The object of research is a complex system of three subsystems, which function independently of each other and are in a working or failed state. There is a need to analytically model and manage the Markov random process in the system, varying the intensity of their development-restoration and degradation-destruction flows. In the study, an analytical method for solving Kolmogorov equations of the eighth order for an asymmetric Markov chain was devised.

The corresponding Kolmogorov equations of the eighth order have an ordered transition probability matrix. The distribution of the eight roots of this equation in the complex plane has central symmetry.

The results are analytical solutions for the probabilities of the eight states of the Markov chain in time in the form of ordered determinants with respect to the indices of the eight roots and the indices of the eight states, including the column vector of the initial conditions.

Symmetry has been established in the distribution on the complex plane of eight real, negative roots of the characteristic Kolmogorov equation centered at the point defined as $Re\theta = -a_7/8$, where a_7 is the coefficient of the characteristic equation of the eighth degree at the seventh power. Formulas expressing eight roots of the characteristic Kolmogorov equation have been heuristically derived, one of which is zero, due to the intensities of failures and recovery of three subsystems, the eight states of which in general make up an asymmetric Markov chain.

For structures consisting of three independently functioning processes, the random process of the transition of the structure through eight possible states with a known initial state is determined in time. An analytical solution to Kolmogorov differential equations of the eighth order for an asymmetric state graph is proposed in harmonic form for the purpose of analysis and synthesis of a random Markov process in a triple system

Keywords: state graph, state probabilities, simulation of random processes, distribution of roots

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DEVISING AN ANALYTICAL METHOD FOR SOLVING THE EIGHTH-ORDER KOLMOGOROV EQUATIONS FOR AN ASYMMETRIC MARKOV CHAIN

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1. Introduction

The theory of random Markov processes is fundamental and has been widely used in practical problems. In modern control systems, it is necessary to analytically model and manage a Markov random process in a system consisting of three independently functioning subsystems, varying the intensity of their development-restoration and degradation-destruction flows.

An asymmetric Markov chain with eight states and the corresponding Kolmogorov differential equations of the eighth order are a mathematical model of a number of applied problems in various fields, in particular, technical, military, social, medical, ecological, energy, economic, and others.

Numerous technical tasks for objective and subjective reasons are conventionally solved by approximate numerical methods and illustrated by drawings, graphs, tables in a limited range of system parameter changes. It is known that numerical solutions cannot cover the variety of analytical conclusions. Analytical solutions do not require numerical interpretation but can be used in various applications, that is, illustrating an analytical solution with a numerical example is equivalent to checking the well-known Ferrara formula on a private example, which cannot be productive and appropriate.

Analytical solutions to private equations of algebra of the fourth power and higher open up new opportunities for technical applications. This paper provides an analytical solution to the algebra equation of the eighth degree of a particular form for an asymmetric Markov chain and the corresponding Kolmogorov differential equations.

The methodology for composing the Kolmogorov equations based on the state graph is known and described in [1, 2].

Solving numerous private tasks of technical, military, social, economic, and other domains on the basis of the proposed analytical solution may be the subject of separate consideration with interested organizations. Therefore, research on the development of an analytical method for solving eighth-order Kolmogorov equations for an asymmetric Markov chain is relevant.

2. Literature review and problem statement

Non-stationary queues with losses are considered in [3], which are a type of queuing system where arrival and service rates are not constant over time. The author uses the approximation of cumulative moments to analyze the behavior of these queues and derives approximations for the length of the queue and the probability of losses. The study does not fully address the problem of analyzing non-stationary queues with losses, as it only provides approximations for queue length and loss probability. The author notes that the approximations are based on certain assumptions and may be inaccurate for all types of non-stationary arrival and service processes. The paper assumes that the arrival and service processes are independent and equally distributed, which may not be the case in many real-world scenarios. A comprehensive analysis of the mass service system is not presented, as it focuses only on the length of the queue and the probability of losses. Therefore, further research is needed to devise more accurate and reliable methods for the analysis of non-stationary queues with losses.

Although paper [4] provides a general introduction to the spectral theory of Markov chains, it does not explore specific applications of the theory to fields such as biology, finance, or computer science. Also, the study covers the basics of spectral theory but does not delve into more advanced topics, such as the relationship between spectral theory and other branches of mathematics, such as operator theory or functional analysis. The work's focus on theoretical aspects means that it does not provide a comprehensive overview of the computational techniques that are needed to apply spectral theory to real-world problems. Therefore, readers wishing to apply spectral theory to specific areas or to explore complex topics will need to consult additional resources.

In [5], the authors develop a structure of stochastic modeling to capture traffic flow dynamics and apply it to real traffic scenarios. The work provides a comprehensive overview of the stochastic evolution of dynamic traffic flow and its applications; it is not aimed at solving any specific open problem in the field of traffic flow modeling. It covers advanced modeling techniques such as machine learning or deep learning approaches. It focuses on modeling and simulation but does not provide a comprehensive overview of real-time traffic forecasting methods and their applications.

Paper [6] considers the possibility of using a hidden Markov model (HMM) to predict people's mobility based on GPS tracking data. The authors investigate whether HMM can accurately capture complex patterns of human mobility and predict future locations.

The paper only compares HMM to a simple Markov chain model but does not evaluate its performance against other state-of-the-art models. The authors use a relatively small dataset and do not investigate the scaling of HMM approach to larger datasets or real-world applications. Furthermore, the robustness of the HMM approach to noisy or incomplete GPS tracking data, which is a common problem in real-world applications, is not evaluated.

A special place is occupied by problems that allow analytical solutions [7]. Previously, analytical solutions were obtained in an ordered matrix form, allowing the analysis of Markov random processes with discrete states and discrete time (discrete Markov chains). Mathematical models of discrete Markov chains are constructed in algebraic form according to asymmetric state graphs and are represented by ordered transition probability matrices. The used methodology is applied to solve a number of actual problems.

Mathematical models of continuous Markov chains (random processes with discrete states and continuous time) are built in the differential form of Kolmogorov's equations based on asymmetric state graphs, as described in [8]. The Kolmogorov equations considered there are systems of ordinary, linear, homogeneous differential equations with constant coefficients, the analytical solution to which is limited to the fourth order; the analytical solution to the Kolmogorov equations for an asymmetric graph of two states is obtained. In [9], an analytical solution to the Kolmogorov equation for an asymmetric four-state graph was derived. But the analytical solution to the Kolmogorov equations for an asymmetric eight-state graph remained unresolved.

Analytical solution to eighth-order Kolmogorov equations describing an asymmetric, continuous Markov chain is reduced to analytical solution to the corresponding eighth-order characteristic equation, which is known to be problematic.

It is necessary to analytically model and control a Markov random process in a system consisting of three independently functioning subsystems, varying the intensity of their development-restoration and degradation-destruction flows.

3. The aim and objectives of the study

The purpose of our study is to devise a method for analytical solutions to Kolmogorov equations of the eighth order. This makes it possible to assess the reliability of power systems.

To achieve the goal, the following tasks were set:

– to establish the formulas of the eight roots of the characteristic equation;

– to form analytical solutions to Kolmogorov equations of the eighth order;

– to assess the reliability of the energy system.

4. The study materials and methods

4. 1. The object and hypothesis of the study

The object of research is a complex system of three subsystems that function independently of each other and are in a working or failed state. For example, methods for calculating the reliability of energy systems, railroads, urban electric transport, and industry.

The method implements a comprehensive approach to increasing the efficiency of generating stations in power systems with the possibility of forecasting their state probability (operability or restoration). Implementation of the method will make it possible to make a reasoned decision regarding the use of the full resource of generating stations in power systems without the risk of failure or premature decommissioning.

The intensities of Poisson flows of failures and recovery of each of the subsystems are considered known and constant. The possible discrete states of the system are determined by the number $2^3=8$ and are connected by an asymmetric Markov chain as an asymmetric graph of eight states. Markov random processes in a system with discrete states and continuous time are described by the corresponding Kolmogorov equations. Kolmogorov's equations are formed according to a well-known procedure, using an asymmetric graph of eight states.

4. 2. Description of the method

A system consisting of three independently functioning subsystems (triple system) is under consideration. The possible discrete states of each of the subsystems are known as operational, denoted as ⊕, or inoperable, denoted as ⊖. Then the possible discrete states of the system as a whole, determined in the number of $2^3=8$, take the following form (Fig. 1) and are represented by an asymmetric graph of eight states (Fig. 2).

A mathematical model (Kolmogorov equation) of a homogeneous Markov random process with eight discrete states and continuous time (continuous Markov link) is built. Probabilities of eight states of the system in time (in dynamics) $P_i(t)$ ($i=1, 2, 3, ..., 8$) are analytically found with the known state of the system at the initial time point $P_i(0)$, given and constant intensities of Poisson failure flows (λ_1, λ_2, λ_3) and recovery (μ_1, μ_2, μ_3) of three subsystems. The stated general problem includes a number of individual applied tasks in various fields: technical [10, 11], military, social [12], medical [13], ecological [14], energy [15], economic, and others. Examples of setting some applied problems are given below.

S_1=⊕⊕⊕ S_2=⊖⊖⊖
S_3=⊖⊕⊕ S_4=⊕⊖⊖
S_5=⊕⊖⊕ S_6=⊖⊕⊖
S_7=⊕⊕⊖ S_8=⊖⊖⊕

Fig. 1. Discrete states of the system

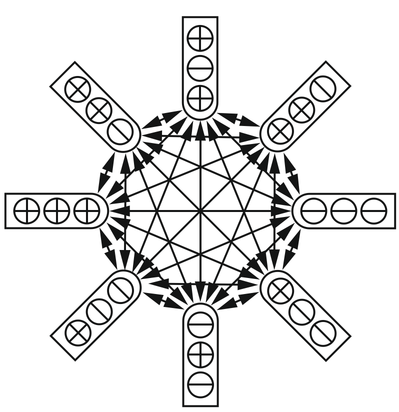

Fig. 2. Asymmetric graph of eight states of a triple system

Task No. 1: technical application.

This technical system consists of three subsystems: energy subsystem, control subsystem, power subsystem. Intensities of failure flows and recovery of subsystems are considered to be statistically determined. The probabilities of eight possible states of the system at the current time and in the future ($t\to\infty$) are found, given the known state of the subsystems at the initial time ($t=0$).

Task No. 2: military application.

The military structure is given, consisting of three types of troops: land, air, and sea. The intensity of losses of troops during military operations is statistically determined. The intensity of the recovery flows is determined by the availability of resources and belongs to the known. Limiting the intensity of loss flows and changing the intensity of recovery flows, it is necessary to determine the probabilities of eight possible states of the military structure both now and in the future with a given state of the troops at the initial time.

Task No. 3: social application.

A social structure consisting of three autonomous groups is given. The intensity of birth and death flows of each of the autonomous groups are considered statistically significant. The probabilities of eight possible states of the social structure at the present time and in the future are found, given the known state of autonomous groups at this moment in time.

Tasks in other areas are stated similarly.

4. 3. Kolmogorov's triple system equation

Kolmogorov's equations are composed according to the constructed asymmetric graph of eight states and in expanded notation take the form:

$$\begin{aligned}
\dot{P}_1 &= -(\lambda_1+\lambda_2+\lambda_3)P_1+\mu_1P_3+\mu_2P_5+\mu_3P_7,\\
\dot{P}_2 &= -(\mu_1+\mu_2+\mu_3)P_2+\lambda_1P_4+\lambda_2P_6+\lambda_3P_8,\\
\dot{P}_3 &= \lambda_1P_1-(\mu_1+\lambda_2+\lambda_3)P_3+\mu_3P_6+\mu_2P_8,\\
\dot{P}_4 &= \mu_1P_2-(\lambda_1+\mu_2+\mu_3)P_4+\lambda_3P_5+\lambda_2P_7,\\
\dot{P}_5 &= \lambda_2P_1+\mu_3P_4-(\lambda_1+\mu_2+\lambda_3)P_5+\mu_1P_8,\\
\dot{P}_6 &= \mu_2P_2+\lambda_3P_3-(\mu_1+\lambda_2+\mu_3)P_6+\lambda_1P_7,\\
\dot{P}_7 &= \lambda_3P_1+\mu_2P_4+\mu_1P_6-(\lambda_1+\lambda_2+\mu_3)P_7,\\
\dot{P}_8 &= \mu_3P_2+\lambda_2P_3+\lambda_1P_5-(\mu_1+\mu_2+\lambda_3)P_8,
\end{aligned} \tag{1}$$

or in matrix form:

$$\frac{d}{dt}P(t)=R\cdot P(t), \tag{2}$$

where $P(t)$ is a column matrix of probabilities of eight triple system states:

$$P(t)=\begin{Vmatrix} P_1(t)\\ P_2(t)\\ P_3(t)\\ P_4(t)\\ P_5(t)\\ P_6(t)\\ P_7(t)\\ P_8(t) \end{Vmatrix};$$

R – block matrix of failure-recovery flow intensities:

$$R=\begin{Vmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ r_{41} & r_{42} & r_{43} & r_{44} \end{Vmatrix}.$$

Here, the symmetrical structure of the Kolmogorov equations of the considered triple system is reflected by the following ordered matrices of the intensities of failure (λ_1, λ_2, λ_3) and recovery (μ_1, μ_2, μ_3) flows:

$$r_{11}=\begin{Vmatrix} -(\lambda_1+\lambda_2+\lambda_3) & 0 \\ 0 & -(\mu_1+\mu_2+\mu_3) \end{Vmatrix},$$

$$r_{22}=\begin{Vmatrix} -(\mu_1+\lambda_2+\lambda_3) & 0 \\ 0 & -(\lambda_1+\mu_2+\mu_3) \end{Vmatrix},$$

$$r_{33}=\begin{Vmatrix} -(\lambda_1+\mu_2+\lambda_3) & 0 \\ 0 & -(\mu_1+\lambda_2+\mu_3) \end{Vmatrix},$$

$$r_{44}=\begin{Vmatrix} -(\lambda_1+\lambda_2+\mu_3) & 0 \\ 0 & -(\mu_1+\mu_2+\lambda_3) \end{Vmatrix},$$

$$r_{12}=\begin{Vmatrix} \mu_1 & 0 \\ 0 & \lambda_1 \end{Vmatrix},\ r_{21}=\begin{Vmatrix} \lambda_1 & 0 \\ 0 & \mu_1 \end{Vmatrix},\ r_{13}=\begin{Vmatrix} \mu_2 & 0 \\ 0 & \lambda_2 \end{Vmatrix},$$

$$r_{31}=\begin{Vmatrix} \lambda_2 & 0 \\ 0 & \mu_2 \end{Vmatrix},\ r_{14}=\begin{Vmatrix} \mu_3 & 0 \\ 0 & \lambda_3 \end{Vmatrix},\ r_{41}=\begin{Vmatrix} \lambda_3 & 0 \\ 0 & \mu_3 \end{Vmatrix},$$

$$r_{23}=\begin{Vmatrix} 0 & \mu_3 \\ \lambda_3 & 0 \end{Vmatrix},\ r_{32}=\begin{Vmatrix} 0 & \mu_3 \\ \lambda_3 & 0 \end{Vmatrix},\ r_{24}=\begin{Vmatrix} 0 & \mu_2 \\ \lambda_2 & 0 \end{Vmatrix},$$

$$r_{42}=\begin{Vmatrix} 0 & \mu_2 \\ \lambda_2 & 0 \end{Vmatrix},\ r_{34}=\begin{Vmatrix} 0 & \mu_1 \\ \lambda_1 & 0 \end{Vmatrix},\ r_{43}=\begin{Vmatrix} 0 & \mu_1 \\ \lambda_1 & 0 \end{Vmatrix}. \tag{3}$$

The orderliness of the given matrices reflects the structural symmetry of the state graph of the triple system. The elements of the main diagonal of the matrix R correspond to eight states of the system, the matrix R is asymmetric with respect to the main diagonal. Given the specified structural asymmetry, the matrix R is degenerate, i.e., $\det(R)=0$.

4. 4. Characteristic equation of the triple system

The characteristic determinant of the square matrix R of the eighth order $\det(R-\nu E)$ is reduced to an algebraic polynomial of the eighth power:

$$\sum_{m=0}^{8} a_{8-m}\nu^{8-m}. \tag{4}$$

In particular, the coefficients a_8, a_7, a_6, ..., a_0 are determined from an algorithm based on symmetric polynomials of ordered determinants [9]:

$$a_8=\begin{vmatrix} -1 & 0 & & 0 & 0 \\ 0 & -1 & & 0 & 0 \\ 0 & 0 & & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & & 0 & 0 \\ 0 & 0 & & 0 & 0 \\ 0 & 0 & & -1 & 0 \\ 0 & 0 & & 0 & -1 \end{vmatrix},$$

$$a_7=\begin{vmatrix} -(\lambda_1+\lambda_2+\lambda_3) & 0 & 0 & & 0 & 0 \\ 0 & -1 & 0 & & 0 & 0 \\ \lambda_1 & 0 & -1 & & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \lambda_2 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 \\ \lambda_3 & 0 & 0 & & -1 & 0 \\ 0 & 0 & 0 & & 0 & -1 \end{vmatrix}+\begin{vmatrix} -1 & 0 & 0 & & 0 & 0 \\ 0 & -(\mu_1+\mu_2+\mu_3) & 0 & & 0 & 0 \\ 0 & 0 & -1 & & 0 & 0 \\ 0 & \mu_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 \\ 0 & \mu_2 & 0 & & 0 & 0 \\ 0 & 0 & 0 & & -1 & 0 \\ 0 & \mu_3 & 0 & & 0 & -1 \end{vmatrix}+$$

$$+\begin{vmatrix} -1 & 0 & \mu_1 & & 0 & 0 \\ 0 & -1 & 0 & & 0 & 0 \\ 0 & 0 & -(\mu_1+\lambda_2+\lambda_3) & & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & \lambda_3 & & 0 & 0 \\ 0 & 0 & 0 & & -1 & 0 \\ 0 & 0 & \lambda_2 & & 0 & -1 \end{vmatrix}+\begin{vmatrix} -1 & 0 & 0 & 0 & & 0 \\ 0 & -1 & 0 & \lambda_1 & & 0 \\ 0 & 0 & -1 & 0 & & 0 \\ 0 & 0 & 0 & -(\lambda_1+\mu_2+\mu_3) & \cdots & 0 \\ 0 & 0 & 0 & \mu_3 & & 0 \\ 0 & 0 & 0 & 0 & & 0 \\ 0 & 0 & 0 & \mu_2 & & 0 \\ 0 & 0 & 0 & 0 & & -1 \end{vmatrix}+$$

$$+\begin{vmatrix} -1 & & 0 & 0 & 0 & 0 \\ 0 & & 0 & \lambda_2 & 0 & 0 \\ 0 & & 0 & \mu_3 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & & -1 & 0 & 0 & 0 \\ 0 & & 0 & -(\mu_1+\lambda_2+\mu_3) & 0 & 0 \\ 0 & & 0 & \mu_1 & -1 & 0 \\ 0 & & 0 & 0 & 0 & -1 \end{vmatrix}+\begin{vmatrix} -1 & 0 & & 0 & \mu_3 & 0 \\ 0 & -1 & & 0 & 0 & 0 \\ 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & \lambda_2 & 0 \\ 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & & -1 & \lambda_1 & 0 \\ 0 & 0 & & 0 & -(\lambda_1+\lambda_2+\mu_3) & 0 \\ 0 & 0 & & 0 & 0 & -1 \end{vmatrix}+$$

$$+\begin{vmatrix} -1 & 0 & & 0 & 0 & 0 \\ 0 & -1 & & 0 & 0 & \lambda_3 \\ 0 & 0 & & 0 & 0 & \mu_2 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & & 0 & 0 & \mu_1 \\ 0 & 0 & & -1 & 0 & 0 \\ 0 & 0 & & 0 & -1 & 0 \\ 0 & 0 & & 0 & 0 & -(\mu_1+\mu_2+\lambda_3) \end{vmatrix},$$

$$a_6=\begin{vmatrix} -\lambda_1-\lambda_2-\lambda_3 & 0 & & 0 & 0 & 0 \\ 0 & -\mu_1-\mu_2-\mu_3 & & 0 & 0 & 0 \\ \lambda_1 & 0 & & 0 & 0 & 0 \\ 0 & \mu_1 & \cdots & 0 & 0 & 0 \\ \lambda_2 & 0 & & 0 & 0 & 0 \\ 0 & \mu_2 & & -1 & 0 & 0 \\ \lambda_3 & 0 & & 0 & -1 & 0 \\ 0 & \mu_3 & & 0 & 0 & -1 \end{vmatrix}+\begin{vmatrix} -\lambda_1-\lambda_2-\lambda_3 & 0 & \mu_1 & & 0 & 0 \\ 0 & -1 & 0 & & 0 & 0 \\ \lambda_1 & 0 & -\mu_1-\lambda_2-\lambda_3 & & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \lambda_2 & 0 & 0 & & 0 & 0 \\ 0 & 0 & \lambda_3 & & 0 & 0 \\ \lambda_3 & 0 & 0 & & -1 & 0 \\ 0 & 0 & \lambda_2 & & 0 & -1 \end{vmatrix}+$$

$$+\begin{vmatrix} -\lambda_1-\lambda_2-\lambda_3 & 0 & 0 & 0 & & 0 \\ 0 & -1 & 0 & \lambda_1 & & 0 \\ \lambda_1 & 0 & -1 & 0 & & 0 \\ 0 & 0 & 0 & -\lambda_1-\mu_2-\mu_3 & \cdots & 0 \\ \lambda_2 & 0 & 0 & \mu_3 & & 0 \\ 0 & 0 & 0 & 0 & & 0 \\ \lambda_3 & 0 & 0 & \mu_2 & & 0 \\ 0 & 0 & 0 & 0 & & -1 \end{vmatrix}+\begin{vmatrix} -\lambda_1-\lambda_2-\lambda_3 & 0 & 0 & 0 & \mu_2 & \\ 0 & -1 & 0 & 0 & 0 & \\ \lambda_1 & 0 & -1 & 0 & 0 & \\ 0 & 0 & 0 & -1 & \lambda_3 & \cdots \\ \lambda_2 & 0 & 0 & 0 & -\lambda_1-\mu_2-\lambda_3 & \\ 0 & 0 & 0 & 0 & 0 & \\ \lambda_3 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \lambda_1 & \end{vmatrix}+$$

$$+\begin{vmatrix} -\lambda_1-\lambda_2-\lambda_3 & & 0 & 0 & 0 & 0 \\ 0 & & 0 & \lambda_2 & 0 & 0 \\ \lambda_1 & & 0 & \mu_3 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ \lambda_2 & & -1 & 0 & 0 & 0 \\ 0 & & 0 & -\mu_1-\lambda_2-\mu_3 & 0 & 0 \\ \lambda_3 & & 0 & \mu_1 & -1 & 0 \\ 0 & & 0 & 0 & 0 & -1 \end{vmatrix}+\begin{vmatrix} -\lambda_1-\lambda_2-\lambda_3 & & 0 & 0 & \mu_3 & 0 \\ 0 & & 0 & 0 & 0 & 0 \\ \lambda_1 & & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & \lambda_2 & 0 \\ \lambda_2 & & -1 & 0 & 0 & 0 \\ 0 & & 0 & -1 & \lambda_1 & 0 \\ \lambda_3 & & 0 & 0 & -\lambda_1-\lambda_2-\mu_3 & 0 \\ 0 & & 0 & 0 & 0 & -1 \end{vmatrix}+$$

$$+\begin{vmatrix} -\lambda_1-\lambda_2-\lambda_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3 \\ \lambda_1 & 0 & 0 & 0 & \mu_2 \\ 0 & \dots 0 & 0 & 0 & 0 \\ \lambda_2 & -1 & 0 & 0 & \mu_1 \\ 0 & 0 & -1 & 0 & 0 \\ \lambda_3 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -\mu_1-\mu_2-\lambda_3 \end{vmatrix} + \begin{vmatrix} -1 & 0 & \mu_1 & 0 & 0 \\ 0 & -\mu_1-\mu_2-\mu_3 & 0 & 0 & 0 \\ 0 & 0 & -\mu_1-\lambda_2-\lambda_3 & 0 & 0 \\ 0 & \mu_1 & 0 & \dots 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_2 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & \mu_3 & \lambda_2 & 0 & -1 \end{vmatrix} +$$

$$+\begin{vmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -\mu_1-\mu_2-\mu_3 & 0 & \lambda_1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & \mu_1 & 0 & -\lambda_1-\mu_2-\mu_3 \dots & 0 \\ 0 & 0 & 0 & \mu_3 & 0 \\ 0 & \mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_2 & 0 \\ 0 & \mu_3 & 0 & 0 & -1 \end{vmatrix} + \begin{vmatrix} -1 & 0 & 0 & 0 & \mu_2 \\ 0 & -\mu_1-\mu_2-\mu_3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & \mu_1 & 0 & -1 & \lambda_3 \dots \\ 0 & 0 & 0 & 0 & -\lambda_1-\mu_2-\lambda_3 \\ 0 & \mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_3 & 0 & 0 & \lambda_1 \end{vmatrix} +$$

$$+\begin{vmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -\mu_1-\mu_2-\mu_3 & \lambda_2 & 0 & 0 \\ 0 & 0 & \mu_3 & 0 & 0 \\ 0 & \mu_1 & \dots 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_2 & -\mu_1-\lambda_2-\mu_3 & 0 & 0 \\ 0 & 0 & \mu_1 & -1 & 0 \\ 0 & \mu_3 & 0 & 0 & -1 \end{vmatrix} + \begin{vmatrix} -1 & 0 & 0 & \mu_3 & 0 \\ 0 & -\mu_1-\mu_2-\mu_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_1 & \dots 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_2 & -1 & \lambda_1 & 0 \\ 0 & 0 & 0 & -\lambda_1-\lambda_2-\mu_3 & 0 \\ 0 & \mu_3 & 0 & 0 & -1 \end{vmatrix} +$$

$$+\begin{vmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -\mu_1-\mu_2-\mu_3 & 0 & 0 & \lambda_3 \\ 0 & 0 & 0 & 0 & \mu_2 \\ 0 & \mu_1 & \dots 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_1 \\ 0 & \mu_2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & \mu_3 & 0 & 0 & -\mu_1-\mu_2-\lambda_3 \end{vmatrix} + \begin{vmatrix} -1 & 0 & \mu_1 & 0 & 0 \\ 0 & -1 & 0 & \lambda_1 & 0 \\ 0 & 0 & -\mu_1-\lambda_2-\lambda_3 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_1-\mu_2-\mu_3 \dots & 0 \\ 0 & 0 & 0 & \mu_3 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & \mu_2 & 0 \\ 0 & 0 & \lambda_2 & 0 & -1 \end{vmatrix} +$$

$$+\begin{vmatrix} -1 & 0 & \mu_1 & 0 & \mu_2 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -\mu_1-\lambda_2-\lambda_3 & 0 & 0 \\ 0 & 0 & 0 & -1 & \lambda_3 \dots \\ 0 & 0 & 0 & 0 & -\lambda_1-\mu_2-\lambda_3 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & \lambda_1 \end{vmatrix} + \begin{vmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & \lambda_2 & 0 & 0 \\ 0 & 0 & \mu_3 & 0 & 0 \\ 0 & 0 \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu_1-\lambda_2-\mu_3 & 0 & 0 \\ 0 & 0 & \mu_1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{vmatrix} +$$

$$+\begin{vmatrix} -1 & 0 & 0 & \mu_3 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 \dots & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & \lambda_1 & 0 \\ 0 & 0 & 0 & -\lambda_1-\lambda_2-\mu_3 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{vmatrix} + \begin{vmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & \lambda_3 \\ 0 & 0 & 0 & 0 & \mu_2 \\ 0 & 0 \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -\mu_1-\mu_2-\lambda_3 \end{vmatrix} +$$

$$+\begin{vmatrix} -1 & 0 & 0 & 0 & \mu_2 \\ 0 & -1 & 0 & \lambda_1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_1-\mu_2-\mu_3 & \lambda_3 \dots \\ 0 & 0 & 0 & \mu_3 & -\lambda_1-\mu_2-\lambda_3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \mu_3 & 0 & 0 \\ \dots -\lambda_1-\mu_2-\mu_3 & 0 & 0 & 0 & 0 \\ \mu_3 & -1 & 0 & 0 & 0 \\ 0 & 0 & -\mu_1-\lambda_2-\mu_3 & 0 & 0 \\ \mu_2 & 0 & \mu_1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{vmatrix} +$$

$$+\begin{vmatrix} 0 & 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \mu_3 & 0 & 0 \\ \dots -\lambda_1-\mu_2-\mu_3 & 0 & 0 & 0 & 0 \\ \mu_3 & -1 & 0 & 0 & 0 \\ 0 & 0 & -\mu_1-\lambda_2-\mu_3 & 0 & 0 \\ \mu_2 & 0 & \mu_1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 0 & \mu_3 & 0 \\ \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \dots -\lambda_1-\mu_2-\mu_3 & 0 & 0 & \lambda_2 & 0 \\ \mu_3 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & \lambda_1 & 0 \\ \mu_2 & 0 & 0 & -\lambda_1-\lambda_2-\mu_3 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{vmatrix} +$$

$$+\begin{vmatrix} 0 & 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 & \lambda_3 \\ 0 & 0 & 0 & 0 & \mu_2 \\ \dots -\lambda_1-\mu_2-\mu_3 & 0 & 0 & 0 & 0 \\ \mu_3 & -1 & 0 & 0 & \mu_1 \\ 0 & 0 & -1 & 0 & 0 \\ \mu_2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -\mu_1-\mu_2-\lambda_3 \end{vmatrix} + \begin{vmatrix} -1 & \mu_2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \mu_3 & 0 & 0 \\ 0 \dots & \lambda_3 & 0 & 0 & 0 \\ 0 & -\lambda_1-\mu_2-\lambda_3 & 0 & 0 & 0 \\ 0 & 0 & -\mu_1-\lambda_2-\mu_3 & 0 & 0 \\ 0 & 0 & \mu_1 & -1 & 0 \\ 0 & \lambda_1 & 0 & 0 & -1 \end{vmatrix} +$$

$$+\begin{vmatrix} -1 & \mu_2 & 0 & \mu_3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 \dots & \lambda_3 & 0 & \lambda_2 & 0 \\ 0 & -\lambda_1-\mu_2-\lambda_3 & 0 & 0 & 0 \\ 0 & 0 & -1 & \lambda_1 & 0 \\ 0 & 0 & 0 & -\lambda_1-\lambda_2-\mu_3 & 0 \\ 0 & \lambda_1 & 0 & 0 & -1 \end{vmatrix} + \begin{vmatrix} -1 & \mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3 \\ 0 & 0 & 0 & 0 & \mu_2 \\ 0 \dots & \lambda_3 & 0 & 0 & 0 \\ 0 & -\lambda_1-\mu_2-\lambda_3 & 0 & 0 & \mu_1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & \lambda_1 & 0 & 0 & -\mu_1-\mu_2-\lambda_3 \end{vmatrix} +$$

$$+\begin{vmatrix} -1 & 0 & 0 & \mu_3 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \mu_3 & 0 & 0 \\ 0 & \dots 0 & 0 & \lambda_2 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -\mu_1-\lambda_2-\mu_3 & \lambda_1 & 0 \\ 0 & 0 & \mu_1 & -\lambda_1-\lambda_2-\mu_3 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{vmatrix} + \begin{vmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & \lambda_3 \\ 0 & 0 & \mu_3 & 0 & \mu_2 \\ 0 & \dots 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & \mu_1 \\ 0 & 0 & -\mu_1-\lambda_2-\mu_3 & 0 & 0 \\ 0 & 0 & \mu_1 & -1 & 0 \\ 0 & \lambda_1 & 0 & 0 & -\mu_1-\mu_2-\lambda_3 \end{vmatrix} +$$

$$+\begin{vmatrix} -1 & 0 & 0 & \mu_3 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3 \\ 0 & 0 & 0 & 0 & \mu_2 \\ 0 & \dots 0 & 0 & \lambda_2 & 0 \\ 0 & -1 & 0 & 0 & \mu_1 \\ 0 & 0 & -1 & \lambda_1 & 0 \\ 0 & 0 & 0 & -\lambda_1-\lambda_2-\mu_3 & 0 \\ 0 & 0 & 0 & 0 & -\mu_1-\mu_2-\lambda_3 \end{vmatrix},$$

$$a_0 = \det(R).$$

The characteristic equation of the triple system is an algebraic equation of the eighth power in a special form:

$$\sum_{m=0}^{8} a_{8-m} \nu^{8-m} = 0. \tag{5}$$

The eight roots of this equation ν_i ($i=1, 2, 3,..., 8$) are subject to analytical determination.

5. Results of investigating the method of analytical solutions to Kolmogorov equations of the eighth order and calculation of power system reliability assessment

5. 1. Analytical solution to the characteristic equation of the triple system

From the analysis of the analytical solution to the characteristic equation of the binominal system [8], it follows that one root is zero, the others are real, different, negative, and distributed in the complex plane in a symmetrical manner with respect to the center:

$$M(\nu_i) = \frac{\sum_{i=0}^{4} \nu_i}{4}, \tag{6}$$

or

$$M(\nu_i) = \frac{\sum_{j=1}^{2} (\lambda_j + \mu_j)}{2}, \tag{7}$$

$$M(\nu_i) = -\frac{a_3}{4}. \tag{8}$$

It was established that the distribution of roots of the characteristic equation of the triple system in the complex plane has a similar property of symmetry:

$$M(\nu_i) = -\frac{a_7}{8}, \tag{9}$$

or

$$M(\nu_i) = \frac{\sum_{j=1}^{3} (\lambda_j + \mu_j)}{2}, \tag{10}$$

that is

$$M(\nu_i) = \frac{\sum_{i=0}^{8} \nu_i}{8}. \tag{11}$$

The central symmetry in the distribution of eight roots of the characteristic equation, one of which is zero, the others are real, different, negative, is illustrated in Fig. 3.

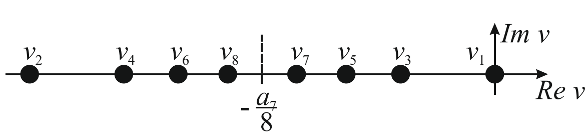

Fig. 3. Scheme of the harmonic distribution of eight roots of the characteristic Kolmogorov equation for a triple system

Hence, the analytical solution to the characteristic Kolmogorov equation for the triple system in harmonic form:

$$\begin{aligned}
\nu_1 &= 0,\\
\nu_3 &= -(\lambda_1+\mu_1),\\
\nu_5 &= \qquad\qquad -(\lambda_2+\mu_2),\\
\nu_7 &= \qquad\qquad\qquad\qquad -(\lambda_3+\mu_3),\\[6pt]
\nu_2 &= -(\lambda_1+\mu_1)-(\lambda_2+\mu_2)-(\lambda_3+\mu_3),\\
\nu_4 &= \qquad\qquad -(\lambda_2+\mu_2)-(\lambda_3+\mu_3),\\
\nu_6 &= -(\lambda_1+\mu_1) \qquad\qquad -(\lambda_3+\mu_3),\\
\nu_8 &= -(\lambda_1+\mu_1)-(\lambda_2+\mu_2).
\end{aligned} \tag{12}$$

Using Vieta's theorem allowed us to obtain the expression:

$$\begin{aligned}
&a_7 = \nu_1+\nu_2+\nu_3+\nu_4+\nu_5+\nu_6+\nu_7+\nu_8,\\
&a_6 = \nu_1\nu_2+\nu_1\nu_3+\nu_1\nu_4+\nu_1\nu_5+\nu_1\nu_6+\nu_1\nu_7+\nu_1\nu_8+\\
&+\nu_2\nu_3+\nu_2\nu_4+\nu_2\nu_5+\nu_2\nu_6+\nu_2\nu_7+\nu_2\nu_8+\nu_3\nu_4+\\
&+\nu_3\nu_5+\nu_3\nu_6+\nu_3\nu_7+\nu_3\nu_8+\nu_4\nu_5+\nu_4\nu_6+\nu_4\nu_7+\\
&+\nu_4\nu_8+\nu_5\nu_6+\nu_5\nu_7+\nu_5\nu_8+\nu_6\nu_7+\nu_6\nu_8+\nu_7\nu_8,\\
&\dots\dots\dots\dots\\
&a_0 = \nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7\nu_8,
\end{aligned} \tag{13}$$

or by a direct substitution that the given set of roots is an analytical solution to the original characteristic Kolmogorov equation for the considered triple system.

5. 2. Analytical solution to the system of Kolmogorov differential equations for the triple system

Following the results of work [9], according to the found real and different roots, the analytical solution to the analyzed Kolmogorov differential equations is given in the following harmonic form, using determinants ordered by the indices of the eight states of the system and the indices of the eight roots:

$$P_i(t) = \sum_{k=1}^{8} \frac{\Delta_i(\nu_k)}{\prod_{\substack{s=1\\k\neq s}}^{8} (\nu_k - \nu_s)} e^{\nu_k t}, \tag{14}$$

$$\Delta_i(\nu_k) =$$

$$= \begin{vmatrix}
-\lambda_1-\lambda_2-\lambda_3-\nu_k & \cdots & -P_1(0) & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
\cdots & \cdots & -P_i(0) & \cdots & \cdots\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
0 & \cdots & -P_8(0) & \cdots & -\mu_1-\mu_2-\lambda_3-\nu_k
\end{vmatrix}, \tag{15}$$

$(i=1, 2, 3,..., 8)$.

Here, the ordered determinant $\Delta_i(\nu_k)$ corresponds to the probability of the ith state of the system $P_i(t)$. It is formed by the characteristic determinant $\det(R-\nu E)$ of the square matrix of intensities of failure-recovery flows R of the eighth order, which contains the given initial conditions in the i-th column $P_i(0)$ ($i=1, 2, 3,..., 8$) and depends on the k-th root ν_k ($k=1, 2, 3,..., 8$).

5. 3. Results of energy system reliability assessment

The illustration of the obtained analytical results is carried out on the example of the time reliability assessment of the energy system in a hypothetical region of Ukraine, which has thermal power plants (TPP), nuclear power plants (NPP), hydroelectric power plants (HPP) as sources of electrical energy. The considered energy system includes three independent generating subsystems:

1. TPP.
2. NPP.
3. HPP.

The possible states of each of the generating subsystems are known as ⊕ – operational or ⊖ – failed. The intensities of the Poisson flows of failures $(\lambda_1, \lambda_2, \lambda_3)$ and recovery (μ_1, μ_2, μ_3) of each of the generating subsystems are considered to be statistically established and equal:

$$\lambda_1 = 2;\ \lambda_2 = 1;\ \lambda_3 = 0.5;$$

$$\mu_1 = 1.5;\ \mu_2 = 0.5;\ \mu_3 = 2. \quad (16)$$

At the initial time point (t=0), the state of the power system is considered known and is taken as:

$$P_1(0) = 1;\ P_3(0) = 0;$$

$$P_5(0) = 0;\ P_7(0) = 0;$$

$$P_2(0) = 0;\ P_4(0) = 0;$$

$$P_6(0) = 0;\ P_8(0) = 0. \quad (17)$$

That is, three generating subsystems are operational. It is necessary to establish the expected state of the power system in the future, that is, at $t \to \infty$:

$$P_1(\infty);\ P_3(\infty);$$

$$P_5(\infty);\ P_7(\infty);$$

$$P_2(\infty);\ P_4(\infty);$$

$$P_6(\infty);\ P_8(\infty). \quad (18)$$

The roots of the characteristic equation take the following values:

$$\begin{aligned}
&\nu_1 = 0,\\
&\nu_3 = -3.5,\\
&\nu_5 = -1.5,\\
&\nu_7 = -2.5,\\
&\nu_2 = -3.5 - 1.5 - 2.5 = -7.5,\\
&\nu_4 = -1.5 - 2.5 = -4,\\
&\nu_6 = -3.5 - 2.5 = -6,\\
&\nu_8 = -3.5 - 1.5 = -5. \quad (19)
\end{aligned}$$

It is obvious that:

$$e^{v_1 t} = 1,\ \lim_{t\to\infty} e^{v_k t} = 0\ (k = 2,3,4,5,6,7,8), \quad (20)$$

which allows us to significantly simplify the calculation formulas:

$$\prod_{s=2}^{8}(\nu_1 - v_s) = 7.5\cdot 3.5\cdot 4\cdot 1.5\cdot 6\cdot 2.5\cdot 5,$$

$$P_1(\infty) = \frac{1}{\prod_{s=2}^{8}(\nu_1 - v_s)} \times$$

$$\times \begin{vmatrix}
-1 & 0 & \mu_1 & 0 & \mu_2 & 0 & \mu_3 & 0\\
0 & -(\mu_1+\mu_2+\mu_3) & 0 & \lambda_1 & 0 & \lambda_2 & 0 & \lambda_3\\
0 & 0 & -(\mu_1+\lambda_2+\lambda_3) & 0 & 0 & \mu_3 & 0 & \mu_2\\
0 & \mu_1 & 0 & -(\lambda_1+\mu_2+\mu_3) & \lambda_3 & 0 & \lambda_2 & 0\\
0 & 0 & 0 & \mu_3 & -(\lambda_1+\mu_2+\lambda_3) & 0 & 0 & \mu_1\\
0 & \mu_2 & \lambda_3 & 0 & 0 & -(\mu_1+\lambda_2+\mu_3) & \lambda_1 & 0\\
0 & 0 & 0 & \mu_2 & 0 & \mu_1 & -(\lambda_1+\lambda_2+\mu_3) & 0\\
0 & \mu_3 & \lambda_2 & 0 & \lambda_1 & 0 & 0 & -(\mu_1+\mu_2+\lambda_3)
\end{vmatrix},$$

$$P_2(\infty) = \frac{1}{\prod_{s=2}^{8}(\nu_1 - v_s)} \times$$

$$\times \begin{vmatrix}
-(\lambda_1+\lambda_2+\lambda_3) & -1 & \mu_1 & 0 & \mu_2 & 0 & \mu_3 & 0\\
0 & 0 & 0 & \lambda_1 & 0 & \lambda_2 & 0 & \lambda_3\\
\lambda_1 & 0 & -(\mu_1+\lambda_2+\lambda_3) & 0 & 0 & \mu_3 & 0 & \mu_2\\
0 & 0 & 0 & -(\lambda_1+\mu_2+\mu_3) & \lambda_3 & 0 & \lambda_2 & 0\\
\lambda_2 & 0 & 0 & \mu_3 & -(\lambda_1+\mu_2+\lambda_3) & 0 & 0 & \mu_1\\
0 & 0 & \lambda_3 & 0 & 0 & -(\mu_1+\lambda_2+\mu_3) & \lambda_1 & 0\\
\lambda_3 & 0 & 0 & \mu_2 & 0 & \mu_1 & -(\lambda_1+\lambda_2+\mu_3) & 0\\
0 & 0 & \lambda_2 & 0 & \lambda_1 & 0 & 0 & -(\mu_1+\mu_2+\lambda_3)
\end{vmatrix},$$

$$P_3(\infty) = \frac{1}{\prod_{s=2}^{8}(\nu_1 - v_s)} \times$$

$$\times \begin{vmatrix}
-(\lambda_1+\lambda_2+\lambda_3) & 0 & -1 & 0 & \mu_2 & 0 & \mu_3 & 0\\
0 & -(\mu_1+\mu_2+\mu_3) & 0 & \lambda_1 & 0 & \lambda_2 & 0 & \lambda_3\\
\lambda_1 & 0 & 0 & 0 & 0 & \mu_3 & 0 & \mu_2\\
0 & \mu_1 & 0 & -(\lambda_1+\mu_2+\mu_3) & \lambda_3 & 0 & \lambda_2 & 0\\
\lambda_2 & 0 & 0 & \mu_3 & -(\lambda_1+\mu_2+\lambda_3) & 0 & 0 & \mu_1\\
0 & \mu_2 & 0 & 0 & 0 & -(\mu_1+\lambda_2+\mu_3) & \lambda_1 & 0\\
\lambda_3 & 0 & 0 & \mu_2 & 0 & \mu_1 & -(\lambda_1+\lambda_2+\mu_3) & 0\\
0 & \mu_3 & 0 & 0 & \lambda_1 & 0 & 0 & -(\mu_1+\mu_2+\lambda_3)
\end{vmatrix},$$

$$P_4(\infty) = \frac{1}{\prod_{s=2}^{8}(\nu_1 - v_s)} \times$$

$$\times \begin{vmatrix}
-(\lambda_1+\lambda_2+\lambda_3) & 0 & \mu_1 & -1 & \mu_2 & 0 & \mu_3 & 0\\
0 & -(\mu_1+\mu_2+\mu_3) & 0 & 0 & 0 & \lambda_2 & 0 & \lambda_3\\
\lambda_1 & 0 & -(\mu_1+\lambda_2+\lambda_3) & 0 & 0 & \mu_3 & 0 & \mu_2\\
0 & \mu_1 & 0 & 0 & \lambda_3 & 0 & \lambda_2 & 0\\
\lambda_2 & 0 & 0 & 0 & -(\lambda_1+\mu_2+\lambda_3) & 0 & 0 & \mu_1\\
0 & \mu_2 & \lambda_3 & 0 & 0 & -(\mu_1+\lambda_2+\mu_3) & \lambda_1 & 0\\
\lambda_3 & 0 & 0 & 0 & 0 & \mu_1 & -(\lambda_1+\lambda_2+\mu_3) & 0\\
0 & \mu_3 & \lambda_2 & 0 & \lambda_1 & 0 & 0 & -(\mu_1+\mu_2+\lambda_3)
\end{vmatrix},$$

One similarly finds $P_5(\infty)$, $P_6(\infty)$, $P_7(\infty)$, $P_8(\infty)$.

Determinants are filled with numbers and calculated using standard programs:

$$P_1(\infty)=0.114;\ P_3(\infty)=0.152;$$

$$P_5(\infty)=0.229;\ P_7(\infty)=0.029;$$

$$P_2(\infty)=0.076;\ P_4(\infty)=0.057;$$

$$P_6(\infty)=0.038;\ P_8(\infty)=0.305. \quad (21)$$

Verification of calculation results:

$$\sum_{i=1}^{8} P_i(\infty)=1, \quad (22)$$

$$0.114+0.076+0.152+0.057+$$

$$+0.0229+0.038+0.029+0.305=1.$$

The most probable states in the future are the eighth or fifth, i.e., TPP and NPP are in failure state, HPP is operational, $P_8(\infty)=0.305$.

Or the NPP is out of order, TPP and HPP are operational, $P_5(\infty)=0.229$.

6. Discussion of results based on the analytical method for solving Kolmogorov equations of the eighth order for an asymmetric Markov chain

An asymmetric Markov chain with eight states and the corresponding Kolmogorov differential equations of the eighth order are a mathematical model of a number of applied problems in various fields.

Analogous Kolmogorov equations of the fourth order are considered in [8], in which the formula for the partial solution of an algebraic equation of the fourth power is given and the pattern in the distribution of four roots in the complex plane is shown in a figure. The results of solving the Kolmogorov equations of the eighth order for the asymmetric Markov chain are explained in Fig. 2 and formula (2). A similar regularity in the distribution of roots (central symmetry) is established for Kolmogorov equations of the eighth order describing a symmetric Markov chain with eight discrete states explained by formulas (12) to (14).

Traditionally, systems of high-order differential equations are studied by approximate numerical methods in a limited range of parameter changes. Analytical solutions similar to those obtained for the discussed Kolmogorov equations of the eighth order could not be found in the literature that we know of.

The peculiarity of the proposed method and the obtained analytical results is caused by the use of symmetrization of the mathematical description.

The limitations of the study are related to the dimensionally of the studied system, which is determined by the formula 2^n, where n is the number of subsystems, and the cumbersomeness of the analytical presentation of the results.

The disadvantage of this study is the cumbersomeness of the analytical presentation of the results, which is overcome by the use of symmetrization of the mathematical description, in particular, the introduction of special matrices and ordered determinants.

The development of this scientific area implies considering complex systems of high order: $2^4=16$, $2^5=32$, $2^6=64$, etc. The problems of this area of research are related to the mathematical description of these systems and representation of analytical results.

7. Conclusions

1. Based on the concept of harmonization, an algorithm for analytical modeling of random processes in a continuous, homogeneous Markov chain of an asymmetric structure and eight states was formulated using the method of mathematical induction. The asymmetric structure of a continuous Markov chain imitating a triple system is reflected in the matrix form of eighth-order Kolmogorov differential equations. Using elementary symmetric polynomials, the characteristic determinant of the Kolmogorov equations is transformed into an algebraic polynomial of the eighth power, the coefficients of which reflect the asymmetric structure of the state graph of the triple system in the intensity of failure and restoration flows. The property of central symmetry in the distribution of eight roots of the characteristic equation of a continuous Markov chain of an asymmetric structure has been established. Relying on the property of central symmetry, formulas expressing the eight roots of the characteristic Kolmogorov equation in the intensity of failure and restoration flows were derived.

2. An analytical solution to eighth-order Kolmogorov differential equations for an asymmetric state graph is proposed in harmonic form for the purpose of analysis and synthesis of a random Markov process in a triple system.

3. Analytical results of the time reliability assessment of the energy system, which has TPP, NPP, and HPP as a source of electric energy, were obtained.

Conflicts of interest

The authors declare that they have no conflicts of interest in relation to the current study, including financial, personal, authorship, or any other, that could affect the study, as well as the results reported in this paper.

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Data availability

All data are available, either in numerical or graphical form, in the main text of the manuscript.

Use of artificial intelligence

The authors confirm that they did not use artificial intelligence technologies when creating the current work.

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