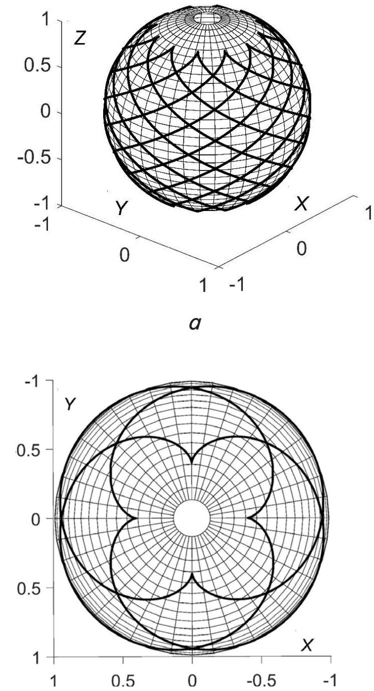
a *b*

Fig. 5. Spherical involutes constructed from equation (6): *a* – *a*=0.5; *b* – *a*=0.85



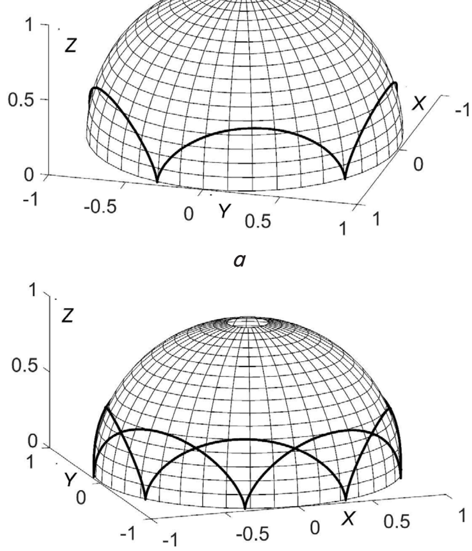
a *b*

Fig. 3. The involute of a circle on a plane and its spherical analog: *a* – symmetric turns of the involute of a circle in the plane; *b* – symmetric turns of a spherical involute of a circle



b

Fig. 6. Spherical involutes rotated relative to the OZ axis by the same angle: *a* – *a*=0.5, 12 turns; *b* – *a*=0.4, 2 turns



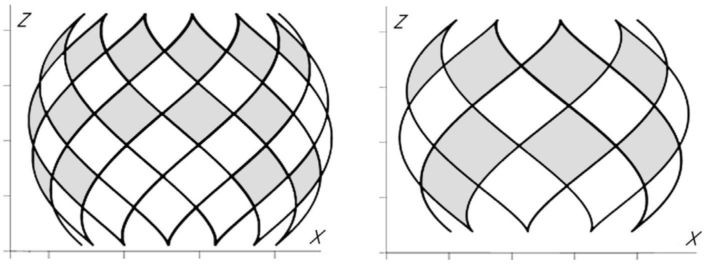
b

Fig. 4. Spherical cycloids constructed according to equations (8): *a* – *n*=5, arcs do not intersect; *b* – *n*=4.5, arcs intersect

Using various combinations of the spherical involute, it is possible to form a grid on the surface of the sphere for its parqueting with elements of the same type, both square (Fig. 7) and diamond-shaped (Fig. 5, b). At a=0.707, the formed grid will have the form of curvilinear squares. By turning the spherical involute around the OZ axis by a certain angle, you can get squares of different sizes (Fig. 7).

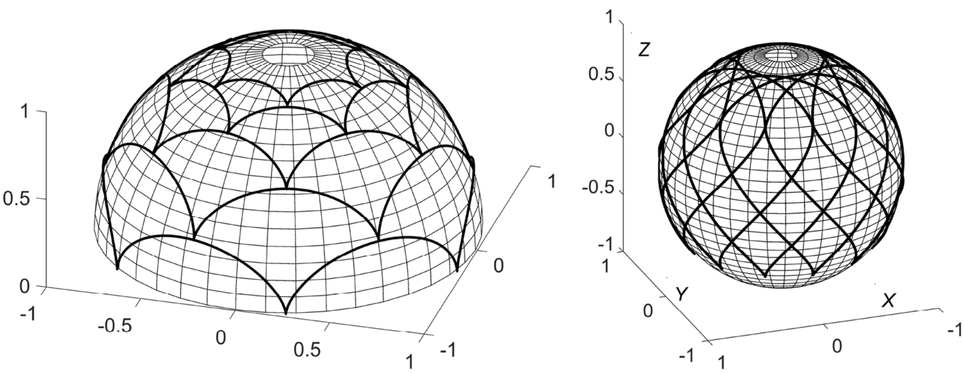
Regarding the formation of a spherical cycloid, it is possible to set the condition that the circle does not roll along the equator, as shown in Fig. 2, a, and on another parallel. Such curves are shown in Fig. 8, a, which also form a certain pattern on the surface of the sphere. In Fig. 8, b, the pattern is formed by a combination of involutes and cycloids.

The obtained parametric equations of spherical involutes and cycloids make it possible to combine them in an orderly manner to obtain patterns.



a *b*

Fig. 7. Grid on the sphere in the form of curvilinear squares at a=0.707: a — rotation of the involute by increasing the angle of rotation by $\pi/7$; b — rotation of the involute by increasing the angle of rotation by $\pi/5$



a *b*

Fig. 8. Formation of patterns on the sphere using spherical cycloids and involutes: a — spherical cycloids; b — combination of cycloids and involutes

6. Discussion of results of studying the procedure for constructing spherical involutes and cycloids

The results obtained in the previous chapter can be explained by analogies of the construction of curves on a plane and on the surface of a sphere. The essence of this analogy lies in the physical techniques of curves on a plane and their spherical prototypes (analogs).

For some known plane curves, there are ways to physically construct them using tools or devices. For example, a circle is built using a compass. Obviously, with its help, you can build a circle on one of the hemispheres of the sphere. An involute on a plane can be constructed by unwinding a thread that is conventionally wound on a circle. Then the end of the stretched string will execute the involute. It is obvious that such a scheme is also suitable for the physical construction of a spherical involute. At the same time, the stretched thread will be tangent to the circle and will wrap around the sphere, as shown in Fig. 3, *b*. It should be noted that the stretched thread is an arc of a large circle corresponding to a straight line on the plane (Fig. 3, *a*). This is logical because the arc of a great circle is the shortest distance between two points on the sphere.

A feature of the proposed method is the transfer of techniques for constructing flat curves to obtain their spherical analogs. In journal [11] it is noted that the properties of the spherical ellipse were invented for the first time by Fuss (1755–1825), a Swiss by origin. A spherical ellipse was constructed by analogy with an ellipse on a plane. The difference between the studies reported in this paper is that other known flat curves were considered, and their spherical analogs were found.

When rolling a flat sector μ around a cone, a set of circles was created that go around this cone. The spherical involute of the circle was obtained on the basis of the established relationship between the independent variable of the base of the cone – the angle α and the independent variable of the circle tangent to the cone – the angle α in the form $\alpha=-r\cdot\gamma/R=a\cdot\gamma$. With any other dependence $\alpha=\alpha(\gamma)$, another spherical curve will be obtained. Thus, substituting the arbitrary dependence $\alpha=\alpha(\gamma)$ in equation (5) instead of $(-r\cdot\gamma/R)$ will make it possible to obtain various spherical curves. For any curve, both flat and spatial, an important characteristic is the length of the arc, that is, a natural parameter. It is determined by integrating the root expression, which is a significant obstacle to obtaining the original function. Therefore, thanks to different dependences $\alpha=\alpha(\gamma)$, different spherical curves can be obtained, but obtaining the expression of the length of the arc with this approach is possible in rare cases, and then only with a purposeful choice of the dependence $\alpha=\alpha(\gamma)$. For the involute of a circle in the plane, the expression for the arc length has a simple form. This also applies to the spherical involute.

Thanks to our research results, it was possible to build a geometric model of the formation of spherical analogs of plane curves using the example of well-known curves – involute of a circle and cycloid. The basis for achieving the stated goal was the hypothesis according to which spherical curves can be obtained by similar constructions on the plane and on the surface of the sphere. At the same time, the analogy was made both for the physical construction of curves with the help of a tool, and for the construction of curves by analytical methods.

A cycloid is the trajectory of a point on a circle that rolls in a straight line. The involute of a circle is the trajectory of a straight line that rolls around a circle. This is a partial case of mutual rolling of a cylinder and a plane as a result of their cross section. Thanks to the generalization and replacement of the cylinder with a cone, a physical model for obtaining spherical curves was created, which became the basis for the mathematical notation of these curves. In Fig. 1, a plane in the form of a sector (cone sweep) rolls along the cone. The arc point of this sector executes a spherical curve – an analog of the involute of a circle (in this case – the base of a cone). If the cone is rolled along the plane (Fig. 2), then the point of the circle – the base of the cone – will execute a spherical curve, which is analogous to a cycloid. Based on this approach, the parametric equations (6) of the spherical involute and (8) of the spherical cycloid were built. These equations include the constant a, which is subject to a restriction: it must be less than unity. The disadvantage is that the construction of a grid on the surface of the sphere in the form of equal curvilinear squares is possible on a limited area of the sphere. The development of the current study is to find spherical analogs of other known plane curves, which will help overcome this drawback.

Parametric equations of spherical cycloids and involutes were derived in our work. The analytical description of the spherical ellipse [14] is supplemented by the analytical description of two more spherical curves.

It is known that some spherical mechanisms are formed by analogy with flat ones. For example, a gear between parallel axes, in which the profile of the teeth is outlined along the involute of a circle, is a prototype for the formation of a bevel gear, in which the axes intersect. At the same time, the profile of the bevel gear teeth is a spherical involute. Based on the rolling of the cone along its sweep, you can create a suitable gear. Spherical cycloids can also be used, provided that our results are made public and available to the general public, including specialists in the design of spherical mechanisms.

7. Conclusions

1. The qualitative indicator of the current research is the derivation of the parametric equations of the spherical involute. This result differs from the known ones in that it gives a new way of constructing a spherical involute, namely the transition from manual construction to construction by means of computer graphics.

2. The qualitative indicator of our study is the derivation of the parametric equations of the spherical cycloid. This result differs from the known ones in that it provides a new way of constructing a spherical cycloid, namely, the transition from manual construction to computer graphics construction.

3. The confirmation of our theoretical results is the construction of spherical curves; their application for applying grids with different shapes of cells to the sphere is also shown. This result is a qualitative indicator of the research as it makes it possible to construct spherical curves, which are analogs of flat ones, using modern methods involving computer graphics software products.

Conflicts of interest

The authors declare that they have no conflicts of interest in relation to the current study, including financial, personal, authorship, or any other, that could affect the study and the results reported in this paper.

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Data availability

All data are available in the main text of the manuscript.

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