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DEVISING A METHOD TO DESIGN SUPERSONIC NOZZLES OF ROCKET ENGINES BY USING NUMERICAL ANALYSIS METHODS

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The object of research is supersonic nozzles of liquid-propellant rocket engines. The work considers the problem of the lack of an effective method for profiling the supersonic contour of the nozzle, which will generate maximum thrust. For its solution, a method is proposed, the essence of which is approximating the nozzle contour with a power-law polynomial and determining the values of its coefficients by solving a multidimensional minimization problem using numerical modeling methods. The expression for the axial component of the thrust with the opposite sign at the specified values of atmospheric pressure and radius at the nozzle section was chosen as the objective function in this paper.

Using the proposed method, contours of optimal nozzles were obtained based on polynomials of powers 2, 3, and 4, which were compared with nozzles obtained by the generally accepted Rao method. The maximum value of the relative deviation modulus calculated during the comparison did not exceed 3 %, which allows us to assert the correctness of the obtained results. The existence of such a discrepancy is explained by the difference in the numerical modeling method used. In contrast to the method of characteristics common in similar problems, the method of finite volumes of the Godunov type was used in the work. This has made it possible to reduce the sensitivity of the calculation to initial and boundary conditions and make decisions regardless of the flow regime of combustion products. In addition, the use of the extended cells method for the integration of finite volumes at the boundary of the calculation domain significantly reduced the total time of solving the problem of profiling the contour of the optimal nozzle

Keywords: supersonic nozzle contour optimization, liquid-propellant rocket engines, extended cells method

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1. Introduction

A supersonic nozzle is an integral part of a liquid rocket engine. It makes it possible to accelerate combustion products to supersonic speeds by converting their internal energy into kinetic energy. Thrust is created as a result of this process. The more thrust the nozzle can generate, the more payload the launch vehicle can carry. From this point of view, profiled nozzles are the most effective [1, 2]. Different methods can be used to design their contour. The choice of the appropriate method depends on the requirements for accuracy and simplicity of calculation [1–3].

In general, nozzle design methods are divided into two groups:

1) approximate, based on the analysis of families of already designed nozzles;

2) methods involving the solution of a variational or optimization problem.

For the first group, the desired contour is built taking into account already existing ones or approximated with the help of proven dependences. The advantage of the methods of the first group is the speed of obtaining results, so they are usually used in sketch projects, when it is necessary to obtain the approximate geometry of the product for a preliminary assessment of its performance. For more accurate results, more complex methods of the second group are used. Nozzles built with their help generate 2–3 % more thrust [1, 2]. In addition, the methods of the second group during the solution use numerical modeling to obtain the fields of physical parameters during the operation of the nozzle. The disadvantage of the methods of the second group is the lack of an efficient method for modeling the flow of combustion products, which leads to an increase in the duration of calculations and the sensitivity of the solution results to the presence of parameter discontinuity surfaces in the calculation area. Thus, the task to devise a more effective method for designing the supersonic part of the nozzle without reducing the accuracy of the results is urgent.

2. Literature review and problem statement

Works [4, 5] report the results of using arcs of circles to design the contour of the maximum thrust nozzle. The considered method is based on a geometric approach and does not take into account the real pattern of the flow of combustion products inside the nozzle. The reason for this is the fundamental absence in the algorithm of the process method of numerical simulation of the flow of combustion products. Also, the results of the method deteriorate with a decrease in the degree of expansion of the projected engines, so its application should be limited to the design of nozzles operating at high altitudes. An option to overcome these difficulties may be to use a more complex approach that will take into account the distribution of all physical parameters along the camera. This is what was used in work [6], where the method of characteristics was used to profile optimal nozzles. Its accuracy significantly depends on the correctness of the initial and boundary conditions, and small errors in them can lead to a significant decrease. In addition, if there are shock waves in the gas flow, then certain modifications must be made to the method, which complicate the solution algorithm. In order to solve these problems, study [7] used the software package of the Rocflam family [8]. This made it possible to make the calculations more stable and carry them out regardless of the flow pattern of combustion products in the nozzle. But the issue related to the efficiency of calculations remained unresolved. The reason for this is the use of structured curvilinear non-orthogonal meshes in the Rocflam package, which greatly complicate the solution of the equations that describe the flow of combustion products.

An alternative approach to the design of maximum thrust nozzles in application to hypersonic engines is proposed in works [9, 10]. There, the point inside the nozzle is used as the initial reference point, where the Mach number, the flow angle, and the relative mass flow must be specified. Each of these parameters has a different effect on the resulting contour, so there are objective difficulties associated with the need to determine their correct values before starting the design of the nozzle contour.

Our review of the literature [4–10] reveals that the existing methods of designing maximum thrust nozzles have their limitations, which negatively affect their speed and versatility. Thus, it is advisable to conduct a study to devise a more effective method for designing a supersonic nozzle of maximum thrust.

3. The aim and objectives of the study

The purpose of this study is to devise a fast method for designing a supersonic maximum thrust nozzle by solving an optimization problem using an effective method of numerical modeling of the flow in a nozzle to determine its initial parameters. This will make it possible to reduce the time spent on searching for the optimal nozzle contour at the stage of designing a liquid rocket engine chamber.

To achieve the goal, the following tasks were set:

– to state the problem of optimizing the contour of the supersonic nozzle;

– to choose a method for modeling the flow of combustion products in the engine chamber;

– to verify the proposed design method by comparing its results with the results obtained using commonly known methods.

4. The study materials and methods

The object of our research is supersonic nozzles of liquid-propellant rocket engines.

The main hypothesis of the study is the property of unimodality of the thrust functionality relative to the parameters that determine the contour of the supersonic nozzle. Given this, it is possible to solve the problem of multidimensional optimization in order to find the contour of the nozzle, which will generate maximum thrust.

The assumptions adopted in the study are as follows:

– axial symmetry of the flow of combustion products inside the nozzle;

– absence of significant influence of viscosity and chemical reactions on the flow of combustion products;

– the search for the optimal nozzle contour was carried out for a prototype engine, the geometry of which is known;

– ambient pressure for given engine operating conditions is a constant value.

According to the assumptions, a model of a inviscid ideal compressible gas of constant chemical composition was used to describe the flow of combustion products in the rocket engine chamber. It consists of a system of unsteady Euler equations (1), which is closed by the Mendeleev-Clapeyron equation of state [11]. The equations of system (1) are written in the integral form for an elementary volume with axial symmetry of a cylindrical coordinate system:

$$
\begin{gathered}
\frac{d}{dt}\int_V U\mathrm{d}t+\oint_A\big(F(U)+G(U)\big)\mathrm{d}A=S,\\
U=(\rho, r\rho u, r\rho v, r\rho E)^T,\\
F=\left(r\rho u, r\left(\rho u^2+p\right), r\rho v u, r(\rho E+p)u\right)^T,\\
G=\left(r\rho v, r\rho u v, r\left(\rho v^2+p\right), r(\rho E+p)v\right)^T,\\
S=(0,0,p,0)^T,\\
E=e+\frac{1}{2}\left(u^2+v^2\right),\\
p=(k-1)e\rho,
\end{gathered}
\tag{1}
$$

where t is time; V – elementary volume; U is a vector of variables; A is the area of the lateral surface of the elementary volume; F is the flow vector in the axial direction; G is the flow vector in the radial direction; S – source term; r – radius; ρ – density; p – pressure; u – velocity in the axial direction; v – speed in the radial direction; E – total specific energy; e – specific internal energy; k is the adiabatic heat capacity ratio.

The problem of determining the parameters of the flow of combustion products in the engine chamber, the scheme of which is shown in Fig. 1, was closed by setting the initial and boundary conditions.

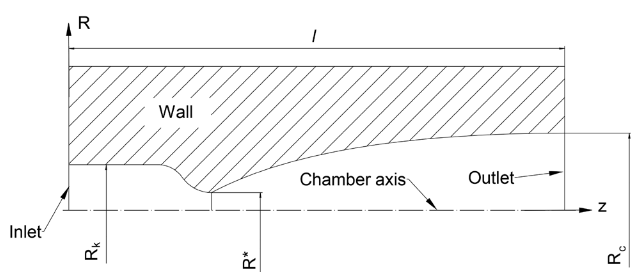

Fig. 1. A typical diagram of a chamber of a liquid rocket engine

The parameters of the stationary gas, calculated for the specified atmospheric pressure and temperature, were used as initial conditions. The flow parameters at the entrance to the chamber were calculated according to dependences (2). For this purpose, fixed values of the stagnation density and pressure and values of the components of the velocity vector extrapolated from the calculation area were used. In system (2), the index 0 denotes the stagnation flow parameter, the *inlet* index is a parameter calculated at the entrance to the chamber, and the *domain* index is a parameter obtained from the calculation domain. At the exit from the chamber, the gradients of all parameters normal to the boundary were set to zero (3). The slip conditions (4) were set on the axis and on the wall:

$$\begin{cases} s_0 = s_{inlet} = \dfrac{p_0}{\rho_0^k} = \text{const}, \\ H_0 = H_{inlet} = \dfrac{k}{k-1}\dfrac{p_0}{\rho_0} = \text{const}, \\ u_{inlet} = u_{domain}, \\ v_{inlet} = v_{domain}, \\ \rho_{inlet} = \left(\dfrac{k-1}{k}\left(H_{inlet} - \dfrac{\left(u_{inlet}^2 + v_{inlet}^2\right)}{2}\right)\right)^{\frac{1}{k-1}}, \\ p_{inlet} = s_{inlet}\rho_{inlet}^k, \end{cases} \quad (2)$$

where s is entropy; p – pressure; ρ – density; H – enthalpy; k is the adiabatic heat capacity ratio; u – velocity in the axial direction; v is the speed in the radial direction:

$$\frac{d\varphi_{outlet}}{dn} = 0, \quad (3)$$

where φ_{outlet} is a physical parameter of the flow at the exit from the chamber; n is the direction normal to the boundary:

$$\begin{cases} \dfrac{dV_\tau}{d\tau} = 0 \\ V_n = 0 \end{cases}, \quad (4)$$

where V_τ is the velocity in the direction tangent to the wall; τ is the direction tangent to the wall; V_n is the velocity in the direction normal to the wall; n is the direction normal to the wall.

Table 1 gives the values of parameters of the gas flow at the entrance to the chamber, with the help of which the simulation was performed.

Table 1

Parameters of the gas flow at the inlet to the chamber

Parameter	Value	Dimensionality
Full pressure	5.85	MPa
Density	4.48	kg/cm^3
Adiabiatic heat capacity ratio	1.21	–

The coordinate descent algorithm [12] was used to determine the optimal values for a set of contour-defining parameters. According to the algorithm, a coordinate variable was determined from this set at each of its iterations. Then, while fixing other coordinates, the objective function was optimized along the hyperplane corresponding to the selected variable using any linear search method. In this work, the method of the golden section was used [12]. The search for optimal parameter values was considered complete when the one-dimensional objective function optimization problem was solved for each of them separately. At the same time, the calculated values make it possible to obtain the contour of the supersonic part of the maximum thrust nozzle.

Numerical simulation of the flow of combustion products in the chamber of a liquid rocket engine was carried out in a program written in the C++ language in the Qt Creator integrated development environment (Finland).

5. Results of devising a method for profiling supersonic nozzles of maximum thrust

5. 1. Statement of the problem on optimizing the supersonic nozzle contour

To obtain the contour of the supersonic part of the nozzle, it is necessary to solve the minimization problem for the objective function of the following form (5):

$$f(\mathbf{X}) = -P(y(x)) \to \min, \quad (5)$$

where f is the objective function; X is a vector of optimized parameters; P is the module of the axial component thrust; $y(x)$ is the exact contour that is being sought; x is a coordinate along the axis of the nozzle.

In our work, the exact contour of the nozzle is replaced by a polynomial of power n (6):

$$y(x) \approx R(a_i, n, l, x) = \sum_{i=0}^{n-1} a_i x^i, x \in [0, l], \quad (6)$$

where $R(a_i, n, l, x)$ is a polynomial approximating the contour; n is the power of the polynomial; a_i – polynomial coefficients of the polynomial; l is the length of the nozzle.

This choice is due to the following reasons:

– power polynomials do not require time-consuming calculations;

– they are easy to study using methods of mathematical analysis [13].

After substituting (6) into (5), the objective function will take the following form (7):

$$f(\mathbf{X}) = -P(R(a_i, n, l, x)). \quad (7)$$

The coefficients of the polynomial and the nozzle length in (7) are parameters to be optimized. For further calculations, they were written in the form of a vector (8):

$$\mathbf{X} = \left(\left\{a_i, i = \overline{0 \ldots n}\right\}, l\right). \quad (8)$$

In the problem under consideration, the nozzle contour was designed for one prototype engine – RD-107 [14]. That has made it possible to establish the relationship equation (9) for the unknown coefficients of the polynomial:

$$\begin{cases} \left.\dfrac{dR}{dx}\right|_{x=l} = \theta_a, \\ \left.R\right|_{x=l} = R_a, \\ \left.R\right|_{x=0} = R^*, \end{cases} \quad (9)$$

where R is a polynomial approximating the contour; θ_a is the angle of inclination of the contour to the axis of the nozzle on the nozzle section; R_a is the radius of the nozzle exit; R^* is the throat radius.

If we solve (9) with respect to the first three coefficients of the polynomial a_0, a_1, a_2 and substitute the result in (8), then the vector of optimized parameters will take the form:

$$\mathbf{X} = \left(\left\{a_i, i = \overline{3 \ldots n}\right\}, l\right). \quad (10)$$

In addition, the geometric parameters of the projected contour in the solved problem were imposed structural restrictions, expressed in the form of a system of inequalities (11):

$$\begin{cases} \theta^*_{\min} \le \theta^* \le \theta^*_{\max}, \\ l_{\min} \le l \le l_{\max}, \end{cases} \quad (11)$$

where θ^* is the angle of inclination of the contour to the axis of the nozzle in the critical section; indices min, max indicate, respectively, the minimum and maximum values that the design parameters can take.

To fully determine the objective function (7), it is necessary to derive an expression for the modulus of the axial component thrust.

In the general case, the thrust of the nozzle can be calculated as the equivalent of the pressure forces applied to its side surface [1–3]:

$$P = \int_F (p_{in} - p_e)\,\mathrm{d}F, \quad (12)$$

where P is the thrust of the nozzle; F is the side surface of the nozzle; dF is an elemental section of the lateral surface of the nozzle oriented in space; p_{in} is the static pressure of combustion products inside the nozzle; p_e is the atmospheric pressure at a given altitude.

The transformation of formula (12) was carried out according to [15]. As a result, we get an expression for determining the axial component thrust of a nozzle with axial symmetry:

$$P = 2\pi \int_{x_1}^{x_2} p_{in} R(x) R'(x)\,\mathrm{d}x - \pi p_e \left(R_2^2 - R_1^2\right) = P_{in} - P_e, \quad (13)$$

where P is the modulus of the axial component thrust; x_1 is the value of the coordinate of the beginning of the nozzle on the horizontal axis; x_2 is the coordinate value of the end of the nozzle on the horizontal axis; $R(x)$ is a polynomial approximating the contour; $R'(x)$ is the derivative of the polynomial along the horizontal coordinate; R_1 is the throat radius of the nozzle; R_2 is the exit radius of the nozzle; P_{in} – the internal component of thrust; P_e is the external component of thrust.

In (13), two components of thrust are clearly present: internal, which depends on the static pressure of combustion products, and external, which depends only on atmospheric pressure. In connection with the assumptions about the presence of a prototype engine and the invariance of the ambient pressure, expression (13) can be simplified by getting rid of the external thrust component, which in this case becomes a constant independent of the nozzle contour. Then the expression for determining the axial thrust component of a nozzle with axial symmetry will take the following form (14):

$$P = \int_{x_1}^{x_2} p_{in} R(x) R'(x)\,\mathrm{d}x = P_{in}. \quad (14)$$

To calculate the thrust according to formula (14), it is necessary to know the geometry of the nozzle and the distribution of static pressure along the wall of the nozzle.

Substituting (14) into (5) allows us to obtain the final expression for the objective function, relative to which the minimization problem will be solved.

5. 2. The choice of the method for modeling the flow of combustion products

The distribution of static pressure along the nozzle wall can be obtained as a result of modeling the flow of combustion products inside the chamber of a liquid rocket engine.

For the numerical solution of the system of equations (1), there was an explicit finite volume method of the Godunov type. It is known [16] that methods of a higher order of accuracy are more effective than methods of the first order. Therefore, integration over time was carried out using the explicit Runge-Kutta method of the third order of accuracy [17], and WENO-reconstruction of the third order of accuracy was used to reconstruct the flow parameters at the faces of the finite volume [18]. The Riemann problem at the faces of the finite volumes was solved using Lax-Friedrichs relations [19].

For calculations, the flow area was divided into finite volumes of a toroidal shape with a rectangular cross-section – an orthogonal rectilinear grid in the plane (x0R) was used. Solving problems (1) to (4) on such grids is much faster than when using curvilinear or unstructured systems of finite volumes.

However, the use of such meshes leads to the appearance of fractional finite volumes, one part of which is inside the calculation area, and the other – outside it. An arbitrary section of such a region is shown in Fig. 2. Fractional volume is highlighted in red on it, regular volume is highlighted in blue.

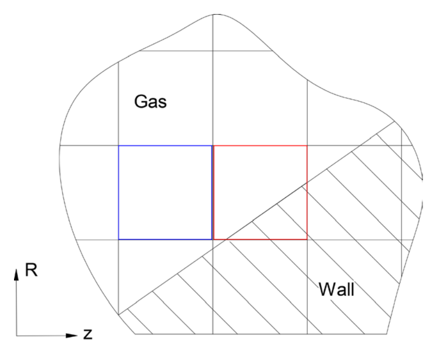

Fig. 2. Fractional and regular finite volumes

With the explicit numerical integration of the system of equations (1) in fractional volumes, only that part of them that is in the gas region will be used. This will cause a decrease in the stability of the solution due to a decrease in the size of the volume. In addition, the task of setting boundary conditions on the sides of the fractional volume is significantly complicated since they partially intersect the wall line, and not parallel to it. To solve these problems, the extended cells method of setting boundary conditions [20, 21] was used in our work, which does not have these shortcomings.

5. 3. Verification of the proposed method

In the current paper, several nozzles were profiled using the considered method using polynomials of powers 2, 3, and 4. Verification of the results was performed by comparison with nozzles built by the Rao method [22]. In both methods, a dimensionless form of parameter representation was used in order to ensure the scalability of the obtained results.

Table 2 gives the unknown and optimized parameters of the problem being solved, taking into account the equations of connections (6) depending on the selected polynomial.

Table 3 gives numerical values of the lower and upper limits of the areas of permissible values of the optimized parameters in dimensionless form. They were chosen based on structural considerations. specificity of engine operating conditions and according to (11).

Table 2

Problem parameters

Polynomial	Unknown parameters	Optimized parameters
$a_2x^2+a_1x+a_0$	l	θ*
$a_3x^3+a_2x^2+a_1x+a_0$	a_3, l	θ*, l
$a_4x^4+a_3x^3+a_2x^2+a_1x+a_0$	a_4, a_3, l	θ*, l, a_4

Table 3

Threshold values of optimized parameters

Optimized parameter	Lower limit	Upper limit
θ*	0.26	0.96
l	7	13
a	–0.0001	0.00005

The results of nozzle profiling by both methods and their relative deviation from each other are shown in Fig. 3–5. Table 4 gives the maximum value of the relative deviation modulus depending on the power of the polynomial as a percentage.

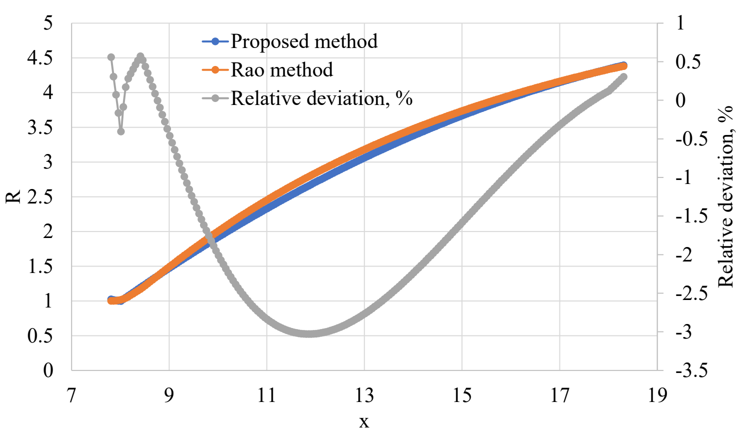

Fig. 3. Results of using a polynomial of the 2nd power

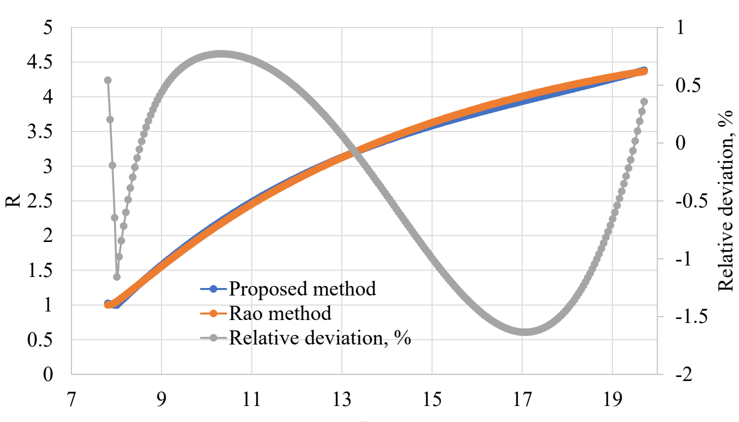

Fig. 4. Results of using a polynomial of the 3rd power

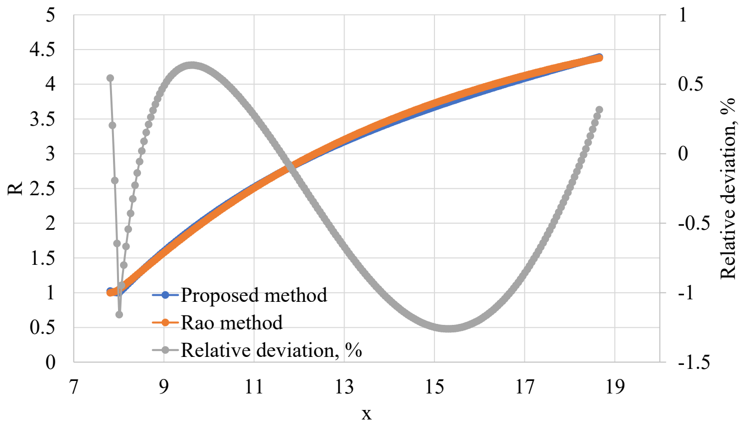

Fig. 5. Results of using a polynomial of the 4th power

Table 4

Maximum value of the modulus of relative deviation depending on the power of the polynomial

Polynomial power	Maximum relative deviation modulus, %
2	3.03
3	1.64
4	1.26

From Fig. 3–5 and Table 4, one can see that the deviation of results of the proposed method from the generally accepted one does not exceed 3 % when using the smallest polynomial with the smallest exponent. At the same time, the use of a polynomial of the 3rd power allows us to significantly reduce the discrepancy – by 1.85 times.

6. Discussion of results of devising a method for profiling supersonic nozzles of maximum thrust

In this work, the optimization problem was stated for profiling the contour of the supersonic nozzle, which will provide maximum thrust. To solve this problem, it is necessary to find such values of the set of parameters to be optimized (10), at which the objective function (7) will reach a minimum under the condition that the axial component of thrust is calculated according to formula (14). At the same time, relations-equalities (9) and relations-inequalities (11) must be fulfilled.

The use of the extended cells method has made it possible to avoid the problem of fractional volumes (Fig. 2) at the borders of the calculation area while preserving the advantages of using orthogonal rectangular grids.

A comparison of the results of the proposed method with the generally accepted one (Fig. 3–5, Table 4) showed a good convergence between them. This allows us to assert the possibility of using the considered method for designing the supersonic contour of the maximum thrust nozzle of a liquid rocket engine. In addition, from Table 4, it can be concluded that the use of a polynomial of the 3rd power is the best choice in terms of the deviation from the generally accepted method with moderate complexity of calculations. And a further increase in the power of the polynomial will lead to an increase in the complexity of the task by an order of magnitude with a slight increase in the accuracy of the solution.

The differences between the proposed method and existing methods [4–10, 22] are as follows:

– use of the finite volume method for modeling the flow of combustion products in the engine chamber. This makes it possible to reduce the sensitivity of the calculation to the initial and boundary conditions compared to the method of characteristics and to make decisions even in the presence of shock waves in the calculation area;

– the use of the extended cells method for the integration of finite volumes at the boundary of the calculation area. Owing to this, the time of modeling the flow of combustion products is reduced, which reduces the total time of solving the problem of designing an optimal nozzle.

It should be noted that the current study considered the case of an inviscid flow of combustion products of constant chemical composition. Therefore, the application of the proposed method is limited to gas flows in which there is no significant influence of viscosity or chemical reactions on the parameters. The disadvantage of the proposed method is the assumption of axial symmetry of the flow of combustion products. The significant influence of three-dimensional phenomena on the processes in the nozzle will make it impossible to obtain correct results.

In future studies, it is planned to increase the order of accuracy of modeling the flow of combustion products in the chamber and to use the proposed approach to design nozzles for which existing methods cannot be applied, in particular dual-bell nozzles.

7. Conclusions

1. The problem of optimizing the contour of the supersonic nozzle has been stated. To this end, it was approximated by a power-law polynomial of length l, the coefficients of which, together with the length, formed a vector of optimized parameters. An expression for the axial component of thrust with the opposite sign was used as the objective function for solving the minimization problem. To simplify the task of finding the optimal nozzle contour, its solution was carried out for a prototype engine with a known chamber geometry – RD-107.

2. The explicit method of finite volumes was chosen as the method for modeling the flow of combustion products. In order to increase the simulation speed, the calculations were performed on orthogonal rectilinear grids. To overcome the problems associated with the appearance of fractional volumes when using such meshes, the extended cells method was used.

3. The proposed method was verified by comparing nozzles designed using it with nozzles built on the basis of Rao's method. The differences in the results are 3 %, 1.6 %, and 1.3 % when polynomials of the second, third, and fourth powers are used to approximate the nozzle contour, respectively. The magnitude of the error is determined by different approaches in modeling the flow of combustion products in the chamber.

Conflicts of interest

The authors declare that they have no conflicts of interest in relation to the current study, including financial, personal, authorship, or any other, that could affect the study and the results reported in this paper.

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Data availability

All data are available in the main text of the manuscript.

Use of artificial intelligence

The authors confirm that they did not use artificial intelligence technologies when creating the presented work.

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