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CONSTRUCTION OF HOMOGENEOUS SOLUTIONS OF THE TORSION PROBLEM FOR A RADIALLY INHOMOGENEOUS TRANSVERSELY ISOTROPIC CYLINDER

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The torsion problem for a radially inhomogeneous transversely isotropic cylinder of small thickness was investigated by the method of asymptotic integration of elasticity theory equations. It is assumed that the side part of the cylinder is stress-free, and boundary conditions are set at the ends of the cylinder, leaving the cylinder in equilibrium. The elastic moduli are thought to be arbitrary continuous functions of the variable along the cylinder radius. The formulated boundary value problem is reduced to a spectral problem containing a small parameter characterizing the thin-walledness of the cylinder. Homogeneous solutions are built, i.e. any solutions of the equilibrium equation satisfying the condition of no stresses on the side surfaces. It is shown that the solution of the torsion problem consists of a penetrating solution and a boundary layer character solution similar to Saint-Venant's edge effect in the theory of inhomogeneous plates. The penetrating solution determines the internal stress-strain state of a radially inhomogeneous cylinder. The stress state determined by the penetrating solution is equivalent to the torsional moments of stresses acting in the cross-section perpendicular to the cylinder axis. Solutions having the boundary layer character are localized at the ends of the cylinder and decrease exponentially with distance from the ends. These solutions are absent in applied shell theories. Asymptotic formulas for displacement and stresses are built, which make it possible to calculate the three-dimensional stress-strain state of a radially inhomogeneous transversely isotropic cylinder of small thickness. Based on the obtained asymptotic expansions, it is possible to assess the applicability of applied theories and build a refined applied theory for radially inhomogeneous cylindrical shells

Keywords: torsion problem, elastic moduli, penetrating solution, boundary layer, torsional moment

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1. Introduction

One of the properties of materials affecting the stress-strain state of elastic bodies is their inhomogeneity. In the study of inhomogeneous bodies, the real properties of materials are taken into account. Investigating radially inhomogeneous transversely isotropic cylinders based on three-dimensional equations of elasticity theory is a laborious task. Analysis of the torsion problem of a radially inhomogeneous transversely isotropic cylinder is reduced to the study of boundary value problems for a second-order partial differential equation with variable coefficients. These coefficients include elastic moduli, which are arbitrary continuous functions of the cylinder radius. This complicates the construction of torsion problem solutions. Despite the significant mathematical difficulties that arise, this more adequately takes into account the mechanical and geometric structure of a radially inhomogeneous transversely isotropic cylinder. Along with this, new qualitative and quantitative effects arise. Therefore, research devoted to the study of the stress-strain state of a radially inhomogeneous transversely isotropic cylinder based on elasticity theory equations is relevant.

2. Literature review and problem statement

A number of studies [1, 2] are devoted to investigating the problems of elasticity theory for a radially inhomogeneous cylinder. In [3], the problem of elasticity theory for a radially inhomogeneous cylinder was solved using the variational asymptotic method. In [4], the Almansi-Michell problem for an inhomogeneous anisotropic cylinder was studied numerically and analytically. In [5], the stress-strain state of an inhomogeneous orthotropic cylinder with a given inhomogeneity is considered. The influence of the inhomogeneity of the stress-strain state of the cylinder is investigated. In [6, 7], based on the spline collocation method and the finite element method, the three-dimensional stress-strain state of a radially inhomogeneous cylinder was studied and the numerical results obtained were compared. In [8], analysis of the stress-strain state of a radially inhomogeneous cylinder subjected to uniform internal pressure is considered. In [9], mechanical and thermal stresses in a radially inhomogeneous hollow cylinder under radially symmetric loads were examined with the assumption that temperature distribution is a function of radial direction. In [10–13], the problems of elasticity theory for a radially inhomogeneous cylinder of small thickness are investigated based on the asymptotic integration method. Analysis of the stress-strain state determined by homogeneous solutions is carried out. In [14], the axisymmetric problem of elasticity theory for a radially layered cylinder was studied and the existence of weakly damped boundary-layer solutions was shown. In [15], the axisymmetric problems of elasticity theory for a transversely isotropic cylinder of small thickness were investigated by the method of homogeneous solutions. A comparison of the asymptotic solution with solutions obtained from applied theories is given.

Torsion of an elastic cylinder is a classical problem of elasticity theory [16–19]. In [20], an analytical solution is given to the torsion problem of a radially inhomogeneous cylinder when the elastic moduli are power functions of the radial coordinate. In [21], the torsion problem of a radially layered cylinder is studied and a possible violation of Saint-Venant's principle in its classical formulation is shown. In [22], based on the method of homogeneous solutions, the torsion problem of a radially inhomogeneous transversely isotropic cylinder of small thickness is examined when the elastic moduli are quadratic functions of the radial coordinate.

Note that studies of the torsion problem of radially inhomogeneous transversely isotropic cylinders of small thickness, when the elastic moduli are arbitrary positive continuous functions, are not considered. There are no asymptotic formulas for displacement and stresses to determine the stress-strain state in a radially inhomogeneous cylinder of small thickness with any predetermined accuracy.

3. The aim and objectives of the study

The aim of the work is to build a homogeneous solution to the torsion problem and reveal the features of the stress-strain state for a radially inhomogeneous transversely isotropic cylinder of small thickness. Based on the analysis carried out, it is possible to assess the applicability of existing applied theories for radially inhomogeneous cylinder shells and build a new refined applied theory.

To achieve the aim, the following objectives should be accomplished:

– to formulate a boundary value problem for a radially inhomogeneous transversely isotropic cylinder;

– to obtain asymptotic formulas for displacements and stresses;

– to study the nature of stress-strain states corresponding to various types of homogeneous solutions;

– to perform numerical analysis.

4. Materials and methods

Equations of elasticity theory describing the torsion problem for a radially inhomogeneous cylinder in a cylindrical coordinate system are given. Considering that the formulated boundary value problems include a small parameter characterizing the thickness of the cylinder, the asymptotic integration method is used to build the solution. This method is one of the effective methods for studying the spatial stress-strain state of inhomogeneous elastic bodies of small thickness.

5. Results of the study on the construction of homogeneous solutions to the torsion problem for a radially inhomogeneous transversely isotropic cylinder

5. 1. Setting boundary value problems

The torsion problem of a radially inhomogeneous transversely isotropic hollow cylinder of small thickness is considered. The cylinder is referred to the cylindrical coordinate system r, φ, z:

$$r_1 \le r \le r_2,\ 0 \le \phi \le 2\pi,\ -l_0 \le z \le l_0.$$

The equilibrium equation, in the absence of mass forces in the cylindrical coordinate system r, φ, z, has the following form [16]:

$$\frac{\partial \sigma_{r\phi}}{\partial r} + \frac{\partial \sigma_{\phi z}}{\partial z} + \frac{2\sigma_{r\phi}}{r} = 0, \quad (1)$$

where, $\sigma_{r\varphi}$, $\sigma_{\varphi z}$ are components of the stress tensor, which are expressed through the components of the displacement vector $\upsilon_\varphi=\upsilon_\varphi(r, z)$ as follows [15]:

$$\sigma_{\phi z} = \tilde{G}_1 \frac{\partial \upsilon_\phi}{\partial z}, \quad (2)$$

$$\sigma_{r\phi} = \tilde{G}\left(\frac{\partial \upsilon_\phi}{\partial r} - \frac{\upsilon_\phi}{r}\right). \quad (3)$$

New dimensionless variables ρ and ξ are introduced:

$$\rho = \frac{1}{\varepsilon}\ln\left(\frac{r}{r_0}\right),\ \ \xi = \frac{z}{r_0}, \quad (4)$$

where $\varepsilon = \frac{1}{2}\ln\left(\frac{r_2}{r_1}\right)$ is a small parameter characterizing the thickness of the cylinder, $r_0 = \sqrt{r_1 r_2}$, $\rho \in [-1;1]$, $\xi \in [-l;l]$, $l = \frac{l_0}{r_0}$.

(2), (3) in the new dimensionless variables ρ, ξ have the following form:

$$\sigma_{\phi\xi} = G_1 \frac{\partial u_\phi}{\partial \xi}, \quad (5)$$

$$\sigma_{\rho\phi} = \frac{G}{\varepsilon} e^{-\varepsilon\rho}\left(\frac{\partial u_\phi}{\partial \rho} - \varepsilon u_\phi\right). \quad (6)$$

Here, $u_\phi = \frac{\upsilon_\phi}{r_0}$, $\sigma_{\rho\phi} = \frac{\sigma_{r\phi}}{G_0}$, $\sigma_{\phi\xi} = \frac{\sigma_{\phi z}}{G_0}$, $G_1 = \frac{\tilde{G}_1}{G_0}$, $G = \frac{\tilde{G}}{G_0}$ are dimensionless quantities; G_0 is some characteristic parameter having an elastic modulus dimension.

Substituting (2), (3) in (1) taking into account (4) leads to the displacement equilibrium equation as follows:

$$\frac{\partial}{\partial \rho}\left[G\left(\frac{\partial u_\phi}{\partial \rho} - \varepsilon u_\phi\right)\right] + \varepsilon G\left(\frac{\partial u_\phi}{\partial \rho} - \varepsilon u_\phi\right) +$$
$$+\varepsilon^2 G_1 e^{2\varepsilon\rho}\frac{\partial^2 u_\phi}{\partial \xi^2} = 0. \quad (7)$$

The elastic moduli $G=G(\rho)$, $G_1=G_1(\rho)$ are thought to be arbitrary positive continuous functions of the variable ρ.

The side surface is assumed to be stress-free, i.e.

$$\sigma_{\rho\phi} = \frac{Ge^{-\varepsilon\rho}}{\varepsilon}\left(\frac{\partial u_\phi}{\partial \rho} - \varepsilon u_\phi\right) = 0,\ \text{at}\ \rho = \pm 1, \quad (8)$$

and boundary conditions are set at the ends of the cylinder:

$$u_\phi\big|_{\xi=-l} = 0,\ \ \sigma_{\phi\xi}\big|_{\xi=l} = f(\rho), \quad (9)$$

where $f(\rho)$ is a smooth function.

5. 2. Building homogeneous solutions

A homogeneous solution is any solution of the equilibrium equation (7) satisfying the condition (8).

The solution of (7) is sought in the following form:

$$u_\phi(\rho,\xi)=\upsilon(\rho)m(\xi), \tag{10}$$

where:

$$m''(\xi)-\mu^2 m(\xi)=0. \tag{11}$$

Here μ is a spectral parameter.

Substituting (10) in (7), (8), taking into account (11) leads to the equation:

$$\left[G\left(\upsilon'(\rho)-\varepsilon\upsilon(\rho)\right)\right]'+ \\ +\varepsilon G\left(\upsilon'(\rho)-\varepsilon\upsilon(\rho)\right)+\varepsilon^2\mu^2 G_1 e^{2\varepsilon\rho}\upsilon(\rho)=0, \tag{12}$$

$$G\left(\upsilon'(\rho)-\varepsilon\upsilon(\rho)\right)=0 \text{ at } \rho=\pm1. \tag{13}$$

(12), (13) can be represented as:

$$F\upsilon=\mu^2\upsilon. \tag{14}$$

Here:

$$F\upsilon=\begin{cases}-\dfrac{1}{\varepsilon^2 G_1 e^{2\varepsilon\rho}}\left[G\left(\upsilon'(\rho)-\varepsilon\upsilon(\rho)\right)\right]'-\\ -\dfrac{G}{\varepsilon G_1 e^{2\varepsilon\rho}}\left(\upsilon'(\rho)-\varepsilon\upsilon(\rho)\right);\\ G\left(\upsilon'(\rho)-\varepsilon\upsilon(\rho)\right)\Big|_{\rho=\pm1}=0,\end{cases}$$

$$\left.G\left(\upsilon'(\rho)-\varepsilon\upsilon(\rho)\right)\Big|_{\rho=\pm1}=0\right\}.$$

It is proved that F is a symmetric operator in the space $L_2(-1;1)$ with weight $G_1e^{2\varepsilon\rho}$. For an arbitrary function $\upsilon(x)\epsilon D_F$, $w(x)\epsilon D_F$, we can write:

$$(F\upsilon,w)-(\upsilon,Fw)=\int_{-1}^{1}(wF\upsilon-\upsilon Fw)G_1e^{2\varepsilon\rho}d\rho=$$

$$=\int_{-1}^{1}G_1e^{2\varepsilon\rho}\begin{Bmatrix}\dfrac{\upsilon}{\varepsilon^2G_1e^{2\varepsilon\rho}}\times\dfrac{d}{d\rho}\left[G\left(w'-\varepsilon w\right)\right]+\\ +\dfrac{G\upsilon}{\varepsilon G_1e^{2\varepsilon\rho}}\left(w'-\varepsilon w\right)-\\ -\dfrac{w}{\varepsilon^2G_1e^{2\varepsilon\rho}}\times\dfrac{d}{d\rho}\left[G\left(\upsilon'-\varepsilon\upsilon\right)\right]-\\ -\dfrac{Gw}{\varepsilon G_1e^{2\varepsilon\rho}}\left(\upsilon'-\varepsilon\upsilon\right)\end{Bmatrix}d\rho. \tag{15}$$

By virtue of (13) from (15) we get:

$$(F\upsilon,w)-(\upsilon,Fw)=$$
$$=-\frac{1}{\varepsilon^2}\int_{-1}^{1}G\left(w'-\varepsilon w\right)\upsilon'd\rho+\frac{1}{\varepsilon^2}\int_{-1}^{1}G\left(\upsilon'-\varepsilon\upsilon\right)w'd\rho+$$
$$+\frac{1}{\varepsilon}\int_{-1}^{1}G\left(\upsilon w'-w\upsilon'\right)d\rho=0,$$

i.e. $(F\upsilon, w)=(\upsilon, Fw)$.

All eigenvalues of $\lambda_k(F)$ are real, and their corresponding eigenfunctions can be considered orthogonal [23]:

$$\int_{-1}^{1}G_1\upsilon_k(\rho)\upsilon_n(\rho)e^{2\varepsilon\rho}d\rho=0,(n\neq k). \tag{16}$$

To determine the solution of (12), (13) for $\varepsilon\to0$, the asymptotic integration method based on three iteration processes was applied [24, 25]. A solution corresponding to the first iteration process was built. $\mu=0$ is the eigenvalue of the problem (14). The solutions corresponding to the eigenvalues $\mu=0$ are as follows:

$$u_\phi^{(1)}(\rho,\xi)=\left(E_0+B_0\xi\right)e^{\varepsilon\rho}. \tag{17}$$

The stresses corresponding to this solution have the following form:

$$\sigma_{\rho\phi}^{(1)}=0, \tag{18}$$

$$\sigma_{\phi\xi}^{(1)}=B_0G_1(\rho)e^{\varepsilon\rho}. \tag{19}$$

The constant E_0 corresponds to the displacement of the cylinder as a rigid body. Therefore, we can assume $E_0=0$. Thus:

$$u_\phi^{(1)}(\rho,\xi)=B_0\xi e^{\varepsilon\rho}. \tag{20}$$

The second iteration process is absent here, i.e. there is no solution having the edge effect character.

According to the third iteration process, the solution of (12), (13) is sought as follows:

$$\upsilon=\upsilon_0+\varepsilon\upsilon_1+..., \tag{21}$$

$$\mu=\varepsilon^{-1}\left(\mu_0+\varepsilon\mu_1+...\right). \tag{22}$$

After substituting (21), (22) in (12), (13) in zero approximation, we get:

$$A_0\upsilon_0=\mu_0^2\upsilon_0, \tag{23}$$

where:

$$A_0\upsilon_0=\left\{-\frac{1}{G_1}\left(G\upsilon_0'(\rho)\right)',G\upsilon_0'(\rho)\Big|_{\rho=\pm1}=0\right\}.$$

The problem (23) coincides with the problem for vortex solution of an inhomogeneous transversely isotropic plate [25, 26]. A_0 is a positive operator in the space $L_2(-1; 1)$ with weight $G_1(\rho)$. Therefore, all eigenvalues $\lambda_k(A_0)=\mu_{0k}^2$ are non-negative.

To determine υ_1 and μ_1, we get:

$$\left(G\upsilon_1'(\rho)\right)'+G_1\mu_0^2\upsilon_1(\rho)=\left(G\upsilon_0(\rho)\right)'-\\ -G\upsilon_0'(\rho)-2\left(\mu_0\mu_1+\rho\mu_0^2\right)G_1\upsilon_0(\rho), \tag{24}$$

$$G\left(\upsilon_1'(\rho)-\upsilon_0(\rho)\right)\Big|_{\rho=\pm1}=0. \tag{25}$$

The solution of (24) is sought as follows:

$$\upsilon_{1n}(\rho)=\int_{-1}^{\rho}\upsilon_{0n}(x)dx+\sum_{k=0}^{\infty}\alpha_{nk}\upsilon_{0k}(\rho). \tag{26}$$

(26) satisfies the boundary conditions (25).

After substituting (26) in (24) taking into account (16), we finally get:

$$\mu_{1n}=-\frac{1}{2\mu_{0n}\left\|\upsilon_{0n}\right\|^2}\times\times\left[\begin{array}{l}\mu_{0n}^2\int_{-1}^{1}G_1\upsilon_{0n}(\rho)\left(\int_{-1}^{\rho}\upsilon_{0n}(x)\mathrm{d}x\right)\mathrm{d}\rho+\\+\int_{-1}^{1}G\,\upsilon_{0n}'(\rho)\upsilon_{0n}(\rho)\mathrm{d}\rho+\\+2\mu_{0n}^2\int_{-1}^{1}\rho G_1\upsilon_{0n}^2(\rho)\mathrm{d}\rho\end{array}\right].\quad(27)$$

$$\alpha_{nk}=\frac{1}{\left(\mu_{0k}^2-\mu_{0n}^2\right)\left\|\upsilon_{0k}\right\|^2}\times\times\left[\begin{array}{l}\int_{-1}^{1}G\,\upsilon_{0n}'(\rho)\upsilon_{0k}(\rho)\mathrm{d}\rho+\\+2\mu_{0n}^2\int_{-1}^{1}\rho G_1\upsilon_{0n}(\rho)\upsilon_{0k}(\rho)\mathrm{d}\rho+\\+\mu_{0n}^2\int_{-1}^{1}G_1\left(\int_{-1}^{\rho}\upsilon_{0n}(x)\mathrm{d}x\right)\upsilon_{0k}(\rho)\mathrm{d}\rho\end{array}\right],\ n\neq k.\quad(28)$$

$$\alpha_{nn}=-\frac{1}{\left\|\upsilon_{0n}\right\|^2}\int_{-1}^{1}G_1\left(\rho\upsilon_{0n}^2(\rho)+\upsilon_{0n}(\rho)\left(\int_{-1}^{\rho}\upsilon_{0n}\mathrm{d}x\right)\right)\mathrm{d}\rho,\quad(29)$$

where $\left\|\upsilon_{0n}\right\|^2=\int_{-1}^{1}G_1\upsilon_{0n}^2(\rho)\mathrm{d}\rho.$

Substituting (28), (29) in (21) leads to the following expression:

$$\upsilon_n(\rho)=\upsilon_{0n}(\rho)+\\+\varepsilon\left\{\begin{array}{l}\int_{-1}^{\rho}\upsilon_{0n}(x)\mathrm{d}x-\frac{1}{\left\|\upsilon_{0n}\right\|^2}\int_{-1}^{1}G_1\left(\begin{array}{l}\rho\upsilon_{0n}^2(\rho)+\\+\upsilon_{0n}(\rho)\int_{-1}^{\rho}\upsilon_{0n}(x)\mathrm{d}x\end{array}\right)\mathrm{d}\rho\times\upsilon_{0n}(\rho)+\\+\sum_{\substack{k=0\\k\neq n}}^{\infty}\frac{1}{\left(\mu_{0k}^2-\mu_{0n}^2\right)\left\|\upsilon_{0k}\right\|^2}\left[\begin{array}{l}\int_{-1}^{1}G\upsilon_{0n}'(\rho)\upsilon_{0k}(\rho)\mathrm{d}\rho+\\+2\mu_{0n}^2\int_{-1}^{1}\rho G_1\upsilon_{0n}(\rho)\upsilon_{0k}(\rho)\mathrm{d}\rho+\\+\mu_{0n}^2\int_{-1}^{1}G_1\left(\int_{-1}^{\rho}\upsilon_{0n}(x)\mathrm{d}x\right)\upsilon_{0k}(\rho)\mathrm{d}\rho\end{array}\right]\upsilon_{0k}(\rho)\end{array}\right\}+\\+O\left(\varepsilon^2\right).\quad(30)$$

The solutions corresponding to the third iteration process are as follows:

$$u_\phi^{(3)}(\rho,\xi)=\sum_{n=1}^{\infty}\upsilon_n(\rho)m_n(\xi),\quad(31)$$

where $m_n(\xi)=C_n\mathrm{ch}(\mu_n\xi)+D_n\mathrm{sh}(\mu_n\xi)$; C_n, D_n are arbitrary constants.

For stresses, we get:

$$\sigma_{\rho\phi}^{(3)}=\sum_{n=1}^{\infty}\frac{G(\rho)e^{-\varepsilon\rho}}{\varepsilon}\left(\upsilon_n'(\rho)-\varepsilon\upsilon_n(\rho)\right)m_n(\xi),\quad(32)$$

$$\sigma_{\phi\xi}^{(3)}=\sum_{n=1}^{\infty}G_1(\rho)\upsilon_n(\rho)m_n'(\xi).\quad(33)$$

The total solution of (7), (8) will be the sum of solutions (20), (31):

$$u_\phi(\rho,\xi)=B_0\xi e^{\varepsilon\rho}+\sum_{k=1}^{\infty}\upsilon_k(\rho)\,m_k(\xi).\quad(34)$$

Based on (18), (19), (32), (33) for stresses we get:

$$\sigma_{\rho\phi}=\sum_{k=1}^{\infty}\frac{G(\rho)e^{-\varepsilon\rho}}{\varepsilon}\left(\upsilon_k'(\rho)-\varepsilon\upsilon_k(\rho)\right)m_k(\xi),\quad(35)$$

$$\sigma_{\phi\xi}=B_0G_1e^{\varepsilon\rho}+\sum_{k=1}^{\infty}G_1\upsilon_k(\rho)m_k'(\xi).\quad(36)$$

The obtained asymptotic formulas (34)–(36) make it possible to calculate the stress-strain state of a radially inhomogeneous transversely isotropic cylinder of small thickness with any predetermined accuracy.

5. 3. Analysis of the stress-strain state determined by homogeneous solutions

For torsional moments $M_{k\rho}$ of stresses acting in the section ξ=const, we get:

$$M_{kp}=2\pi\varepsilon\int_{-1}^{1}\sigma_{\phi\xi}e^{3\varepsilon\rho}\mathrm{d}\rho.\quad(37)$$

After substituting (36) in (37), the following expression is obtained:

$$M_{kp}==2\pi\varepsilon\int_{-1}^{1}\left[\begin{array}{l}B_0G_1e^{\varepsilon\rho}+\\+\sum_{k=1}^{\infty}G_1\upsilon_k(\rho)m_k'(\xi)\end{array}\right]e^{3\varepsilon\rho}\mathrm{d}\rho==2\pi\varepsilon B_0\int_{-1}^{1}G_1e^{4\varepsilon\rho}\mathrm{d}\rho+2\pi\varepsilon\sum_{k=1}^{\infty}\omega_k m_k'(\xi),\quad(38)$$

where:

$$\omega_k=\int_{-1}^{1}G_1\upsilon_k(\rho)e^{3\varepsilon\rho}\mathrm{d}\rho.$$

Multiplying both parts of (12) by $e^{\varepsilon\rho}$ and integrating the resulting expression within [–1; 1], we get:

$$\int_{-1}^{1}\left[G\left(\upsilon_k'(\rho)-\varepsilon\upsilon_k(\rho)\right)\right]'e^{\varepsilon\rho}\mathrm{d}\rho+\\+\int_{-1}^{1}\varepsilon G\left(\upsilon_k'(\rho)-\varepsilon\upsilon_k(\rho)\right)e^{\varepsilon\rho}\mathrm{d}\rho+\\+\varepsilon^2\mu_k^2\int_{-1}^{1}G_1e^{3\varepsilon\rho}\upsilon_k(\rho)\mathrm{d}\rho=0.\quad(39)$$

Integrating by parts and using the boundary condition (13) from (39) we get:

$$\varepsilon^2\mu_k^2\int_{-1}^{1}G_1e^{3\varepsilon\rho}\upsilon_k(\rho)\mathrm{d}\rho=0,\quad(40)$$

i.e.

$$\omega_k = 0. \quad (41)$$

Substituting (41) in (38) leads to the equality:

$$M_{kp} = 2\pi\varepsilon\, B_0 \int_{-1}^{1} G_1 e^{4\varepsilon\rho} \mathrm{d}\rho. \quad (42)$$

The constant B_0 in the absence of external forces on the side surfaces is proportional to the torsional moments M_{kp} of stresses acting in the section ξ=const. The first part of the solution (34) determines the internal stress-strain state of the cylinder.

The stress state corresponding to the third group of solutions has the boundary layer character, and the first terms of its asymptotic expansion are equivalent to Saint-Venant's edge effect in the theory of transversely isotropic inhomogeneous plates [25, 26].

Substituting (34), (36) in (9) yields the following:

$$\sum_{k=1}^{\infty} \upsilon_k(\rho) m_k(\xi)\Big|_{\xi=-l} = f_1(\rho), \quad (43)$$

$$\sum_{k=1}^{\infty} G_1 \upsilon_k(\rho) m_k'(\xi)\Big|_{\xi=l} = f_2(\rho), \quad (44)$$

where:

$$f_1(\rho) = \frac{M\, l\, e^{\varepsilon\rho}}{2\pi\varepsilon \int_{-1}^{1} G_1 e^{4\varepsilon\rho} \mathrm{d}\rho};$$

$$f_2(\rho) = f(\rho) - \frac{M G_1 e^{\varepsilon\rho}}{2\pi\varepsilon \int_{-1}^{1} G_1 e^{4\varepsilon\rho} \mathrm{d}\rho}.$$

Multiplying (43) by $G_1 \upsilon_n(\rho) e^{2\varepsilon\rho}$, (44) by $\upsilon_n(\rho) e^{2\varepsilon\rho}$ and integrating within [–1; 1], we get:

$$\begin{cases} ch(\mu_n l)\cdot C_n - sh(\mu_n l)\cdot D_n = h_{1n}, \\ \mu_n sh(\mu_n l)\cdot C_n + \mu_n ch(\mu_n l)\cdot D_n = h_{2n}. \end{cases} \quad (45)$$

Here:

$$h_{1n} = \frac{M\, l}{2\pi\varepsilon \|\upsilon_n\|^2} \int_{-1}^{1} G_1 \upsilon_n(\rho) e^{3\varepsilon\rho} \mathrm{d}\rho \cdot \left(\int_{-1}^{1} G_1 e^{4\varepsilon\rho} \mathrm{d}\rho \right)^{-1};$$

$$h_{2n} = \frac{1}{\|\upsilon_n\|^2} \int_{-1}^{1} f(\rho) \upsilon_n(\rho) e^{2\varepsilon\rho} \mathrm{d}\rho - \frac{h_{1n}}{l}.$$

After solving (45), the following are determined:

$$C_n = \frac{h_{2n} sh(\mu_n l) + h_{1n}\mu_n ch(\mu_n l)}{\mu_n ch(2\mu_n l)}; \quad (46)$$

$$D_n = \frac{h_{2n} ch(\mu_n l) - h_{1n}\mu_n sh(\mu_n l)}{\mu_n ch(2\mu_n l)}. \quad (47)$$

The unknown constants C_n and D_n included in (31) are determined by the formula (46), (47).

5. 4. Numerical analysis

As an example, the problem of the stress-strain state of a radially inhomogeneous and homogeneous cylinder of small thickness is considered. The area occupied by the cylinder $\Gamma=\{r\epsilon[1; 1.5], \varphi\epsilon[0; 2\pi], z\epsilon[-1.5; 1.5]\}$. The parameter characterizing the thickness of the cylinder is ε=0.2. It is assumed that the side surface of the cylinder is stress-free, and boundary conditions are set at the ends of the cylinder:

$$\upsilon_\phi = 0 \text{ at } z = -1.5, \quad (48)$$

$$\sigma_{\phi z} = C_0 r \text{ at } z = 1.5. \quad (49)$$

Fig. 1–4 shows the stress distributions $\sigma_{r\varphi}$ along the thickness (along the center line) and distributions of $\sigma_{\varphi z}$, υ_φ, $\sigma_{r\varphi}$ along the line connecting the points (r_1; $-l$) and (r_2; l) for a homogeneous and radially inhomogeneous cylinder.

For a radially inhomogeneous cylinder, the following cases are considered:

a) the elastic moduli vary linearly with the radius (increasing elastic moduli): $G=A_0 r$, $G_1=B_0 r$.

b) the elastic moduli vary inversely with the radius (decreasing elastic moduli): $G=A_0/r$, $G_1=B_0/r$.

The stress distributions $\sigma_{r\varphi}$ along the thickness (along the center line) for homogeneous and radially inhomogeneous cylinders are qualitatively the same, but vary quantitatively. The distribution $\sigma_{r\varphi}$ for homogeneous and radially inhomogeneous cylinders occurs according to a law close to quadratic. The parabola branches point downward. At r=1.25, the stress $\sigma_{r\varphi}$ takes on the highest value (Fig. 1).

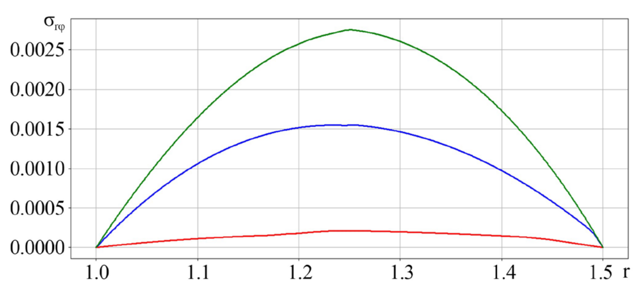

Fig. 1. Distributions of $\sigma_{r\varphi}$ along the thickness (along the center line)

The distribution of $\sigma_{\varphi z}$ along the line connecting the points (r_1; $-l$) and (r_2; l) for a homogeneous and radially inhomogeneous cylinder is qualitatively the same, but differs quantitatively (Fig. 2).

The distribution of υ_φ along the line connecting the points (r_1; $-l$) and (r_2; l) for a homogeneous and radially inhomogeneous cylinder occurs according to the same laws and only quantitatively depends on the degree of inhomogeneity (Fig. 3).

At $z\epsilon[-1.5; 0.956]$, the values of $\sigma_{r\varphi}$ for a radially inhomogeneous cylinder, the elastic moduli of which vary linearly with the radius, are greater than the values corresponding to a homogeneous and radially inhomogeneous cylinder, the elastic moduli of which vary inversely with the radius. At $z\epsilon[-1.021; 1.5]$, the values of $\sigma_{r\varphi}$ for a radially inhomogeneous cylinder, the elastic moduli of which vary linearly with the radius, are less than the values obtained for a homogeneous and inhomogeneous cylinder, the elastic moduli of which vary inversely with the radius (Fig. 4).

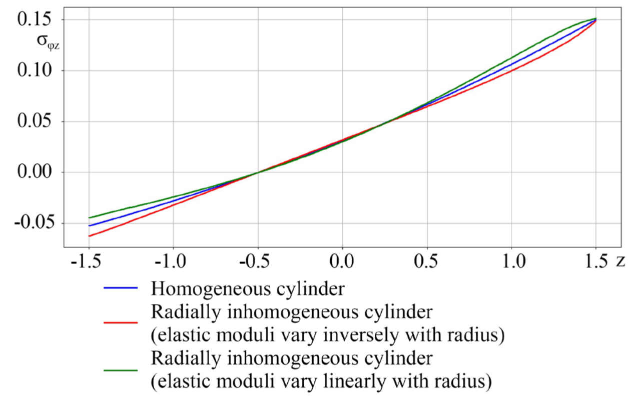

Fig. 2. Distributions of $\sigma_{\varphi z}$ along the line connecting the points (r_1; $-l$) and (r_2; l)

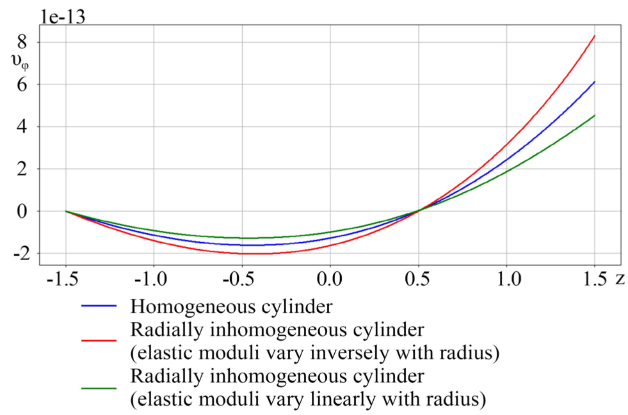

Fig. 3. Distributions of υ_φ along the line connecting the points (r_1; $-l$) and (r_2; l)

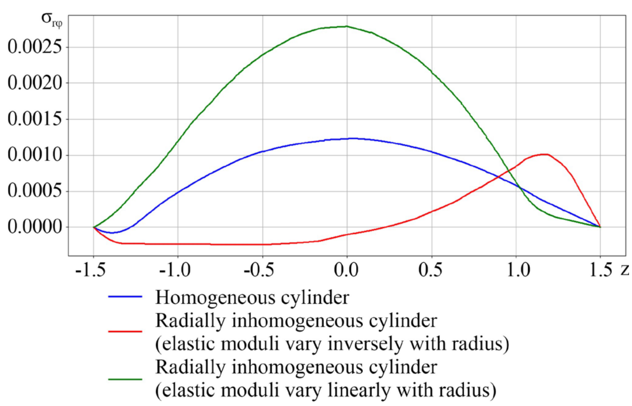

Fig. 4. Distributions of $\sigma_{r\varphi}$ along the line connecting the points (r_1; $-l$) and (r_2; l)

From the analysis of the numerical results, it follows that the inhomogeneity of the material can have a significant effect on the stress-strain state of the cylinder.

6. Discussion of the results of the study on the construction of homogeneous solutions to the torsion problem for a radially inhomogeneous transversely isotropic cylinder

Asymptotic analysis of the torsion problem for a radially inhomogeneous transversely isotropic cylinder of small thickness was carried out. Assuming that the elastic moduli are arbitrary continuous functions, boundary value problems (7) and (8) along the radius are formulated. Given that the formulated boundary value problems include a small parameter characterizing the thickness of the cylinder, the asymptotic integration method is used to build the solution. This method is one of the effective methods for investigating the three-dimensional stress state of inhomogeneous bodies of finite dimensions. Using the asymptotic integration method, homogeneous solutions are built, i.e. any solutions of the equilibrium equation (7) that satisfy homogeneous boundary conditions (8). Analysis of stress-strain states corresponding to various types of homogeneous solutions was carried out. It is shown that some penetrating solution (20) corresponds to the first iteration process. The stress state determined by the penetrating solution (20) is equivalent to the torsional moments M_{kp} of stresses acting in the section ξ=const. The second iteration process determining the solution having the edge effect character does not exist here. The solutions corresponding to the third iteration process have the boundary layer character. These solutions are localized at the ends of the cylinder, and with distance from the ends they decrease exponentially. Solutions having the boundary layer character are absent in applied shell theories. The first terms of its asymptotic expansion are equivalent to Saint-Venant's edge effect in the theory of transversely isotropic inhomogeneous plates. It was found that homogeneous solutions consist of two types: penetrating solutions (20) and boundary-layer solutions (31). The obtained asymptotic formulas (34)–(36) are suitable for $\varepsilon\to 0$. The asymptotic integration method is successfully applied and has no drawbacks in solving the problem of elasticity theory for radially inhomogeneous cylinders of small thickness. The division of the stress-strain state into internal and boundary-layer solutions is valid only for radially inhomogeneous transversely isotropic cylinders of small thickness. The obtained asymptotic formulas for displacement and stresses, which are characteristic of radially inhomogeneous thin cylinders, are not suitable for radially inhomogeneous transversely isotropic thick cylinders. For a highly radially inhomogeneous transversely isotropic cylinder of small thickness, i.e. when the elastic modulus values do not vary within the same order and differ greatly, two different small parameters appear. One of them characterizes the thickness of the cylinder, and the other characterizes the relative stiffness of the layers (for example, a radially multilayer cylinder of small thickness with alternating rigid and soft layers). As a result of two small parameters, a weak boundary layer appears, which significantly changes the pattern of the stress-strain state away from the cylinder ends. The processes of determining the internal solution and the weak boundary layer are not separated. In such cases, the asymptotic integration method cannot be applied to solve the torsion problem of a highly radially inhomogeneous cylinder of small thickness.

In the case of $G=G_1$, all the solutions obtained are completely consistent with the solutions for a radially inhomogeneous isotropic cylinder. When G and G_1 are constants, the solutions obtained coincide with the solutions for a transversely isotropic cylinder [15]. The asymptotic formulas (34)–(36) make it possible to calculate the stress-strain state of a small thickness with any predetermined accuracy.

Further, based on the obtained asymptotic expansions for displacement and stresses, the definition of the applicability of existing applied theories and the construction of new more refined applied theories for torsion of a radially inhomogeneous transversely isotropic cylindrical shell can be considered.

7. Conclusions

1. Using the asymptotic integration method, the formulated boundary value problem for a radially inhomogeneous transversely isotropic cylinder of small thickness was built. The solution corresponding to the first iteration process is shown to be a penetrating solution. The second iteration process, which determines the solution having the edge effect character, is absent here. The third iteration process determines solutions that have the boundary layer character and are localized at the ends of the cylinder. Solutions corresponding to the third iteration process are absent in applied theories. The general solution consists of the penetrating solution and solutions having the boundary layer character.

2. Asymptotic formulas for displacement and stresses are obtained, which make it possible to calculate a three-dimensional stress-strain state in a radially inhomogeneous transversely isotropic cylinder of small thickness with any predetermined accuracy.

3. On the basis of asymptotic analysis, the features of the stress-strain state in a radially inhomogeneous cylinder were studied. It is shown that the penetrating solution is determined through the torsional moments M_{kp} of stresses acting in the section ξ=const. The stress state corresponding to the third group of solutions has the boundary layer character and the first terms of its asymptotic expansion are equivalent to Saint-Venant's edge effect in the theory of transversely isotropic inhomogeneous plates.

4. Based on the numerical analysis performed, displacement and stress distributions for homogeneous and radially inhomogeneous transversely isotropic cylinders were determined. Indicating the qualitative and quantitative differences in these distributions for homogeneous and radially inhomogeneous cylinders, the effect of material inhomogeneity on the stress-strain state is estimated.

Conflict of interest

The author declares that he has no conflict of interest in relation to this research, whether financial, authorship or otherwise, that could affect the publication of this paper.

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Data availability

The manuscript has no associated data.

Use of artificial intelligence

The authors confirm that they did not use artificial intelligence technologies when creating the presented work.

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