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RESULTS OF THE ANALYTICAL SOLUTION OF THE PROBLEM OF RADIAL VIBRATIONS OF DISKS OF VARIABLE THICKNESS

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An analytical solution is obtained for the problem of radial vibrations of disks of variable thickness. A disk is considered that is rigidly fixed along the inner circular contour (ρ=0.2) and free on the outer contour (ρ=1). The thickness of the disk varies according to the law $H=H_0(\rho^{\nu+\mu}+C\rho^{\nu-\mu})^2$, where H_0, C, μ are arbitrary constants; ν is the Poisson's ratio. The exact solution of the problem is known only for H=const and $H=1/\rho^3$. However, these solutions are not sufficient to study the vibrations of disks of other configurations. The proposed law of thickness variation $H(\rho)$ allows us to obtain exact solutions to the problem at any value of the constant coefficients H_0, C, μ, ν. By varying the values of these coefficients within a single given function, it is possible to set the disk profile of the desired appearance. The methods used to obtain these solutions are based on appropriate mathematical transformations of the original equation.

The problem of disk oscillations is solved for four variants of thickness change. The natural frequencies for the first three forms of vibration are calculated. Comparison of the natural frequencies found for the three cases of the disk profile gently sloping indicates an increase in their values with an increase in the bending of the disk thickness. Based on the obtained eigenfunctions, the stresses were calculated and the nature of their distribution along the radial coordinate of the disk was determined.

The strength of the disks under resonant radial vibrations was evaluated using a special criterion. It is found that the most limiting, i.e., destructive principal stress $\sigma_1=\sigma_r$ at the first (main) form of vibration should be chosen from the ratio $\sigma_r\approx0.79\ [\sigma_{-1}]$, where $[\sigma_{-1}]$ is the endurance limit of the disk material under uniform loading. The results obtained can be used to predict the stress-strain state of disks of variable profile under their radial vibrations

Keywords: disk of variable thickness, radial vibrations, natural frequencies, waveforms, stresses

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1. Introduction

Disks of variable thickness are one of the most important structural elements of turbines for various purposes, engines, and other high-speed machines and mechanisms, in particular in aircraft and rocket engineering [1–13]. Of practical importance are special-purpose disks in which radial vibrations are deliberately excited for technological purposes. In addition, disks can be used as active acoustic elements in transducers that are made of piezoceramic or magnetostrictive materials. Ensuring the strength of disks during their operation is related to the analysis of the vibrations that occur in them. The appearance of high-intensity fluctuations may be the cause of destructive stresses.

One of the approaches to the search for enabling an increased resource of oscillating disks is to operate them under special operating conditions [5, 6], or to introduce certain heterogeneity into the characteristics of the material from which the disk is made (anisotropy of the material, variable coefficient of elasticity, variable density). In particular, in order to solve the problem of reducing critical stresses, in some works it is proposed to consider disk models made of orthotropic material [7, 8] or when the disks have a radially stepped shape [9]. In the case when the disk profile is determined by an arbitrary law of thickness change, the corresponding problem of oscillations cannot be solved analytically at present. This applies to all three types of disk oscillation. The relevance of this scientific area relates to the fact that in order to analyze the intrinsic radial vibrations of a disk of variable thickness, it is necessary to have an analytical solution to the corresponding boundary value problem. This requirement is due to the fact that in order to ensure the required operational resource of the structure with disks, it is necessary to have information about the distribution of radial movements and stresses in the disk.

2. Literature review and problem statement

The results of analytical studies on disks of variable thickness under the mode of radial vibrations are limited because of known difficulties of a mathematical nature that arise when trying to solve the differential equation of the problem. Our review of current literature [2–13] revealed that the solutions to the problems of oscillations of disks of variable thickness are extremely necessary for specialists in the field of mechanics and applied acoustics. This is confirmed by the presence of a significant number of various practical applications of disks as structural elements of machines.

In work [2], a survey analysis of disk vibrations in turbomachines of different designs under conditions of different environments was carried out. The causes of possible vibrations, which include high rotation speed, cavitation, and resonant oscillations, have been studied. It is noted that the presence of these factors can cause structural wear, damage to machine components, and the appearance of cracks on the surface. However, the work does not clearly separate the types of oscillations.

In [3], the object of research is a rotating disk of variable thickness. In the introductory part of the work, examples of the practical application of such disks in various fields are given. The distribution of shear stresses and deformations was evaluated, various functions of the disk thickness were determined, and a comparison of stresses depending on the type of thickness was given. It is noted that extreme tangential stresses can be the cause of disk destruction. The determination of natural frequencies and displacements for disks whose thickness varies according to hyperbolic, exponential, and parabolic laws is carried out numerically. But the analytical solution to this problem is not given.

Work [4] reports the study of damage in the disks of GTE compressors, which are used in military fighters and training aircraft. It is noted that damage in such elements is associated with the presence of highly deformed low-cycle fatigue. That is, we are talking about the study of low-cycle fatigue cracks in the places of bolt holes of GTE disks. However, the work does not indicate which type of oscillations are caused by fatigue damage. To clarify this question, it is necessary to have the results of the appropriate mathematical analysis.

In [5], the magnetoelastic problem for a rotating disk of variable thickness is considered. The operation of the disk is considered under special conditions (magnetic field, high temperature), and thermal stresses are determined using a numerical method. This work cannot be an aid to solving the problem in relation to the analytical solution to the problem.

As in the previous work, in paper [6], a disk of variable thickness was considered under special operating conditions, that is, at temperature and density. In particular, the analysis of the distribution of radial stresses was carried out and special approaches to the solution of the differential equation of radial vibrations were determined. At the same time, the disk itself is defined by a hyperbolic profile and is made of a rubber-like material. The available calculation method (based on the Seth transformation theory) cannot be effectively used for the research tasks.

In work [7], in order to achieve higher working speeds of disk rotation, an orthotropy of reinforcement is proposed. This approach, according to the authors, makes it possible to influence the stress distribution and in this way, it is possible to increase the service life of the rotating disks. The criterion of disk destruction based on the Tsai-Wu hypothesis was considered and the possible values of disk rupture velocities were investigated on its basis. This reinforced disk is designed on the basis of numerical calculations. Disk oscillations are not considered in the paper but reinforcement is useful because even in the case of oscillations it will act as a strengthening factor.

An anisotropic disk of constant thickness is considered in [9]. The phenomenon of anisotropy implemented in this disk makes it possible to reduce radial stresses. In the study, disks made of isotropic material are considered. Therefore, the calculation methods given are not appropriate for the case study.

In theoretical paper [10], a disk with a hyperbolic profile was considered and the maximum rotational speeds were calculated. Two methods of analytical approximation were used for calculations. The estimated distributions of stresses and radial displacements of the disk are given. Disk oscillations are not considered in the work, however, in the case of radial vibrations, the stress distribution may have a similar form.

In [13], the rotor system of a steam turbine with a system of disks of variable thickness was modeled. This simulation is performed using the finite element method. The finite element method, as an alternative to analytical methods, is not considered in the paper.

Thus, radial vibrations of disks are not considered in the aforementioned and other works. It is known, among other things, that the analytical solution to this type of problem in Bessel functions is found only when the thickness of the disk is constant or determined by a hyperbolic dependence [14]. For all other cases of variable thickness, the solution to the differential equation of the problem is not found. Hence the statement of the problem. Due to the noted circumstances regarding the difficulties of the analytical solution of the problem, researchers are forced to use approximate numerical methods. Violating the above-mentioned permanent idea about the peculiar uniqueness of the dependence is possible only by the real development of methods or approaches that allow obtaining accurate analytical solutions to this problem in cases that differ from the variant of hyperbolic dependence. It is desirable that the new solutions have a simple form and also be valid for significantly different thickness profiles of the disks. If the general solution of the differential equation, as a mathematical model of oscillations of a given type, is found, then it is possible to find the frequencies without difficulty and construct the forms of the natural oscillations, as well as study the stress distribution.

3. The aim and objectives of the study

The purpose of our study is a closed analytical solution to the problem of radial vibrations of disks of variable thickness. The thickness of the disk changes according to a specially chosen law. This will make it possible to find out the nature of the distribution of shapes, frequencies, and stresses during free oscillations of real disks.

To achieve the goal, the following tasks were set:

– using the factorization method, transform the original differential equation into a form that makes it possible to decompose it into a system of two equations of the first order;

– by extracting the first derivative, convert the equation to a form that has known solutions;

– to find the solution to the original equation by the method of symmetries;

– to consider the problem of radial vibrations of a disk of selected profiles, which is rigidly fixed along the inner contour.

4. The study materials and methods

The object of our study is a disk of variable thickness. The mathematical model of the research is the differential equation of small oscillations of a disk of variable thickness. The problem is stated within the framework of the Kirchhoff-Lagrange hypothesis, subject to the following assumptions. It is assumed that the disk is a body of revolution and that there is a plane of symmetry that is perpendicular to the central axis of the disk. It is accepted that the inclination of the side surfaces of the disk to this plane can be neglected. The thickness of the disk is small and is no more than 1/5 of its diameter. It is assumed that any point of this plane of symmetry oscillates in the same plane. In addition, any point of this plane moves only along the corresponding radius and at the same time there are no movements, both in the tangential direction and in the direction perpendicular to this plane of the disk.

Appropriate disk profile configurations are chosen to provide for an accurate analytical solution.

The method of successive trial calculations was used to solve the frequency equations.

The research results are intended for structural materials that comply with Hooke's law.

5. Results of investigating the problem of radial vibrations of the disk

5. 1. Scheme of the factorization method for solving equations

The equation of the eigenforms of the radial vibrations of a disk of variable thickness takes the form [14]:

$$R''+R'\left(\left(H'/H\right)+\left(1/\rho\right)\right)+ \\ +R\left(\left(\left(\nu H'\right)/\left(\rho H\right)\right)-\left(1/\rho^2\right)+\lambda^2\right)=0, \quad (1)$$

where $R=R(\rho)$ – radial movements;

$\rho=r/a$ – relative radial coordinate;

r – variable radius;

a – disk radius;

$H=H(\rho)$ – thickness of the disk;

ν – Poisson's ratio;

$$\lambda^2=\left(a^2\omega^2\gamma\left(1-\nu^2\right)\right)/\left(gE\right); \quad (2)$$

$\omega=2\pi f$ – circular frequency;

f – natural frequency;

γ – specific gravity;

g – acceleration of gravity;

E is the modulus of elasticity.

The solution to equation (1) must satisfy the boundary conditions of the problem. If the disk is rigidly fixed along the inner contour ($\rho<1$), then there should be no movements on this contour. This corresponds to the condition $R(\rho_0)=0$. If the outer contour of the disk $\rho=1$ ($r=a$) is not fixed, then there are no radial stresses σ_r on it. This condition should be written in the form $(R'+\nu(R/\rho))_{\rho=1}=0$, that is $R'(1)+\nu R(1)=0$. After substituting the general solution into these conditions in the standard way, the eigenvalues λ are found and the eigenfrequencies are calculated according to formula (2) and the forms of eigenoscillations are constructed. It is appropriate to recall that only one case of variable thickness is widely known, namely, the hyperbolic dependence $H=H_0/\rho^n$, in which equation (1) has an exact analytical solution in Bessel functions [14]. At the same time, for these boundary conditions, when the disk is fixed along the contour $\rho_0<1$, the function $H(\rho)$ can only have an indicator of practical value $n\geq 0$. Since the mentioned dependence $H=H_0/\rho^n$ significantly limits the possibilities for optimal design of geometric profiles of disks, in accordance with the stated purpose of the research, we present the following results.

The equation can be rewritten in the form:

$$R''+2\left(D'/D\right)R'+ \\ +R\left(2\left(\nu/\rho\right)\left(D'/D\right)-\left(\left(1+\nu\right)/\left(\rho^2\right)\right)+\lambda^2\right)=0, \quad (3)$$

where

$$2\left(D'/D\right)=\left(H'/H\right)+\left(1/\rho\right);$$

$$D^2=F=\rho H; \quad H=D^2/\rho. \quad (4)$$

For some $D(\rho)$, equation (3) can be rewritten as:

$$\left(d/d\rho+S\left(\rho\right)\right)^2 R-\left(i\lambda\right)^2 R=0. \quad (5)$$

In this case, the general solution to equation (3) can be represented as the sum of solutions R_1 and R_2 to the following two first-order equations:

$$R_1'+SR_1+i\lambda R_1=0; \quad R_2'+SR_2-i\lambda R_2=0. \quad (6)$$

After opening expression (5) and comparing it with equation (3), $S=D'/D$; $S'+S^2=(2\nu/\rho)(D'/D)-((1+\nu)/\rho)$ is obtained. Hence, we have the equation for determining $D(\rho)$:

$$\rho^2 D''-2\nu\rho D'+D\left(1+\nu\right)=0, \quad (7)$$

and also, according to (6), the solutions R_1 and R_2 and their sum:

$$R=\left(1/D\right)\left(A\sin\lambda\rho+B\cos\lambda\rho\right). \quad (8)$$

The solution to equation (7) is not difficult, as it resembles Euler's equation:

$$D=\left(\left(1+2\nu\right)/2\right)\begin{pmatrix} C_1\sin\sqrt{\left(3/4\right)-4\nu^2}\ln\rho+ \\ +C_2\cos\sqrt{\left(3/4\right)-4\nu^2}\ln\rho \end{pmatrix}. \quad (9)$$

As can be seen from these results, the factorization method made it possible to obtain the solution $R(\rho)$ in elementary functions for $D(\rho)$, the structure of which includes two independent constants C_1 and C_2. These circumstances confirm the simplicity and effectiveness of the method since within one simple solution there are disk thicknesses of different profiles found, which depend on C_1 and C_2.

5. 2. Scheme of the derivative extraction method for solving equations

After taking into account in equation (3) $R=W(\rho)/D$, a new equation is obtained:

$$W''+W\begin{pmatrix}-(D''/D)+(2\nu/\rho)(D'/D)-\\-((1+\nu)/(\rho^2))+\lambda^2\end{pmatrix}=0. \quad (10)$$

At:

$$-(D''/D)+(2\nu/\rho)(D'/D)-(1+\nu)/(\rho^2)=$$
$$=-(p(p-1))/(\rho^2)-q^2, \quad (11)$$

then, instead of equation (10), we get:

$$W''+W(\lambda^2-q^2-((p)(p-1)/p^2))=0, \quad (12)$$

where q^2, p are arbitrary constants.

The solution to this equation is known [15, 16] and therefore it can be rewritten in the form:

$$W=\sqrt{\rho}\left[AJ_{p-1/2}\left(\sqrt{\lambda^2-q^2}\rho\right)+BY_{p-1/2}\left(\sqrt{\lambda^2-q^2}\rho\right)\right]. \quad (13)$$

Condition (11) is an equation for determining the function $D(\rho)$ and the presence or absence of the coefficient q in this equation predetermines its type. When $q=0$, you need to consider the Euler equation in the form:

$$D=\rho^{0.5(1+2\nu)}\left(C_1\rho^{\sqrt{\nu^2+p(p-1)-3/4}}+C_2\rho^{-\sqrt{\nu^2+p(p-1)-3/4}}\right). \quad (14)$$

If $p=0$ or $p=1$, then, in this case, solutions (14) and (9) coincide. For all other values of p, the solution (14) remains unchanged. If $q^2\neq 0$ is set in equation (11), then its solution can be found in the Bessel functions [16]:

$$D=\rho^{0.5(1+2\nu)}\left[C_1I_\mu(q\rho)+C_2K_\mu(q\rho)\right], \quad (15)$$

where $\mu=(\nu^2+p(p-1)-3/4)^{1/2}$ – order of Bessel functions.

The results (14) and (15) for the functions $D(\rho)$, which determine the variable configuration of the disk thickness $H(\rho)=D^2/\rho$, are more informative, compared to result (9), which is obtained by the factorization method. Thus, result (14) for $D(\rho)$ has three independent constants (C_1, C_2, p) in its structure, and (15) has four (C_1, C_2, p, q). This made it possible to manage both the thickness of the disk and its movements $R(\rho)$ more flexibly durirong oscillations. In the solutions for $R(\rho)=W/D$ and $D(\rho)$, which are expressed in terms of Bessel functions, they are replaced by elementary ones, selecting the necessary value of the coefficient p for such replacement. It is known, for example, that Bessel functions with half-integer index based on tabular relations can be expressed in terms of trigonometric and hyperbolic functions. For solution (13), after taking $p_1=(0;1)$; $p_2=(-1;2)$; $p_3=(-2;3)$, we obtained:

$$\left.\begin{aligned}W_1=W_0=A\sin k\rho+B\cos k\rho,\\ k=\sqrt{\lambda^2-q^2};W_2=W_0'-W_0/\rho;\\ W_3=\frac{1}{\rho^2}\left[(3-k^2\rho^2)W_0-3\rho W_0'\right].\end{aligned}\right\} \quad (16)$$

It is well known that operations with elementary functions are more convenient for calculations, therefore, if necessary, the solution (15) for $D(\rho)$ can be expressed in terms of such functions. For this, $\mu=n/2$ (n=1; 3; 5; ...) is set and the value of p is determined. If the disk is solid, then in this case, for the thickness $H(\rho)=D/\rho$, it is necessary to set $H(0)\neq 0$ and $H(0)\neq\pm\infty$. To construct $H(\rho)$, the function $D(\rho)$ in the form (14) is set, and when $p=-1/2$ or $p=3/2$, $H=(C_1\rho^{2\nu}+C_2)^2$ is obtained. For movements according to (13), the expression $R=W/D=(J_{\pm1}(\lambda\rho)+BY_{\pm1}(\lambda\rho))/(C_1\rho^{2\nu}+C_2)$ is obtained.

5. 3. Scheme of the method of symmetries for solving equations

Equation (1) with parameters $H(\rho)$ and $R(\rho)$ can be transformed into its symmetry, that is, into a new equation with parameters $H_1(\rho)$ and $R_1(\rho)$. It acquired this form due to the fact that equation (1) at $\rho H=F=D^2$ is written in the form $R''+(F'/F)R'+R((\nu/\rho)(F'/F)-(1+\nu)/\rho^2)+\lambda^2)=0$. Its symmetry:

$$R_1''+\frac{F_1'}{F_1}R_1'+R_1\left(\frac{\nu}{\rho}\frac{F_1'}{F_1}-\frac{1+\nu}{\rho^2}+\lambda^2\right)=0$$

was found by fulfilling the following dependences:

$$R_1=R/V;\ \ F_1=FV^2;\ \lambda_1{}^2=\lambda^2+m^2;\ \ \rho H_1=F_1=D_1^2. \quad (17)$$

In this case, $V(\rho)$ is determined from the Bessel equation [16]:

$$V''+\left(\left(\left(F\rho^{-2\nu}\right)'\right)/\left(F\rho^{-2\nu}\right)\right)V'-m^2V=0, \quad (18)$$

where m^2 – arbitrary constant.

At $m=0$, we obtained from (18):

$$V=C_1\int(1/F)\rho^{2\nu}\mathrm{d}\rho+C_2. \quad (19)$$

The solution to the equation for $R(\rho)$, as stated above, is known for $F=\rho^n$ and is written in the form:

$$R=\rho^{(1-n)/2}\left(AJ_\psi(\lambda\rho)+BY_\psi(\lambda\rho)\right), \quad (20)$$

where

$$\psi=0.5\sqrt{(1-n)^2-4(\nu n-1-\nu)}=$$
$$=0.5\sqrt{(1-n)^2+4\nu(1-n)+4}.$$

The results of the analytical construction of functions $D_1(\rho)$ and $R_1(\rho)$ through the targeted transformation of equation (1) by the three given methods indicate that this research has achieved its goal. So, in particular, the dependences for $D_1(\rho)$ were obtained, which have more than one independent constant in their structure, which increases the possibilities for optimal selection of variable thickness of the disk. At the same time, the solutions to the problem about the oscillations of the disk $R_1(\rho)$ can be obtained in a closed form. Moreover, due to additional independent constants, it becomes possible to represent $D_1(\rho)$ or $R_1(\rho)$ in elementary functions in cases where these expressions are expressed in special functions, for example, in Bessel functions.

5. 4. Applied implementation of theoretical results

The problem of radial vibrations of a disk, which is rigidly fixed on a shaft along its inner diameter, has been considered (Fig. 1). The relative dimensions of the disk are the outer radius ρ=1 and the fixing radius ρ_0. The thickness of the disk $H(\rho)$ is chosen based on the functions $D(\rho)$ found above according to the ratio $H=D^2/\rho$. In order to ensure reliability and simplify calculations from the calculation expressions obtained above for $D(\rho)$, $R(\rho)$, $W(\rho)$, only those expressed through elementary functions, Bessel functions of zero, integer or fractional-integer order were selected. Fig. 2 shows the normalized plots of functions $H(\rho)$, which are constructed when $\rho_0=0.2$ and $\eta=H(\rho_0)/H(\rho=1)=5$; $\nu=1/3$. To construct the $H(\rho)$ curves, relation (15) was used, and therefore the function $H=H_0\rho^{2\nu}(\rho^{\mu}+C\rho^{-\mu})^2$, where $H_0=C_1^2$; $C=C_2/C_1$; $\mu=(\nu^2+p(p-1)-3/4)^{1/2}$, is valid for the disk thickness. The solutions to equation (1) for these $H(\rho)$ according to expression (13) will take the form $R=(AJ_{p-0.5}(\lambda\rho)+BY_{p-0.5}(\lambda\rho))/(\rho^{\nu}(\rho^{\mu}+C\rho^{-\mu}))$. Depending on the values of the numerical parameter p, the value of μ changes, and the functions $H(\rho)$ and $R(\rho)$ also change along with it. In each of the individual cases, the constant C is selected from given initial conditions, for example, from the requirement to ensure the necessary ratio $\eta=H(\rho_0)/H(\rho=1)$. For example, with a value of η=5, the values of numerical and functional quantities are obtained (Fig. 2):

1) $p=1/2$; $\mu=i\sqrt{1-\nu^2}$; $C=-0.265$;

$$H_1=H_0\rho^{2\nu}\left(\sin\left(\sqrt{1-\nu^2}\ln\rho\right)+C\left(\cos\left(\sqrt{1-\nu^2}\ln\rho\right)\right)\right)^2;$$

$$R=\left(AJ_0(\lambda\rho)+BY_0(\lambda\rho)\right)/\sqrt{H}.$$

2) $p=0$ or $p=1$;

$$\mu=i\sqrt{3/4-\nu^2};\ C=-0.271;$$

$$H_2=H_0\rho^{2\nu}\begin{pmatrix}\sin\left(\sqrt{3/4-\nu^2}\ln\rho\right)+\\+C\left(\cos\left(\sqrt{3/4-\nu^2}\ln\rho\right)\right)\end{pmatrix}^2;$$

$$R=\left(A\sin(\lambda\rho)+B\cos(\lambda\rho)\right)/\sqrt{H}.$$

3) $p=3/2$;

$$\mu=\nu;$$

$$C=-1.532;$$

$$H_3=H_0\left(\rho^{2\nu}+C\right)^2;\ R=\left(AJ_1(\lambda\rho)+BY_1(\lambda\rho)\right)/\sqrt{H}.$$

4) $p=2$;

$$\mu=\sqrt{\nu^2+5/4};\ C=1.352;$$

$$H_4=H_0\rho^{2\nu}\left(\rho^{\mu}+C\rho^{-\mu}\right)^2;$$

$$C=-\left(\left(\rho_0^{\nu+\mu}-\sqrt{\eta}\right)/\left(\rho_0^{\nu-\mu}-\sqrt{\eta}\right)\right);$$

$$\mu=\sqrt{\nu^2+5/4};$$

$$R=\left(AJ_{3/2}(\lambda\rho)+BY_{3/2}(\lambda\rho)\right)/\sqrt{H}.$$

As a result, the following was found:

$$R=\frac{A(\sin\lambda\rho-\lambda\rho\cos\lambda\rho)+B(\cos\lambda\rho+\lambda\rho\sin\lambda\rho)}{\rho^{3/2+\nu+\sqrt{\nu^2+5/4}}+C\rho^{3/2+\nu-\sqrt{\nu^2+5/4}}}=$$
$$=A\left[\frac{(\sin\lambda\rho-\lambda\rho\cos\lambda\rho)+(B/A)(\cos\lambda\rho+\lambda\rho\sin\lambda\rho)}{\rho^{3/2+\nu+\sqrt{\nu^2+5/4}}+C\rho^{3/2+\nu-\sqrt{\nu^2+5/4}}}\right].\quad(21)$$

5) $H_5=H_0/\rho^3$; $R=\rho^{3/2}\left(AJ_2(\lambda\rho)+BY_2(\lambda\rho)\right)$.

From the given disk thicknesses $H(\rho)$ in the form of curves 1÷4, for which simple and accurate solutions $R(\rho)$ of the main equation are obtained, one can choose those that are necessary based on relevant technological considerations. Comparing the trajectory of these curves with a typical profile (Fig. 1), it was concluded that the most suitable for practical modeling or application are profiles based on curves 3, 4. It is noted that the thickness H_5, which corresponds to plot 5, is given for comparison, and that has proven unsuitability of such a profile for practical implementation. It is at $H=1/\rho^3$ and $\nu=1/4$ that a solution can be found through the Bessel functions of the whole order (J_2, Y_2), which is given in [14]. In that work, the authors tried to use the function $H=1/\rho^3$ and they found that due to its rapid decrease with increasing ρ, it cannot be the basis for practical implementation in calculations and design. That is why the authors decided to apply a rather cumbersome approximate method for determining the frequencies and forms of oscillations of a disk whose profile is similar to curves 3 or 4.

For the final solution to the disk oscillation problem, it is necessary that the solution $R(\rho)$ of equation (1) satisfies the boundary conditions that were given above, i.e.:

$$R'(\rho_0)=0;\ \left(R'+(\nu/\rho)R\right)_{\rho=1}=0.\quad(22)$$

In this case, according to (21) solution $R(\rho)$:

$$R=A\begin{pmatrix}(\sin\lambda\rho-\lambda\rho\cos\lambda\rho)+\\+\frac{B}{A}(\cos\lambda\rho+\lambda\rho\sin\lambda\rho)\end{pmatrix}/\left(\rho^{m+\mu}+C\rho^{m-\mu}\right),$$

where

$m=3/2+\nu$.

After entering R and R' into conditions (22), two equations with unknowns A and B are obtained

$$\left.\begin{array}{l}A\left[\sin(\lambda\rho_0)-\lambda\rho_0\cos(\lambda\rho_0)\right]+\\+B\left[\left(\cos(\lambda\rho_0)+\lambda\rho_0\sin(\lambda\rho_0)\right)\right]=0;\\A\left[(\lambda^2+\beta)\sin\lambda-\lambda\beta\cos\lambda\right]+\\+B\left[(\lambda^2+\beta)\cos\lambda+\lambda\beta\sin\lambda\right]=0,\end{array}\right\}\quad(23)$$

where

$$\beta=\nu-\frac{m+\mu+C(m-\mu)}{1+C}=$$
$$=\frac{C(\mu-3/2)-(\mu+3/2)}{C+1}.$$

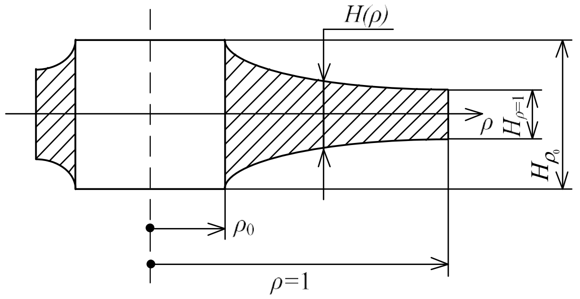

Fig. 1. Disk profile sketch

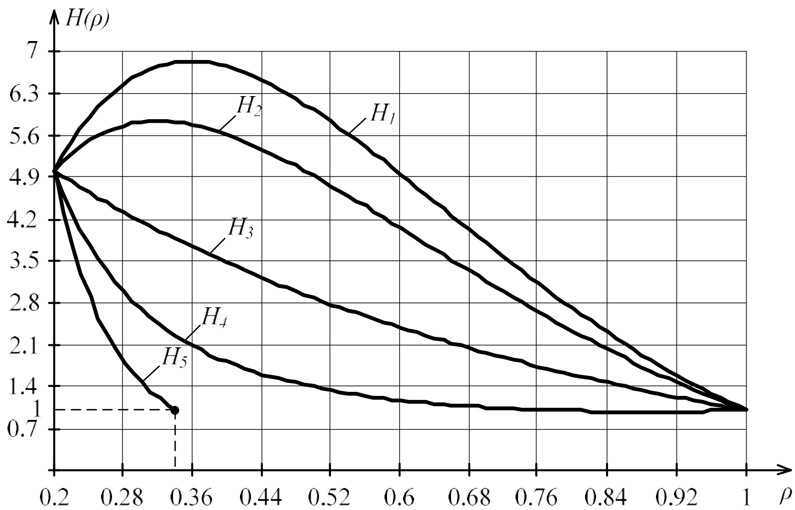

Fig. 2. Graphical interpretation of normalized plots of functions $H(\rho)$ at η=5

This system of equations will have a solution if its determinant is taken equal to zero. This requirement, which is expressed by the frequency equation, led to the following expression – an equation with an unknown frequency parameter λ:

$$\mathrm{tg}\left[\lambda\left(1-\rho_0\right)\right]= =\left[\lambda\left[\beta\left(1-\rho_0\right)-\lambda^2\rho_0\right]\right]/\left[\beta\left(1+\lambda^2\rho_0\right)+\lambda^2\right]. \qquad (24)$$

Given (ρ_0, ν, η), the coefficients C and β are determined, and then the unknown eigen(frequency) numbers λ_1, λ_2, ... can be found from equation (24). With known frequency numbers λ, one or the other of the equations of system (23) can be used to construct the functions of the eigenforms of oscillations. For example, from the second equation of system (23) we find:

$$B/A=-\begin{pmatrix}\left(\lambda^2+\beta\right)\sin\lambda- \\ -\lambda\beta\cos\lambda\end{pmatrix}/\begin{pmatrix}\left(\lambda^2+\beta\right)\cos\lambda+ \\ +\lambda\beta\sin\lambda\end{pmatrix}. \qquad (25)$$

After introducing the B/A ratio into the general solution $R(\rho)$, a function is obtained to construct its own form for the given frequency parameter λ. From the frequency equation (24), several first roots λ_i (i=1, 2, ...) were found for three different bending characteristics $\eta=H(\rho_0)/H(1)$ while maintaining other dimensions of the disk:

a) η=5; ρ_0=0.2; ν=1/3. After calculations, C=1.352 was determined; β=–1.325, hence λ_1=2.62419047; λ_2=6.4531778; λ_3=10.2228777;

b) η=3; ρ_0=0.2; ν=1/3. As a result of the calculation, the value C=0.785 was determined; β=–1.641. Hence the roots of the frequency equation (24) λ_1=2.4190727; λ_2=6.3935865; λ_3=10,1851395 were obtained;

c) η=1; ρ_0=0,2; ν=1/3. After calculations, C=0.322 was obtained; β=–2.098. Then the frequency parameters for this disk bend will be equal to λ_1=2.04143306; λ_2=6.306195; λ_3=10.1301076.

For comparison, the case for a disk of constant thickness is considered. When $H(\rho)$=const, the solution to equation (1) is found in the form [16] $R=AJ_1(\lambda\rho)+BY_1(\lambda\rho)= =A(J_1(\lambda\rho)+\gamma Y_1(\lambda\rho))$, where $\gamma=B/A$. From the boundary conditions (22), the frequency equation:

$$J_1\left(\lambda\rho_0\right)Y_0\left(\lambda\right)-Y_1\left(\lambda\rho_0\right)J_0\left(\lambda\right)- -\left(\left(1-\nu\right)/\lambda\right)\begin{bmatrix}J_1\left(\lambda\rho_0\right)Y_1\left(\lambda\right)- \\ -Y_1\left(\lambda\rho_0\right)J_1\left(\lambda\right)\end{bmatrix}=0$$

is obtained. When ρ_0=0,2; and ν=1/3, the roots of this equation will be λ_1=2.2249572; λ_2=6.100853; λ_3=9.9657101. The forms of the disk oscillations $R(\rho)$ are plotted at $\gamma=-J_1(0.2\lambda)/Y_1(0.2\lambda)$.

Comparisons of λ_i, which were found for three cases of disk profile flatness, indicate an increase in the values of the natural (resonant) frequencies (the frequency indicator is the number λ_i) with an increase in the curvature η. Using (25), the form of natural oscillations can be constructed and, if necessary, the magnitude and nature of the stress distribution in the disk during its radial movements can be estimated. The normal stresses in the disk during its radial vibrations are calculated according to the formulas from [14]:

$$\sigma_r=\frac{1}{a}\left(\left(E\right)/\left(1-\nu^2\right)\right)\left(R'+\left(\nu/\rho\right)R\right);$$

$$\sigma_\theta=\frac{1}{a}\left(\left(E\right)/\left(1-\nu^2\right)\right)\left(\nu R'+\left(1/\rho\right)R\right),$$

where σ_r are radial stresses directed along the disk radius; σ_θ – tangential stresses directed in the perpendicular direction; a is the disk radius. After introducing the eigenfunctions $R(\rho)$ and $R'(\rho)$ into the stress formulas, which have an arbitrary amplitude coefficient A (21), the corresponding stress distribution $\sigma_r(\rho)$ and $\sigma_\theta(\rho)$ were constructed. From these plots, the most vulnerable zones of disks during their resonances are determined. The expression $EA/a(1-\nu^2)$ is taken to be equal to one, and if A has the dimension of length, then this expression will have the dimension of the modulus of elasticity E.

Taking into account the fact that the resonance oscillations on the basic form at the frequency number λ_1 acquire a special practical significance during operational loads of turbine disks, for the sake of saving space, the oscillations and the stressed state of the disks are considered only at the basic frequencies. Fig. 3 shows the normalized first forms of disk oscillations corresponding to the characteristics of η=5; 3; 1 and also shows the basic form of oscillations of a disk of constant thickness. Amplitude coefficients A are chosen according to η with values of A=0.853; 0.729; 0.739; 1,805. Fig. 4, 5 show, respectively, plots of stresses $\sigma_r(\rho)$ and $\sigma_\theta(\rho)$ in disks during resonant radial vibrations at the main frequencies, that is, at λ_1. During oscillations on the basic form, the main tension-compression stresses $\sigma_1=\sigma_r$ and $\sigma_2=\sigma_\theta$ acted in phase at each point (infinitesimal element) of the disk, except for the free edge at $\rho=1$. Strength assessment for this type of resonant oscillations, when biaxial tension-compression occurred, can be performed on the basis of appropriate strength hypotheses.

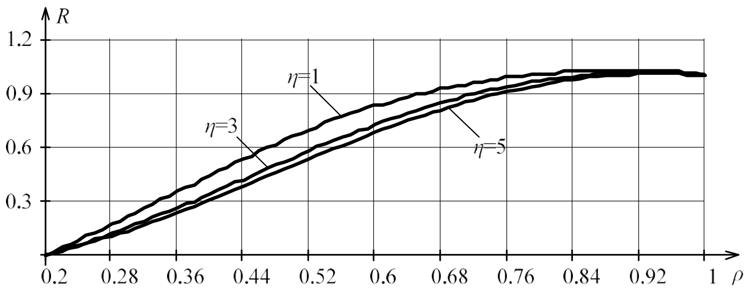

Fig. 3. The first form of natural oscillations of the disk $R(\rho)$ with different types of disk bending η

Owing to these hypotheses, the strength under complex (plane or volumetric) stress can be estimated, based only on the strength data under simple uniaxial loading. The form of implementation of the mentioned hypotheses in calculation practice is determined by the corresponding dependences between the main stresses σ_1 and the equivalent stress σ_{eq}, which to one degree or another should be equal to the stress σ_0 under uniaxial loading. For disks oscillating under the resonance mode, the limit stress is assumed to depend on:

$$\sigma_{eq}=\sigma_{-1}=\left(\sigma_1^2+\sigma_2^2-\sigma_1\sigma_2\right)/\left(\sigma_1-\nu\sigma_2\right), \quad (26)$$

where σ_{-1} is the endurance limit for uniform tension-compression.

This dependence was used in the analysis of the results of fatigue tests on round plates made of a number of metallic materials under bending vibrations.

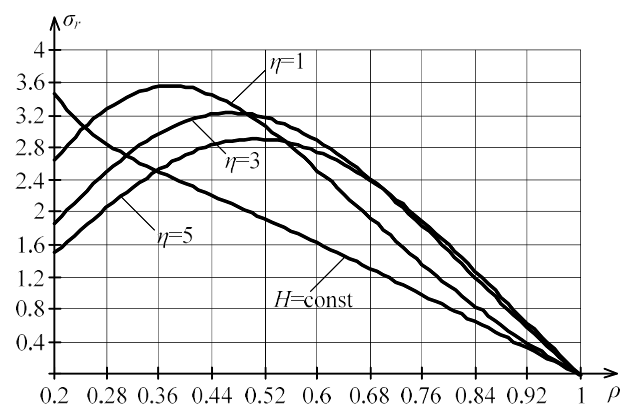

Fig. 4. Radial stresses σ_r for different values of disk curvature η and for the disk at H=const

The results showed a fairly satisfactory agreement between the experimental and calculated data. In addition, in one paper [17], one of the satisfactory tests of relation (26) for a titanium alloy with a titanium nitride coating is given. If $(\sigma_r, \sigma_\theta)$ is entered instead of (σ_1, σ_2) in (26), bearing in mind that $\sigma_1\geq\sigma_2$, then with $\sigma_r=R'+(\nu/\rho)R$; $\sigma_\theta=\nu R'+(R/\rho)$, the dependence will be obtained:

$$\sigma_{eq}=\begin{pmatrix}\left(\nu^2-\nu+1\right)\left(R'+\left(\left(R^2\right)/\left(\rho^2\right)\right)\right)+\\+\left(4\nu-\nu^2-1\right)\cdot\left(\left(RR'\right)/\left(\rho\right)\right)\end{pmatrix}/\Delta,$$

where

$$\Delta=\begin{cases}\left(1-\nu^2\right)R' \text{ when } \rho=\rho_0\div\rho_1;\\ \left(1-\nu^2\right)\dfrac{R}{\rho} \text{ when } \rho=\rho_1\div 1;\end{cases}$$

ρ_1 is the coordinate at which $\sigma_r=\sigma_\theta$, which corresponds to the point of intersection of the curves σ_r and σ_θ according to Fig. 4, 5.

The values of ρ_1 depending on η are as follows: ρ_1=0.7338; 0.6977; 0.6084, if η=5; 3; 1, respectively. ρ_1=0.5491 at H=const was also found. The σ_{eq} curves are shown in Fig. 6. It can be seen that dangerous from the point of view of cyclic (fatigue) strength maximum stresses σ_{eq} for the cases η=5; 3; 1 are distributed at some distance ρ_{max} from the fastening at ρ_0=0.2. It is confirmed that the increase in the thickness of the disk in the zone of its fixation achieved its goal compared to the case of H=const, in which the σ_{eq} maximum coincided with the place of fixation. Radial coordinates ρ_{max}, where σ_{equmax}, namely ρ_{max}=0.5243; 0.4862; 0.3936 for η=5; 3; 1, can find practical value. Based on these data, the location (at ρ_{max}) of possible disk failure due to fatigue can be determined. A further comparison of σ_{eqmax} for different values of η allowed us to conclude that the probability of the threat of fatigue failure of disks with different η is higher where the stresses of σ_{eqmax} are the greatest. It was found that at η=5; 3; 1 σ_{eqmax}=2.36631; 2.86093; 3.16445. As can be seen, an increase in the thickness at the attachment point ρ_0 can lead to a decrease in the value of σ_{eqmax}. If the stress ratio $m=\sigma_2/\sigma_1$ in the dangerous zone at $\rho=\rho_0$ is known, then the limit values of σ_1 or σ_2 can be determined at the place of probable destruction, if the limit value of $\sigma_{eq}=\sigma_1$ is determined. We calculated or found from Fig. 4, 5, for example, the values of m depending on η=5; 3; 1, respectively, m=0.807; 0.809; 0.802. From the criterion dependence (26) one can obtain for the critical values σ_r:

$$\sigma_1=\sigma_{-1}\left(1-\nu m\right)/\left(\sqrt{1+m^2-m}\right).$$

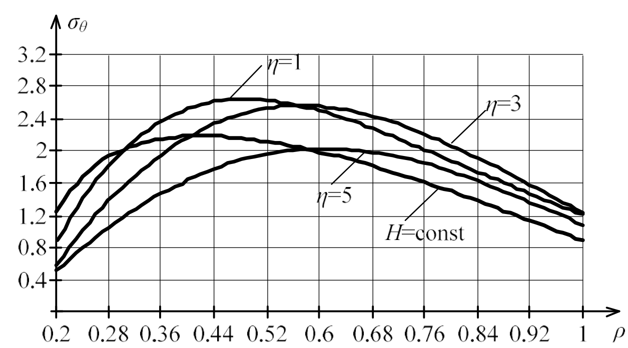

Fig. 5. Tangential stresses σ_θ for different values of disk curvature η and for the disk at H=const

After entering here the values of m for η=5; 3; 1, the limit values of dangerous stresses in disks of this configuration can be obtained $\sigma_{1\eta=5}=\sigma_{-1}(0.795)=79.5\,\%\,\sigma_{-1}$;

$\sigma_{1\eta=3}=\sigma_{-1}(0.7942)=79.42\ \%\ \sigma_{-1}$; $\sigma_{1\eta=1}=\sigma_{-1}(0.7987)=79.87\ \%\ \sigma_{-1}$. These data showed that the limit values of principal stresses σ_1 are smaller compared to the limits of endurance σ_{-1}. Endurance limits are either known for the disk material, or they can be determined experimentally during cyclic stretching-compression on rod samples by known methods.

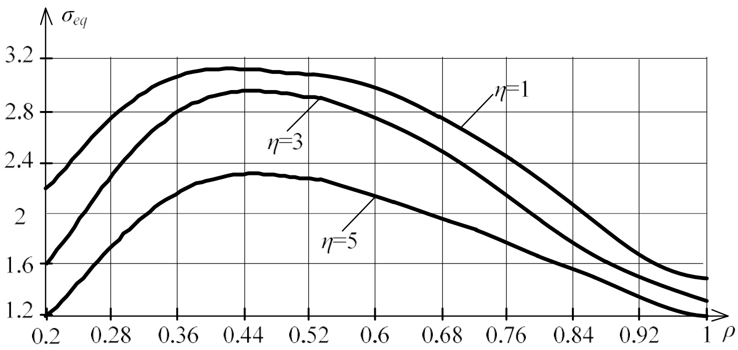

Fig. 6. Stress distribution σ_{eq} for different values of disk curvature η

It follows from this that when designing such disks, the margin of cyclic strength must be determined based not on σ_{-1} but on the values of σ_1, which are smaller than σ_{-1}.

6. Discussion of results of investigating the problem of radial vibrations of the disk

For the second-order differential equation (1), known solutions were found only for H=const and for $H=1/\rho^n$ (H is the thickness of the disk; n is an integer). These solutions are expressed in terms of Bessel functions. However, for variable thickness $H=1/\rho^n$, the disk configuration turned out to be quite steep, which cannot be of practical importance. Fig. 2 showed that the thickness of this type, namely $H=1\ \rho^3$, is unsuitable for practical implementation in the interval $\rho=(0.2...1)$.

The factorization method has made it possible to obtain the solution (8) in elementary functions with $D(\rho)=(H(\rho)\cdot\rho)^{1/2}$ (9), the structure of which included two independent constants. By changing the values of these constants, the desired disk configuration can be achieved.

Results (14) and (15), which were obtained for functions $D(\rho)$ based on the derivative extraction method, turned out to be more informative compared to result (9). Thus, in result (14), three independent constants are introduced for $D(\rho)$, and in result (15) – four. This result can be used for more flexible control over both the thickness of the disk and its movements during oscillations.

Using the method of symmetry, equation (1) with parameters $H(\rho)$ and $R(\rho)$ was transformed into a new equation with parameters $H_1(\rho)$ and $R_1(\rho)$, which does not differ in form from equation (1). The method of extracting the derivative and the method of symmetries in this case led to the same results. It is easy to make sure of this if in the corresponding expressions for $D(\rho)$ and $R(\rho)$ the coefficients (p, μ) are mutually replaced by (ψ, α), simultaneously setting $\psi=p-(1/2)$; $\mu=\alpha$. The equivalence of the methods, by the way, is additionally confirmed by their reliability. The factorization method turned out to be a special case of two more general methods.

From a number of possible profiles (Fig. 2), curves 3 and 4 were chosen for practical implementation because they most correspond to the typical profile of the disk from Fig. 1. Due to the selected configurations and geometric parameters, the calculations were easily performed on the basis of the theoretical solutions obtained above.

According to the boundary conditions, the natural frequencies, shapes, and stresses for the disks of the selected configurations were found (Fig. 3–5). The comparisons of λ_i, which were found for three cases of disk profile flatness, testified to an increase in the values of natural frequencies (the frequency indicator is the number of λ_i) with an increase in the curvature $\eta=H(\rho_0)/H(\rho=1)$.

In contrast to the limited result [14] associated with the law $H=1/\rho^n$, the number of cases of the exact solution to the given problem is expanded in the current work. This is due to the flexible choice of thickness configurations and solutions in the form of trigonometric functions. The results have made it relatively easy not only to determine the frequency indicators of the oscillating system but also to estimate the stressed state of the disks. This became possible owing to the obtained analytical relations, which had a final form convenient for practical use.

Our solutions partially solved the problem of analytical closed-loop solution to the problem of radial vibrations of a disk of variable thickness. Due to this, the possibilities for choosing the necessary structural configurations of disks with predetermined estimated frequency characteristics and strength characteristics have been expanded. Through the expedient variation of arbitrary coefficients that were included in the structure of the disk thickness function, the possibilities of obtaining new profiles have been expanded. At the same time, the analytical solution to the problem remained closed.

An obvious limitation in the practical application of the obtained theoretical results was the direct connection of the exact analytical solution to the problem with a well-defined disk configuration.

Some inconvenience in the practical application of the obtained theoretical results may be cases when the calculated relations for $D(\rho)$ and $R(\rho)$ were expressed in terms of Bessel functions with an arbitrary index, which, as a rule, are not tabulated. In these cases, the calculation is necessary and can probably be performed after representing these functions in the form of their series. The stress analysis could not be considered rigorous because it had to be based on one of the numerous strength hypotheses. It can be mentioned that in cases where the results of previous experiments were not available, energy criteria of the Mises type were used, as a rule. Such an approach can be considered forced, despite its lack of reliability. It should be noted that in any case, the practical implementation of the criteria (strength hypothesis) for the operational control over disks may be associated with technical difficulties. These difficulties are due to the possible lack of means and methods of experimental control and registration of resonant radial vibrations and their frequencies.

A partial disadvantage of the proposed disk thickness function $H(\rho)$ is that at $\rho=0$, that is, in the center of the disk, the thickness is undefined (tends to infinity). And therefore, as in the case of the known function $1/\rho^n$ (n is an integer), the proposed options do not always allow considering a solid disk, but only with a hole in the center.

Further development of our study may be associated with an increase in significant independent constants in the function that determines the thickness of the disk. This will allow expanding the set of new disk profiles, while maintaining the exact analytical solution. Variants with increasing constants are given in results (14) and (15). It is in the direction of practical analysis of these solutions that further research of the stated problem may lie.

7. Conclusions

1. The differential equation of the second order with variable coefficients describing the oscillations of disks of variable thickness was solved by the factorization method. This made it possible to find disk configurations for which the solutions have a closed form. The factorization method made it possible to obtain a solution to the problem in elementary functions for the thickness function, which includes two independent variable constants.

2. By extracting the derivative, the original differential equation is transformed into a form that has known solutions. At the same time, the functions for the thickness for which these solutions are valid contain arbitrary constants in their structure, varying with which the disk profiles change. The results for $D(\rho)=(H(\rho)\cdot\rho)^{1/2}$ can have three or four independent constants. The final solutions also cover the results obtained by the factorization method.

3. The dependences established for $D_1(\rho)=(H_1(\rho)\cdot\rho)^{1/2}$ have more than one independent constant in their composition, which increases the possibilities for optimal selection of disk configuration. At the same time, the solutions to the problem are obtained in a closed form. Moreover, due to additional independent constants, it is possible to represent $D_1(\rho)$ in elementary functions in cases where these expressions were obtained in special functions.

4. The solutions to the corresponding equations for a number of disks of variable thickness $H(\rho)$ and for H=const are presented. From the profiles of disks of variable thickness, those were selected for which the characteristic ratio $\eta=H(\rho_0)/H(\rho=1)$ has a value of 1, 3.5, where ρ_0 is the inner radius of fastening; $\rho=1$ is the outer relative radius. For disks, which are chosen as an example of the practical implementation of new methods, the solution to equations in elementary functions is also provided. Four problems about the radial vibrations of disks at η=1, 3, 5 and H=const, which are rigidly fixed on a shaft with a radius of ρ_0 and free at $\rho=1$, have been solved to the end. At the same time, the frequency equations were built, the natural frequencies of oscillations λ_i were calculated (for serial numbers i=1, 2, 3), the corresponding expressions for the forms of oscillations and their graphic images are given. The comparison of λ_i found for the three cases of flatness of the disk profile η=1, 3, 5 indicates an increase in the values of the natural (resonant) frequencies (the frequency indicator is the number of λ_i) with an increase in the curvature η.

The relationship for calculating the normal radial σ_r and tangential σ_θ stresses is derived. To illustrate the nature of the distribution of these stresses in the disks of this profile, the corresponding dependences $\sigma_r(\rho)$ and $\sigma_\theta(\rho)$ were constructed for the basic shapes. An assessment of the strength of the disks during resonant radial vibrations was carried out. In this case, the most destructive principal stress $\sigma_1=\sigma_r$ should be selected from the ratio $\sigma_r\approx 0.79[\sigma_{-1}]$, $[\sigma_{-1}]$ is the endurance limit of the disk material under uniaxial tension-compression.

Conflicts of interest

The authors declare that they have no conflicts of interest in relation to the current study, including financial, personal, authorship, or any other, that could affect the study and the results reported in this paper.

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Data availability

All data are available in the main text of the manuscript.

Use of artificial intelligence

The authors confirm that they did not use artificial intelligence technologies when creating the current work.

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