In the modern world, where efficiency, stability, and precision play a crucial role, the development and application of optimal control strategies in oscillatory systems hold significant importance. The issues related to the numerical solution of control problems associated with damping oscillatory systems consisting of two objects are considered. To numerically solve the discussed problem, the gradient projection method, based on the formula for the first variation of the functional, and the method of successive approximations, associated with the linearity of boundary problems describing oscillatory processes, are applied. The oscillations of one object are described by a wave equation with first-order boundary conditions, while a second-order ordinary differential equation models the oscillations of the other object. Furthermore, the original and the adjoint boundary value problems are solved using direct methods at each iteration step. An algorithm for the numerical solution of the problem is proposed, and based on this algorithm, a software code for implementation is developed. The numerical results obtained in the study demonstrate that there is convergence in terms of functionality, and the approximately optimal controls found in this process are minimizing sequences in the control space. The mechanism of controlling and regulating the operation of the system according to its input constraints is provided by observed feedback, allowing systems with limited excitation to maintain stability and optimal functioning in conditions of changing external or internal circumstances. The obtained results can also be used to forecast the system's behavior in the future, resource planning, prevention of emergencies, or optimization of production processes

Keywords: system oscillations, control problem, method of straight lines, functional convergence

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FINDING AND IMPLEMENTING THE NUMERICAL SOLUTION OF AN OPTIMAL CONTROL PROBLEM FOR OSCILLATIONS IN A COUPLED OBJECTS SYSTEM

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1. Introduction

In the existing control theory, mathematical formalism largely suppresses the physical content of the control problem. In this regard, the fundamental problem of searching for general objective laws of control arises, which comes down to maximum consideration of the natural properties of an object of the corresponding physical nature.

Oscillatory processes play an extremely important role in modern physical, chemical, biological, technical, and economic sciences. Suffice it to mention that classical and celestial mechanics are, first of all, the sciences of vibrations. If in natural processes oscillations are a reflection of the corresponding natural patterns of interaction between parts of the general system, then in technology oscillatory phenomena can act both as the main operating modes and turn out to be undesirable and contrary to the normal flow of the technological process.

The optimization of oscillatory processes holds significant relevance across various fields, encompassing endeavors such as stabilizing ship motions, managing crane booms, and controlling gas flows in extensive pipelines or power lines [1–4]. Notably, recent studies [5–8] have broached the challenges of damping oscillations within systems described by a combination of wave equations and second-order ordinary differential equations. Specifically, [5] addresses scenarios where a fixed boundary on the left and an object with lumped parameters on the right impact the distributed system. Conversely, [6] tackles cases where control functions and objects with lumped parameters influence the system's ends, interlinked through the boundary conditions of the wave equation. Utilizing the d'Alembert formula, these studies conceptualize wave equation solutions as superpositions of "forward" and "reverse" waves. Moreover, [7] explores boundary control problems involving oscillations of objects with both distributed and lumped parameters, showcasing a diverse range of boundary conditions. Meanwhile, [8] delves into damping oscillations within systems comprising serially connected objects with distributed parameters, with free system boundaries and attachment points for objects with lumped parameters. While akin systems have been studied since the mid-20th century, their practical implementation remains challenged by computational complexities in solving boundary value problems and optimizing control variables [9–11].

Therefore, the problem of controlling nonlinear oscillatory objects and dynamic processes is of significant practical importance. The relevance of research in this direction is justified by the need to find and implement numerical solutions to the optimal control problem of oscillations in the system of objects. Additionally, there is the possibility of observing feedback phenomena in the investigated systems, enabling the response to changes in external conditions or internal parameters and adjusting behavior according to specified constraints.

2. Literature review and problem statement

The utilization of mathematical modeling in solving optimal control problems of oscillatory processes finds ever-expanding applications across various scientific and engineering domains. For instance, tasks related to oscillations and their management may encompass integral efficiency criteria. In [12], such instances are examined, particularly focusing on signal concentration at breakpoints. To enhance precision in defining efficiency and rational utilization of numerical methods, functional analysis techniques are employed, including anisotropic parametrized measures, within the context of semidefinite programming. The proposed resolution approach advocates for leveraging convex optimization methods, such as semidefinite relaxations. Despite the demonstrated advantages of advanced optimal controllers in research and demonstration examples, their practical implementation remains limited. One of the primary reasons for this limitation is that these novel control algorithms are still evaluated individually, hindering the identification of best practices for the widespread adoption of optimal control strategies. Optimal control enables efficient management of motion and vibrations in systems to enhance accuracy, efficiency, and safety. The study [13] aims to address the issue of weakly or negatively damped low-frequency oscillations caused by the cross-transmission of electric power from large wind farms. The study proposes the use of a fast terminal additional damping regulator with a sliding mode. To achieve this, a rotor magnetic circuit controller is developed, which takes into account the relationship between the applied voltage and the rotor magnetic circuit. A simulation model of the system is created in MATLAB/Simulink for autonomous modeling, and a real-time experiment is conducted on the modeling of inter-zonal transmission of a large wind farm based on a digital system. However, this work does not consider all factors influencing low-frequency oscillations in wind farms, such as changes in environmental conditions or the impact of external disturbances. Oscillations can affect the operation of electronic devices, integrated circuits, electrical circuits, and control systems. Optimal management helps minimize the impact of noise, reduce energy consumption, and increase operational reliability. In [14], a gas foil bearing with active gas (AGFB active gas foil bearing) is presented as an author's configuration. The interaction of the gas with the considered structure is modeled using the Reynolds equation, and a dynamic system with a unique source of nonlinearity is created, where linear feedback control using polynomial functions is applied to address the stability issue. In our view, the paper lacks a sufficiently detailed description of the methodology and modeling or control processes, which may hinder the replication and reproduction of the research by other scientists or engineers. In the construction and design of buildings, bridges, dams, and other infrastructure, it is essential to steer vibrations and oscillations. Optimization of control can contribute to the longevity and safety of such structures. The paper [15] examines how optimal controllers can enhance the energy efficiency of buildings by considering forecasts and uncertainties such as weather conditions and room occupancy intensity. This allows for energy savings through more efficient utilization of energy systems within buildings. The paper investigates various controller parameters, including control step, prediction horizon, state-action space, learning algorithm, and network architecture for the value function. The optimal controllers proposed in the paper are evaluated individually, which hinders the identification of optimal methods for widespread use in the construction industry. Additionally, the limited scope of application of these methods in artificially created environments restricts their practical applicability. In energy systems, such as generators, turbines, and power plants, optimal management can improve resource utilization efficiency, reduce equipment wear and tear, and minimize environmental impact. In the field of building construction, there is a growing need for optimizing energy systems, which becomes a key task in the creation of modern buildings and industrial facilities. In this context, energy supply systems are actively evolving to create more intelligent, efficient, and reliable energy supply networks. Modern energy generation and distribution systems, such as microgrid management, face challenges due to the presence of both linear and nonlinear loads, which can lead to distortion of voltage and current waveforms. Essentially, heating and cooling systems in industrial and residential buildings can be considered as sources of harmonic distortions. The paper [16] presents a new concept based on geometric algebra (GA) aimed at determining the power of multivector distortions represented in the form of a bivector. Both bivectors and their relationship with the phase angles of distorted voltage are analyzed. However, the paper does not describe the drawbacks or limitations of the proposed concept, nor does it conduct a comprehensive analysis of its applicability in various scenarios. Additionally, while numerical examples are provided, they may not be sufficiently diverse or detailed to fully understand the effectiveness and scope of applicability of the concept. Optimal control of oscillations plays a pivotal role in stabilizing and managing the aerodynamic characteristics of aircraft and spacecraft.

In [17], a novel approach to aerodynamic feedback control using machine learning is presented, illustrated by a coupled oscillatory system with non-stationary feedback. The obtained results demonstrate that this method can effectively respond to changing conditions and provide optimal control in dynamic scenarios. This automated and versatile feedback control approach opens up new prospects for efficiently managing various tasks related to fluid flow. However, the research findings may only be applicable to a specific type of oscillatory systems with non-stationary feedback. It is possible that the obtained results require further validation on a broader set of test data or under different operating conditions to ensure their generalizability. This limits the applicability of the method in other contexts. In medical technology, optimal control can be applied to manage medical devices like insulin pumps, artificial hearts, and other medical equipment to ensure perfect functionality. In [18], an approach to metabolic control is proposed, based on inducing temporal oscillations in the levels of specific enzymes. This method represents an alternative strategy for enhancing the production of desired metabolites. The objective of the study is to maximize the overall metabolite production by utilizing temporal changes in enzyme concentrations. The results demonstrate that the use of temporal oscillations can significantly improve metabolic processes in experimentally feasible synthetic schemes. However, achieving optimal results requires an accurate kinetic model of central carbon metabolism, which may be challenging for some systems. Considering all these aspects, optimal control problems in oscillatory processes remain relevant and play a significant role in enhancing the efficiency, reliability, and stability of various technologies and systems.

All of this suggests that it is advisable to conduct research to obtain a numerical solution to the problem of optimal control of oscillations in a system of interconnected objects, the processes of which are described by a set of partial and ordinary differential equations.

3. The aim and objectives of the study

The study aims to discover and apply the numerical solution of an optimal control problem for oscillations in a coupled objects system. This will make it possible to determine optimal control actions on the object.

To achieve this aim, the following objectives are accomplished:

– to calculate the gradient of the Lagrange function and replace them with their approximating analogues;

– to present a numerical solution to the problem of controlling oscillatory processes in a system of two objects;

– to develop a numerical solution algorithm, implement it in the form of program code and obtain specific numerical results.

4. Materials and methods

The research focuses on oscillatory systems consisting of two objects. The oscillations of one object are described by a wave equation with first-order boundary conditions, while the oscillations of the other object are described by a second-order ordinary differential equation. To obtain a numerical solution to the problem, the mathematical apparatus of the Pontryagin maximum principle was used. The main assumption put forward in the study is that the developed solution has the potential for practical application. The primary assumption is the solvability of the boundary value problem describing the oscillatory process in the system of interconnected objects. The simplifications adopted in the study include reducing the control problem, described by a set of partial and ordinary differential equations, to a variational problem.

One of the objects is described by a wave equation with boundary conditions of the first kind, while a second-order ordinary differential equation characterizes the other object. To make the proposed system more realistic and applicable to systems with limited excitations, a term that characterizes an object with distributed parameters was introduced.

Consider the oscillation of the system, described by the boundary value problem [9]:

$$u_{tt}(x,t)=u_{xx}(x,t),\ \ 0<x<1,\ \ 0<t\le T, \tag{1}$$

$$u(x,0)=a(x),\ \ u_x(x,0)=b(x),\ \ 0\le x\le 1, \tag{2}$$

$$u(0,t)=p(t),\ \ u(1,t)=y(t),\ 0<t\le T, \tag{3}$$

$$\dot{y}(t)+y(t)=u_x(1,t),\ \ 0<t\le T, \tag{4}$$

$$y(0)=y^0,\ \ \dot{y}(0)=y^1, \tag{5}$$

where $a(x)\in C^2[0,\ 1]$ is an initial state of an object with distributed parameters, $b(x)\in C^1[0,\ 1]$ is an initial speed distribution of the displacement. External force $p(t)\in C^2\ [0,\ T]$, which is chosen as controls, satisfies the matching conditions for the initial and boundary conditions:

$$\begin{cases} p(0)=a(0), \\ p'(0)=b(0), \\ p''(0)=a''(0). \end{cases} \tag{6}$$

External force $p(t)$ and the lumped object affect, respectively, the left and right ends of the distributed object.

It is required without going beyond the specified range $0\le p_{\min}\le p(t)\le p_{\max}$, where $p_{\min}$ and $p_{\max}$ are values given on the basis of technological considerations, to control the external force $p(t)$ so that at the moment of time $t=T$ the system is in a state that differs little from its state of rest. The functional is:

$$F=\int_0^1\left[u^2(x,T)+u_t^2(x,T)\right]\mathrm{d}x, \tag{7}$$

that is, we are dealing with the problem of the best calming of coupled systems by the time T. In particular, if the minimum of the functional (7) is equal to zero, then we can talk about the possibility of complete damping of the oscillations of coupled systems. The control $p(t)$ that satisfies the above conditions will be called an admissible control, and the boundary value problem (1)–(5) will be called a direct boundary value problem [15].

Note that, as a special case of problem (1)–(7), we can formulate the control problem for string vibrations under boundary conditions of the first kind, i.e. process control problems, which are described by the boundary value problem (1)–(3), where $y(t)$ acts as a control function, and $p(t)$ is a fixed function or equal to zero (in the case when the left end of the string is fixed). Problems of this type were considered in [6].

If we exclude from (3)–(5) the variable $y(t)$, then, taking into account the conditions $u(1,0)=y^0$, $u_t(1,0)=y^1$ it was proved in [9] that under conditions (6) and $a''(1)-a'(1)+a(1)=0$ every function $p(t)$ uniquely defines a unique solution $u(x,t)$ of the boundary value problem (1), (2) with the conditions:

$$u(0,t)=p(t),\ \ u_{tt}(1,t)+u(1,t)=u_x(1,t),$$

$$u(1,t)=y^0,\ \ u_t(1,0)=y^1.$$

Using the d'Alembert formula, a solution to the direct boundary value problem is constructed and, under certain additional conditions, an expression is obtained that allows us to determine the control variable. Note that when considering systems with a boundary effect through an object with lumped parameters, conditions (3)–(5) should be replaced by the conditions:

$$u(0,t)=0,\ u(1,t)=y(t),\ 0<t\le T, \tag{8}$$

$$\ddot{y}(t)+y(t)=p(t)+u_x(1,t),\ \ 0<t\le T, \tag{9}$$

$$y(0)=y^0,\ \ \dot{y}(0)=y^1. \tag{10}$$

In this case, the state of the lumped system is represented as:

$$y(t)=y^0\cos(t)+y^1\sin(t)+$$
$$+\int_0^t\left(p(\tau)+u_x(1,\tau)\right)\sin(t-\tau)\mathrm{d}\tau,$$

where $u(x, t)$ describes the states of a distributed system, $p(t)$ is a control function.

For the solution of the above problem, the gradient projection method is used. It is built through the first variations and based on the fact that the optimal control problem can be formulated as a variational problem where it is required to minimize the functional depending on the control and the state of the system. Application of the gradient projection method includes calculating the gradient of the functional concerning the control and then optimizing the control in the direction of the anti-gradient. This method allows us to come to the optimal solution gradually, step by step, iteratively finding a more optimal control at each step.

Further, the method of successive approximations applies in cases where the boundary value problem describing the dynamics of the system is linear. This simplifies the process of finding the optimal control.

5. Results of the study on finding a numerical solution to the problem of optimal control of oscillations in a system of coupled objects

5. 1. Calculation of the gradient of the Lagrange function

Let us compose the Lagrange function of problem (1)–(7):

$$L = F + \int_0^1\int_0^T \psi(x,t)[u_{xx} - u_{tt}]\mathrm{d}x\mathrm{d}t + \int_0^T z(t)\left[u_x(1,t) - \ddot{y}(t) - y(t)\right]\mathrm{d}t, \quad (11)$$

where $\psi(x, t)$, $z(t)$ are Lagrange multipliers. The extrema of the functional L and F coincide if the constraint equations are satisfied.

We calculate the first variation of L. The variation of the Lagrange function (11) has the form:

$$\delta L = \int_0^1 2u(x,T)\delta u(x,T)\mathrm{d}x + \int_0^1 2u_t(x,T)\delta u_t(x,T)\mathrm{d}x + \int_0^1\int_0^T \psi(x,t)[\delta u_{xx} - \delta u_{tt}]\mathrm{d}x\mathrm{d}t + \int_0^T z(t)\left[\delta u_x(1,t) - \delta\ddot{y}(t) - \delta y(t)\right]\mathrm{d}t.$$

We transform the double integral using integration by parts, taking into account the following conditions:

$$\begin{cases} \delta u(x,0) = 0, \\ \delta u_t(x,0) = 0, \\ \delta u(0,t) = \delta p(t), \\ \delta u(1,t) = \delta y(t), \\ \delta y(0) = 0, \\ \delta\dot{y}(0) = 0. \end{cases} \quad (12)$$

After simple transformations, we get:

$$\begin{aligned} \delta L = &\int_0^1\int_0^T \left[\psi_{xx}(x,t) - \psi_{tt}(x,t)\right]\delta u(x,t)\mathrm{d}x\mathrm{d}t + \\ &+\int_0^1 \left[\psi_t(x,T) + 2u((x,T)\right]\delta u(x,T)\mathrm{d}x + \\ &+\int_0^1 \left[-\psi(x,T) + 2u_t(x,T)\right]\delta u_t(x,T)\mathrm{d}x - \\ &-\int_0^T \left[\ddot{z}(t) + z(t) + \psi_x(1,t)\right]\delta y(t)\mathrm{d}t + \\ &+\int_0^T \psi_x(0,t)\delta p(t)\mathrm{d}t - \int_0^T \psi(0,t))\delta u_x(0,t)\mathrm{d}t + \\ &+\int_0^T \left[\psi(1,t) + z(t)\right]\delta u_x(1,t)\mathrm{d}t - \\ &-z(T)\dot{\delta y}(T) + \dot{z}(T)\delta\dot{y}(T), \end{aligned} \quad (13)$$

further, using the condition of stationarity of the Lagrange functions and the randomness in the choice of variations of the phase variables, we equate the coefficients of the corresponding variations to zero, that is considering:

$$\psi_{tt}(x,t) = \psi_{xx}(x,t), \; 0 < x < 1, \; 0 \le t < T, \quad (14)$$

$$\psi(x,T) = 2u_t(x,T), \; \psi_t(x,T) = -2u(x,T), \; 0 \le x \le 1, \quad (15)$$

$$\psi(0,t) = 0, \; \psi(1,t) = -z(t), \; 0 < t \le T, \quad (16)$$

$$\ddot{z}(t) + z(t) = -\psi_x(1,t), \; 0 \le t < T, \quad (17)$$

$$z(T) = 0, \; \dot{z}(T) = 0, \quad (18)$$

from (13) we have:

$$\delta L = \int_0^T \psi_x(0,t)\delta p(t)\mathrm{d}t. \quad (19)$$

It follows that for any $p(t)$ the gradient of the functional is $\psi_x(0,t)$.

So, to calculate the gradient of functional (7) with a fixed control $p(t)$ it is necessary, first of all, to integrate two boundary value problems with the corresponding boundary conditions, that is, initially from the direct problem (1)–(5) we define the functions $u(x, t)$, $y(t)$, then put the resulting $u(x, t)$, $y(t)$ into the adjoint problem (14)–(18) and find $\psi(x, t)$, $z(t)$, and finally calculate $\psi_x(0, t)$.

For the numerical solution of these boundary value problems, in practice, an implicit difference scheme is commonly used in combination with a sweep or the method of straight lines. However, in some cases, it is also possible to use well-known analytical solutions. But at the same time, the question of the convergence of this method for the problems under consideration is far from being investigated. In addition, the question of constructing a difference or differential-difference scheme is rather difficult, which, in addition to accuracy and efficiency, must first of all satisfy the requirement of stability.

Functional (7) and its gradient $\psi_x(0, t)$ are replaced by their approximating counterparts.

5. 2. Numerical solution of the problem of controlling oscillatory processes in a system of two objects

Let us find a solution $y(t)$ to problem (4), (5), which has the following form:

$$y(t)=y^0\cos(t)+y^1\sin(t)+\int_0^t u_x(1,\tau)\sin(t-\tau)d\tau. \quad (20)$$

Taking into account the form of solution (20) and the conditions $u(1, t)=y(t)$, write out the boundary condition that must be satisfied by $u(x, t)$ at the point $x=1$:

$$u(1,t)=y^0\cos(t)+y^1\sin(t)+\int_0^t u_x(1,\tau)\sin(t-\tau)d\tau. \quad (21)$$

If, however, from conditions (4), (5) we exclude $y(t)$, then they can be represented in the form:

$$u(0,t)=p(t),\ u_{tt}(1,t)+u(1,t)=u_x(1,t), \quad (22)$$

$$u(1,0)=y^0,\ u_t(1,0)=y^1. \quad (23)$$

We denote $u_t(x,t)=v(x,t)$ and $\psi_x(x,t)=\varphi(x,t)$. Then, excluding $y(t)$ from (4), (5), the boundary value problem (1)–(5) and functional (7) can be written in the following form:

$$u_t(x,t)=\upsilon(x,t), \quad (24)$$

$$\upsilon_t(x,t)=u_{xx}(x,t),\ 0<x<1,\ \ 0<t\le T, \quad (25)$$

$$u(x,0)=a(x),\ \ \upsilon(x,0)=b(x),\ \ 0\le x\le 1, \quad (26)$$

$$u(0,t)=p(t),\ \ 0<t\le T, \quad (27)$$

$$\upsilon_t(1,t)+u(1,t)=u_x(1,t),\ \ \ 0<t\le T, \quad (28)$$

$$u(1,0)=y^0,\ \ \upsilon(1,0)=y^1, \quad (29)$$

$$F=\int_0^1\left[u^2(x,T)+\upsilon^2(x,T)\right]dx. \quad (30)$$

It can be seen from (26) and (29) that the following conditions must be met: $a(1)=y^0$, $b(1)=y^1$.

Let us find a solution $z(t)$ to problem (17), (18), which has the form:

$$z(t)=-\int_t^T \psi_x(1,\tau)\sin(t-\tau)d\tau. \quad (31)$$

Taking into account the form of solution (31) and the conditions $\psi(1,t)=-z(t)$, write out the boundary condition that the function $\psi(x, t)$ must satisfy in the point $x=1$:

$$\psi(1,t)=\int_t^T \psi_x(1,\tau)\sin(t-\tau)d\tau. \quad (32)$$

Excluding from (16)–(18) the variable $z(t)$ taking into account the equalities:

$$\psi_t(1,t)=\int_t^T \psi_x(1,\tau)\cos(t-\tau)d\tau-\psi_x(1,t)\sin(t-t). \quad (33)$$

(16)–(18) can be represented as:

$$\psi(1,t)=-z(t),\ \ \phi_t(1,t)+\psi(1,t)=\psi_x(1,t), \quad (34)$$

$$\psi(1,T)=0,\ \phi(1,T)=0. \quad (35)$$

Thus, the system (14)–(18) can be written as:

$$\psi_t(x,t)=\phi(x,t), \quad (36)$$

$$\phi_t(x,t)=\psi_{xx}(x,t),\ 0<x<1,\ 0\le t<T, \quad (37)$$

$$\psi(x,T)=2\upsilon(x,T),\ \ \phi(x,T)=-2u(x,T),\ \ 0\le x\le 1, \quad (38)$$

$$\psi(0,t)=0, 0<t\le T, \quad (39)$$

$$\phi_t(1,t)+\psi(1,t)=\psi_x(1,t),\ 0\le t<T, \quad (40)$$

$$\psi(1,T)=0,\ \ \phi(1,T)=0. \quad (41)$$

Now, using the method of straight lines, we construct a finite-dimensional approximation of problem (24)–(29). Let $\{x_i=ih,\ i=0,\ 1,\ \ldots,\ n\}$ is a grid with a step $h=1/n$ on the segment [0, 1]. We introduce the designations $u_i(t)=u(x_i, t)$, $v_i(t)=v(x_i,t)$, $i=1, 2, \ldots, n$. Then, taking into account the conditions $u_0(t)=p(t)$, the boundary value problem (24)–(29) can be approximated by the system of equations:

$$\begin{cases}\dot{u}_i(t)=\upsilon_i(t), i=1,2,\ldots,n,\\ \dot{\upsilon}_1(t)=\dfrac{1}{h^2}\left[p(t)-2u_1(t)+u_2(t)\right],\\ \dot{\upsilon}_i(t)=\dfrac{1}{h^2}\left[u_{i-1}(t)-2u_i(t)+u_{i+1}(t)\right], i=2,3,\ldots,n-1,\\ \dot{\upsilon}_n(t)=\dfrac{1}{h}\left[-u_{n-1}(t)+(1-h)u_n(t)\right],\end{cases} \quad (42)$$

with initial conditions:

$$\begin{cases}u_i(0)=a(x_i), \upsilon(0)=b(x_i), i=1,2,\ldots,n-1,\\ u_n(0)=y^0, \upsilon_n(0)=y^1.\end{cases} \quad (43)$$

Thus, the original problem (24)–(30) is reduced to determining the external force $p(t)$*, without taking it beyond the maximum and minimum possible from the minimum sum condition under conditions (42), (43):

$$F=h\sum_{i=1}^{n}\left[u_i^2(T)+\upsilon_i^2(T)\right]. \quad (44)$$

Taking into account the designations $\psi_i(t)=\psi(x_i,t)$, $\varphi_i(t)=\varphi(x_i,t)$, $i=1, 2, \ldots, n$ and conditions $\psi_0(t)=0$, the boundary value problem (36)–(41) is also approximated:

$$\begin{cases}\dot{\psi}_i(t)=\phi_i(t), i=1,2,\ldots,n,\\ \dot{\phi}_i(t)=\dfrac{1}{h^2}\left[-2\psi_1(t)+\psi_2(t)\right],\\ \dot{\phi}_i(t)=\dfrac{1}{h^2}\begin{bmatrix}\psi_{i-1}(t)-\\ -2\psi_i(t)+\psi_{i+1}(t)\end{bmatrix}, i=2,3,\ldots,n-1,\\ \dot{\phi}_n(t)=\dfrac{1}{h}\left[-\psi_{n-1}(t)+(1-h)\psi_n(t)\right].\end{cases} \quad (45)$$

The boundary conditions for (45) and the approximating analogue $F'(p)$ have the form:

$$\begin{cases}\psi_i(T)=2\upsilon_i(T_i), \phi_i(T)=-2u_i(T), i=1,2,\ldots,n-1,\\ \psi_n(T)=0, \phi_n(T)=0,\end{cases} \quad (46)$$

$$F'(p)=\frac{1}{h}\cdot\left(\psi_1(t)-\psi_0(t)\right). \quad (47)$$

From (44), taking into account the condition $\psi_0(t)=0$, we have:

$$F'(p)=\frac{1}{h}\cdot\psi_1(t). \tag{48}$$

Note that the approach based on obtaining formulas for calculating the first variation of the functional seems to be promising not only for optimizing systems similar to (1), (5), but also for more general ones, considered, for example, in [13]. In this case, the original distributed system can be solved by any numerical method without going over to ordinary differential equations, as it was done in [6, 15].

5. 3. Development and implementation of a numerical solution algorithm to obtain specific numerical results

For the numerical solution of the problem, two methods are used – the gradient projection method and the method of successive approximations [7]. In both methods, a sequence of controls is constructed, starting from some admissible control $p^k(t)$. In the first method, the main work is the transition from controls $p^k(t)$ to the next control $p^{k+1}(t)$. Is associated with the calculation of the gradient of the functional according to formula (48).

The algorithm for solving the problem consists of the following steps:

1. Some admissible control $p^k(t)$ is chosen (its choice may be based on some physical considerations).

2. According to the initial $p^k(t)$ by the Runge-Kutta method (subject to the stability condition, it is also possible by the Euler method), the system of equations (42), (43) is integrated in the "forward direction" and the values of $u_i(T)$, $v_i(T)$, $i=1, 2, ..., n$ are found in the time interval $0\le t\le T$.

It is important to note that due to the linearity of the system of equations (39), in the process of calculation it is possible to store only a table of values of the control function $p(t)$ and values $u_i(T)$, $v_i(T)$, $i=1, 2, ..., n$, for calculating the next approximation. There is no need to remember the conjugate variables since they are used only when calculating the gradient of the functional (44) according to (48). This calculation can be carried out at each integration step of the adjoint system (45), (46). Generally speaking, it is possible not to store the trajectory $u_i(T)$, $v_i(T)$, $i=1, 2, ..., n$ in memory, storing only their final values and then integrating the system (45), (46) and simultaneously (42), (43) in the "reverse direction". However, when integrating systems (42), (43) in the opposite direction, the amount of calculations increases and the counting process, especially for nonlinear systems, often becomes unstable.

3. The values of the approximating sum (44) are calculated.

4. Formulas (46) calculate the values $\psi_i(T)$, $\varphi_i(T)$, $i=1, 2, ..., n$.

5. In the "reverse direction" of time, the system (45), (46) is integrated.

Taking into account the ratios:

$$\begin{cases} p(0)=a(0), \\ p(\Delta t)=a(0)+\Delta t b(0), \\ p(2\Delta t)=a(0)+2\Delta t b(0)+\Delta t^2 a''(0), \end{cases} \tag{49}$$

which are approximating analogues of matching conditions (6), the new control $p^{k+1}(t)$ for values $t_j=j\Delta t$, $j=3, 4, ..., m$, $m=T/\Delta t$ is calculated by the formula:

$$p^{k+1}(t)= \\ =\begin{cases} p_{\min}, \text{if } p^k(t)-\delta p^k(t)<p_{\min}, \\ p_{\max}, \text{if } p^k(t)-\delta p^k(t)>p_{\max}, \\ p^k(t)-\delta p^k(t), \text{if } p_{\min}\le p^k(t)-\delta p^k(t)\le p_{\max}. \end{cases} \tag{50}$$

Here:

$$\delta p^k(t)=\alpha\cdot\frac{\psi_1(t)}{\left|\max\psi_1(t)\right|}, k=0,1,..., \tag{51}$$

where Δt is a time step, k is an iteration number, and the parameter $\alpha>0$ is chosen from one of the methods described in [7].

6. The results are printed.

7. Step for calculating new control $p^{k+1}(t)$ is taken, returning to step 2.

When using the method of successive approximations, which in the case of linear systems already at the second iteration gives the optimal trajectory for any initial approximation, the next control $p^{k+1}(t)$ is selected from the condition for the maximum of the Hamilton functions of problem (1)–(5).

The method of successive approximations is very simple from the computational point of view, since at each step it requires only solving two Cauchy problems: "from left to right" for system (42), (43) and "from right to left" for system (45), (46).

The algorithm for solving the problem consists of the following steps:

1. An initial admissible control $p^k(t)$ is chosen.

2. The Runge-Kutta method integrates the system of equations (42), (43) in the "forward direction" and the values of $u_i(T)$, $v_i(T)$, $i=1, 2,...,n$ are in the time interval $0\le t\le T$.

3. The values of the approximating sum (44) are calculated.

4. Formulas (46) calculate the boundary conditions $\psi_i(T)$, $\varphi_i(T)$, $i=1, 2, ..., n$ for the system of equations (45), (46).

5. In the "reverse direction" of time, the system (45), (46) is integrated.

6. The next control $p^{k+1}(t)$, taking into account the conditions (49) is selected from the condition of maximum control functions H.

To implement this task on a computer, programs have been compiled according to the algorithm described. The software codes that implement the gradient projection method and the successive approximation method differ only in a few small blocks. Four two-dimensional and several one-dimensional arrays were used to store the values of the functions $u_i(t)$, $v_i(t)$, $\varphi_i(t)$ and $\psi_i(t)$ at points $t_j=j\Delta t$, $j=0,0,...,m$. In order to trace the correctness of the computational process, as the program code was compiled, the results of some intermediate calculations were checked, including the behavior of the direct problem (42), (43) with a fixed control. To trace the correctness of the computational process, as the program code was compiled, the results of some intermediate calculations were checked, including the behavior of the direct problem (42), (43) with a fixed control:

$$p(t)=\begin{cases} 1, \text{if } t=0, \\ 1.01, \text{if } t=0.01, \\ 1.0199, \text{if } t=0.02, \\ 2, \text{if } t\ge 0.03. \end{cases} \tag{52}$$

The value of the sum (44) is calculated immediately after the integration of the direct problem (42), (43). The boundary conditions (46) for the adjoint system (45), (46), the values of the functions $\psi_1(t)$, etc. were also checked. The system (39), (40), and the system of equations (45), (46) were integrated with a constant step Δt=0.01, and the output of results was carried out with a step Δt=0.05. The segment [0, 1] is divided into five equal parts with step h=0.2. Function (52) is taken as the initial iteration.

The problem was solved with the following parameter values:

T=0.2,

$p_{\min}$=1,

$p_{\max}$=2,

α=0.1.

The lower limit of the external force is chosen taking into account the conditions $p_{\min}=a(0)$.

In calculations, as functions of $a(x)$, according to [9], we chose the solution of the equation $a''(x)-a'(x)+a(x)=0$, satisfying, for example, the condition $a(0)=1$, $a'(0)=0$. This solution will be denoted by $a_1(x)$. It is easy to check that it has the form:

$$a_1(x)=e^{\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right)-\frac{1}{\sqrt{3}}e^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right), \tag{53}$$

and satisfies the condition $a_1''(1)-a_1'(1)+a(1)=0$, under which problem (24)–(29) has a unique solution. Given that:

$$a_1'(x)=-\frac{2}{\sqrt{3}}e^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right),$$

$$a_1''(x)=-e^{\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right)-\frac{1}{\sqrt{3}}e^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}}{2}\right),$$

conditions $a_1(1)=y^0$, and expression $a_1''(x)$ we have: $y^0=a_1(1)=0.343028$, $a_1''(0)=1$.

Function $b(x)$ is chosen in the form of $b(x)=1-x$. In this case, $y^1=b(1)=0$.

Hence:

$p(0)=a(0)=1$,

$p(\Delta t)=a(0)+\Delta t b(0)=1+0.01\cdot 1=1.01$,

$p(2\Delta t)=a(0)+2\Delta t b(0)+\Delta t 2a''(0)=$
$=1+2\cdot 0.01\cdot 1+0.012\cdot(-1)=1.0199$.

With the indicated data, calculations were carried out. Table 1 shows the results of calculations with initial control (52). In this case, the rate of convergence in terms of the functional was as follows.

With a further increase in the number of iterations, the value of the functional did not change. Fig. 1 shows the approximate optimal controls obtained for some intermediate iterations. It can be seen from the consideration of these graphs that the sequence of controls $p^k(t)$ with an increase in the number of iterations approaches its lower bound and is a minimizing sequence of controls for functional (44).

Table 1

The results of calculations with initial control (52)

No of iterations	$F(p)$	$F'(p)$
0	3.576072	7.267986
1	3.144007	6.674428
2	2.746374	6.079073
3	2.383292	5.483878
4	2.055035	4.886278
5	1.761755	4.285675
6	1.505650	3.681219
7	1.280975	3.071629
8	1.094079	2.459527
9	0.943718	1.841799
10	0.830831	1.227154
11	0.777536	0.899863
12	0.749283	0.749380
13	0.733279	0.691243
14	0.725399	0.666719
15	0.723609	0.661038

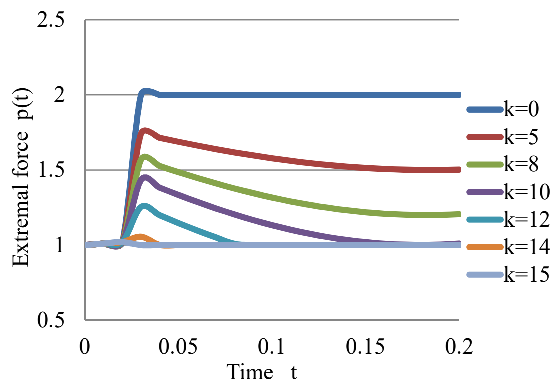

Fig. 1. Approximately optimal controls in intermediate iterations

Note that, taking into account (6), as $a(x)$, we can choose another solution of the equation $a_1''(x)-a_1'(x)+a(x)=0$ that satisfies the condition $a(0)=1$, $a'(0)=0$. We denote this solution by $a_2(x)$. It is easy to check that the solution $a_2(x)$ has the following form:

$$a_2(x)=\frac{2}{\sqrt{3}}e^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right). \tag{54}$$

The solution $a_2(x)$ also satisfies the condition $a''(1)-a'(1)+a(1)=0$, i. e. $a_1(x)$ and $a_2(x)$ are linearly independent solution of the above equation.

Considering:

$$a_2'(x)=e^{\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right)+\frac{1}{\sqrt{3}}e^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right),$$

$$a_2''(x)=e^{\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right)-\frac{1}{\sqrt{3}}e^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right),$$

condition $a_2(1)=y^0$, and expression $a_2''(x)$ we have:

$y^0=a_2(1)=1.450223$, $a_2''(0)=1$, $p_{\min}=a_2(0)=0$.

Calculations were also carried out with these data, under the assumption that:

$$b(x)=e^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right),$$

$$b(1)=1.25593.$$

The qualitative picture of the results has not changed: with an increase in the number of iterations, the control sequence $p^k(t)$ approaches the optimal control, and there is convergence of the functional. Therefore, we do not present them.

We note that by introducing the functions $Y(z)$ and $K(z, s)$ defined by the formulas:

$$Y(z)=2a(1)a_1(z-1)+2b(1)a_2(z-1), \tag{55}$$

$$K(z,s)=\frac{a_2(z)a_1(s)-a_1(z)a_2(s)}{a_1(s)a_2'(s)-a_2(s)a_1'(s)}, \tag{56}$$

and carrying out certain calculations, taking into account some additional conditions, it is possible to construct a solution to the boundary value problem (24)–(29), which can be represented as:

$$u(x,t)=\frac{1}{2}\left[E(x+t)+G(x-t)\right], \tag{57}$$

and a control function according to the formula $p(t)=E(t)/2$, as it was done in [9]. It was also proved there that formula (57) is possible if $E(t)$ is determined at $0\le z\le 3$, and $G(z)$ at $-2\le z\le 1$. Functions $G(z)$ and $E(z)$ have the forms:

$$G(z)=\begin{cases} a(z)+\int_z^0 b(s)\,ds, & 0\le z\le 1,\\ 2p(-z)-E(-z), & -2\le z\le 0,\end{cases}$$

$$E(z)=\begin{cases} a(z)+\int_0^z b(s)\,ds, & 0\le z\le 1,\\ Y(z)-G(2-z)+ & \\ +2\int_1^z K(z-1,s-1)G'(2-s)\,ds, & 1\le z\le 3.\end{cases}$$

In this case, functions $Y(z)$ and $K(z, s)$, taking into account (53) and (54) have the forms:

$$Y(z)=2a(1)e^{\frac{z-1}{2}}\left\{\cos\left(\frac{\sqrt{3}}{2}(z-1)\right)-\frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}(z-1)\right)\right\}+$$
$$+\frac{4}{\sqrt{3}}b(1)e^{\frac{z-1}{2}}\sin\left(\frac{\sqrt{3}}{2}(z-1)\right),$$

$$K(z,s)=\frac{2}{\sqrt{3}}e^{\frac{1}{2}(z-s)}\sin\left(\frac{\sqrt{3}}{2}(z-s)\right).$$

Fig. 2 presents the results of computing the solution of the direct problem (42), (43) with the functions $u(x, t)$, corresponding to the control obtained after 15 iterations. Consideration of Fig. 2 shows that over time, the state of a distributed system practically does not change and approaches a calm state.

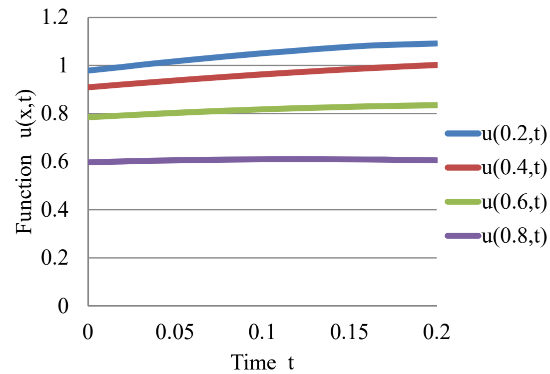

Fig. 2. Change of functions u(x, t) in time for different values of phase coordinates

Fig. 3 shows the dependence of the change in the state of a distributed system in terms of spatial coordinate at different points in time. It can be seen from the graphs in Fig. 3 that the states of the distributed system, whose initial position in the segment $-0\le x\le 1$ is determined by the function $a_1(x)$, coincide over time.

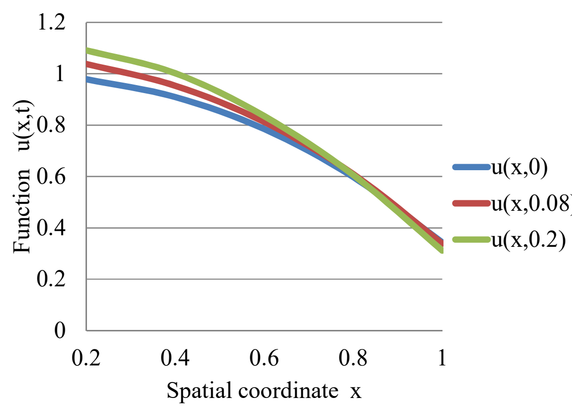

Fig. 3. Change of functions u(x, t) along the spatial coordinate for different moments of time

Fig. 4 shows the results of computing the solution of the adjoint problem (45), (46) with the functions $\psi(x, t)$, corresponding to the optimal control, obtained after 15 iterations.

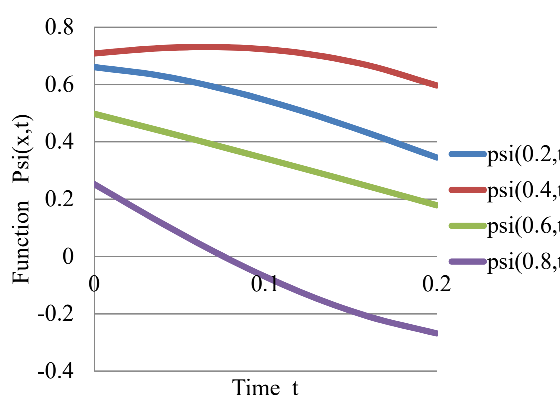

Fig. 4. Change of functions ψ(x, t) in time for different values of phase coordinates

When using the method of successive approximations, which is obtained from the Pontryagin maximum principle, the next control $p^{k+1}(t)$, taking into account conditions (49), is selected from the condition that the Hamiltonian functions are maximal with respect to the variable p. If the process of constructing successive approximations converges, then the iterations continue until subsequent approximations differ from each other within the specified accuracy. The resulting solution will satisfy the maximum principle.

From the linearity of the functions $H=\psi_x(0, t)p(t)$ with respect to the variable $p(t)$, it can be seen that:

$$p(t)=\frac{p_{\max}+p_{\min}}{2}+\frac{p_{\max}-p_{\min}}{2}\cdot \operatorname{sign}\left(\psi_x(0,t)\right). \quad (58)$$

Given that $p(t)$ varies within the interval $p_{\max}$ and $p_{\min}$ and the function H is linear with respect to p, this formula is mathematically valid, that is, if we exclude special controls, in this problem, the optimal control is formally a piecewise constant function. It would seem that the whole solution consists of the optimal selecting sequences of control intervals and finding their junction points, that is, the optimal control will be a boundary, in particular, this includes discontinuous relay controls. When applying the method of successive approximations, a boundary control will be obtained at each iteration – the approximation to the optimal control will be in the class of discontinuous boundary controls.

It should be noted that the formality of writing (55) also lies in the fact that the cancellation of the expression under the mark of "sign" is possible both at individual points of segment $0\le t\le T$ and in its entire sections. In this case, as a rule, the maximum principle is not sufficient to determine the optimal control; additional research is required to identify singular controls [16].

The calculation results show that, indeed, in linear problems, that is, in problems, which equations contain a control of the first degree, the method of successive approximations for any initial control already at the second iteration gives optimal control.

Calculations made with relay prominent initial control:

$$p^0(t)=\begin{cases}1, \text{if } t=0,\\ 1.01, \text{if } t=0.01,\\ 1.0199, \text{if } t=0.02,\\ 2, \text{if } 0.03\le t\le 0.1,\\ 1, \text{if } 0{,}11\le t\le 0.15,\\ 2, \text{if } t\ge 0.16,\end{cases} \quad (59)$$

show that, the control found in the second approximation has the form:

$$p^1(t)=\begin{cases}1, \text{if } t=0,\\ 1.01, \text{if } t=0.01,\\ 1.0199, \text{if } t=0.02,\\ 2, \text{if } t\ge 0.03,\end{cases} \quad (60)$$

and all subsequent approximations are the same. However, we cannot assert that the control (56) found in the second step is optimal, since the control found from the maximum condition for the Hamilton functions differs significantly from the optimal one and does not approach it.

The maximum value of the Hamilton functions and the value of the functional in the first and second approximation, respectively, turned out to be equal to:

$$H\left(p^0(t)\right)=9.939263, \quad F\left(p^0(t)\right)=3.756072,$$

$$H\left(p^1(t)\right)=14.535972, \quad F\left(p^1(t)\right)=2.120949.$$

Thus, the gradient projection method for the considered problem gives significantly more accurate results than the method of successive approximations, although it requires a relatively large number of iterations. It should also be noted that the gradient projection method did not show a tendency to "blur" and, for any initial control, gave a converging sequence of controls.

6. Discussion of the results of solving the problem of optimal control of oscillations in a system of coupled objects

In this paper, topical issues related to the numerical solution of control problems related to the damping of oscillatory systems consisting of two objects were considered. It is important to note that in the works where problems of damping oscillatory systems are investigated, only theoretical results have been obtained, and numerical solutions to these problems, which are of practical interest, have not been considered [9–13]. The use of formulas obtained in these works is associated with significant computational difficulties in constructing and solving boundary value problems and control variables. In this regard, this paper addresses relevant issues related to obtaining numerical solutions to control problems associated with damping oscillatory systems consisting of two objects. One of the objects is described by a wave equation with first-kind boundary conditions (2), (3), while the other is described by an ordinary second-order differential equation (4).

For the efficient numerical solution of this problem, two methods were utilized: the gradient projection method, based on obtaining formulas (19) for the first variation of the functional (7), and the method of successive approximations due to the linearity of the boundary value problems describing the oscillatory processes (1)–(5). The proposed algorithm for the numerical solution of the problem was implemented in software code, enabling practical results to be obtained. These results are presented in Table 1 and Fig. 1–4.

The peculiarity of the proposed method lies in transforming control problems, described by a set of partial and ordinary differential equations, into a variational problem associated with a system of ordinary differential equations, and subsequently solving these problems numerically based on the Pontryagin maximum principle. This approach ensures computational efficiency and allows for rapid and accurate results to be obtained.

The main limitation of the approach presented in the paper is that in the numerical solution of the considered problem it is not possible to vary some initial data in a wide range, it is necessary to take into account the conditions for matching the initial and boundary conditions (6), which narrows the class of admissible controls, for example, the lower limit of the external force $p(t)$ is chosen taking into account conditions $p_{\min}=a(0)$, the initial state of the distributed system, that is, the function $a(x)$ must be a solution to the equation $as-a'+a=0$, the conditions $a(1)=y^0$, $b(1)=y^1$ must be satisfied.

The main drawback of the work, which should be paid attention to, is the significant computational difficulties in constructing and solving boundary value problems and the control variable presented in [5–7]. Nevertheless, these formulas have important theoretical significance in the study of the control of oscillation processes in coupled systems, consisting of objects with distributed and lumped parameters.

The results of numerical calculations obtained in the framework of this work confirm the effectiveness of the proposed algorithm and its applicability for solving the problem of controlling oscillatory systems with feedback. The results obtained can be useful for practical applications in various fields such as engineering, acoustics, and others, where vibration control plays an important role. The numerical solution to the problem of optimal control of oscillations in a system of interconnected objects can be applied, for example, in the design of shock absorbers for automobiles, where it is necessary to optimize the parameters of the system to minimize vibrations and ensure comfortable movement. This approach can also be used in the construction of buildings and bridges to account for dynamic loads and prevent the destruction or deformation of structures under the influence of oscillations.

Thus, this study makes an important contribution to the field of numerical solution of control problems for oscillatory systems and offers effective methods and algorithms for solving such problems. Further development of the study may include refinement of methods, additional experiments, and expansion of the scope of this approach.

7. Conclusions

1. The gradient projection method, based on obtaining the first variation of the functional, yields a convergent minimizing sequence for any initial admissible control. This is particularly useful because in real-world problems, the initial approximation may not be sufficiently close to the optimal solution. This is especially useful, since in real problems the initial approximation may not be close enough to the optimum.

2. Without requiring a large number of calculations, the method of successive approximations, which is obtained from the Pontryagin maximum principle, gives solution already at the second step of iterations. The main advantage of this method lies in its relative savings in computational resources.

3. When employing the method of successive approximations based on the maximum principle, each iteration leads to obtaining a boundary control, which implies approximation to the optimal control within the class of discontinuous (or relay) boundary controls, which is convenient. In the case of using gradient methods, the relay control is approximated by continuous controls. The application of the method of successive approximations is effective when the boundary value problem is linearly described with respect to the phase variables; in this case, the method provides the optimal trajectory already at the second iteration, regardless of the initial approximation, since the solution of the adjoint system is independent of the control.

Conflict of interest

The authors declare that they have no conflict of interest in relation to this research, whether financial, personal, authorship or otherwise, that could affect the research and its results presented in this paper.

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Data availability

Data will be made available on reasonable request.

Use of artificial intelligence

The authors confirm that they did not use artificial intelligence technologies when creating the current work.

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