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DEVELOPMENT OF MATHEMATICAL MODELS OF HEAT CONDUCTIVITY FOR MODERN ELECTRONIC DEVICES WITH ELEMENTS CONTAINING FOREIGN INCLUSIONS

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This paper considers the heat conduction process for an isotropic medium containing a foreign half-through inclusion and heated by a locally concentrated heat flow. Linear and non-linear mathematical models for determining the temperature field have been built to establish the temperature regimes for the effective operation of electronic devices. The coefficient of thermal conductivity of a non-uniform structure is represented as a whole, using asymmetric unit functions, which automatically provides the conditions of ideal thermal contact on the surfaces of materials. This results in solving one heat conduction equation with discontinuous and singular coefficients. A linearizing function was introduced to linearize the nonlinear boundary value problem. Analytical-numerical solutions of linear and nonlinear boundary-value problems have been obtained in a closed form. A linear temperature dependence of the coefficient of thermal conductivity of structural materials was chosen for a heat-sensitive medium. As a result, an analytical-numerical solution was derived, which determines the temperature distribution in this medium. On this basis, a numerical experiment was performed, the results of which are graphically displayed and confirm the adequacy of the constructed mathematical models to a real physical process.

The materials of the plate and inclusion are silicon and silver. The results for these materials based on the linear and non-linear model differ by 7 %. Their slight difference is explained by the fact that the values of the temperature coefficient of thermal conductivity are small. The models built make it possible to analyze the given environments in terms of their thermal resistance. As a result, it becomes possible to improve it, and protect structures from overheating, which could lead to the failure of individual nodes and their elements and the entire electronic device

Keywords: thermal resistance of the structure, foreign half-through inclusion, ideal thermal contact, convective heat exchange

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1. Introduction

Thermal conductivity in microelectronic devices plays an important role in enabling stable operation and their durability. Thermal processes can affect the efficiency, speed, and accuracy of functioning of microelectronic devices, so technologies that provide high thermal conductivity are of great importance and wide application. The speed of heat propagation from one place to another in materials is determined with the use of thermal conductivity methods. In microelectronic devices that function intensively in terms of their speed and high level of accuracy, a significant amount of heat is generated in individual nodes and their elements. As a result, heat generation is a significant factor that leads to a decrease in the reliability of microelectronic devices. The relative influence of temperature is the highest (55 %) compared to humidity (19 %), vibration (20 %), and dust (6 %) [1]. If there is no effective heat dissipation, the device overheats and becomes unusable. One of the main ways to ensure high thermal conductivity in microelectronic devices is the use of materials with high thermal conductivity. Metals, particularly copper and aluminum, which have high thermal conductivity, and silicon-based materials such as silicon and silicon oxide, which have lower thermal conductivity, are widely used. However, silicon-based materials have other important properties, due to which they are chosen for the design of electronic devices. However, in practice, the thermal conductivity of materials can significantly decrease as a result of the influence of various factors, such as geometric parameters, structure, and microstructure of materials. A large number of microstructural defects can significantly reduce thermal conductivity, which leads to overheating of individual nodes and elements of microelectronic devices. One approach to ensure high thermal conductivity in microelectronic devices is to use materials with a low melting point. It can be indium and gallium, which have high thermal conductivity and can be used to coat metal surfaces. Another way is to use technologies to reduce the geometric parameters of microelectronic devices, which makes it possible to increase their thermal capacity and reduce the density of heat losses. This is achieved with the help of nanotechnology and microelectromechanical systems. In addition, effective thermal management can be achieved through the design of microelectronic devices by considering component placement, thermal junction design, and the use of thermal interfaces. The use of thermal materials, such as thermal paste, allows for efficient heat transfer between components, reducing the risk of overheating electronic devices.

As a result, construction of mathematical models of the heat conduction process is an urgent problem since as a result of the operation of modern electronic devices, they are exposed to thermal loads. Due to the heterogeneity of environments and the intensity of heating, significant temperature gradients occur, which contribute to overheating, which leads to the failure of individual elements and assemblies, as well as the device in general. To prevent this, it is necessary to establish acceptable temperature regimes for the effective operation of devices. Without conducting expensive experiments for heterogeneous environments, our research results make it possible to achieve this.

2. Literature review and problem statement

In [2], a method for modeling heat transfer in porous materials with temperature-dependent properties, which is relevant for structures of complex architecture, is considered. This approach can be applied to electronics, as modern electronic devices with components containing foreign half-through inclusions face similar challenges in the field of thermal conductivity. It is important to construct mathematical models based on analytical and numerical methods for predicting the thermal behavior of such devices, which could contribute to their more efficient operation and increased reliability. The use of the modeling method does not make it possible to take into account local thermal disturbances, which often occur in devices with foreign half-through inclusions.

Analytical solutions are given in [3] for the distribution of temperature, displacements, and stresses in layered rectangular plates with a simple support subjected to thermomechanical loads. The properties of the materials of the layers take into account the temperature dependence. The given analytical solutions do not describe local thermomechanical loads, which limits their application in problems with real operating conditions.

Reconstruction of the temperature field from limited observations is important for thermal regulation of electronic equipment. To solve such a problem, a deep learning method combining an adaptive UNet and a small multilayer perceptron (MLP) is reported in [4]. The method makes it possible to transform the problem of temperature field reconstruction into the problem of image-to-image regression. Adaptive UNet reconstructs the general temperature field, while MLP specializes in accurately predicting zones with large temperature gradients. The results of numerical experiments performed using finite element simulation data show that the maximum absolute errors of the reconstructed temperature field are less than 1 °K. The method was also tested for different locations of heat sources and observation points. The disadvantage of this approach is the need for a significant amount of data for training the model, which is not easy to provide under real conditions.

Thermomechanical loads of fixed columns for longitudinal thermal heating with various boundary conditions were analyzed in [5]. The temperature distribution is determined by the differential quadrature method (DQM). A segmental model of a column with a uniform temperature distribution is used to analyze the deflection. The critical load and deflection mode are determined by the transfer matrix method based on the Euler–Bernoulli theory. The results are confirmed by comparison with literature data and FEM. The influence of temperature and material properties on deflection and critical load was investigated. The main drawback of the given approach is the simplified model for determining the temperature distribution, which does not take into account the emergence of significant temperature gradients as a result of the critical temperature load.

Paper [6] presents the basic equations and data set of the thermal model for predicting temperature fields and heating rates when applying localized laser treatments to the Fe-C-Ni alloy. The model takes into account the transient properties of the material and the relationship between temperature and microstructure with an emphasis on the phase dependence of thermal parameters and hysteresis in the phase change. The model provides temperature fields that are consistent with experimental microstructures in the zones of laser exposure. The given model can be applied to other materials that exhibit solid-state transformations during laser processing. Thermophysical parameters are averaged, which leads to errors in the reported results.

In paper [7], a temperature field model was built for controlling the shape of a steel plate during roller hardening. The cooling mechanism was analyzed and the heat transfer coefficients for each surface were obtained. The model is based on the equation of thermal conductivity, which makes it possible to investigate the uniformity of cooling of the plate. A plate shape control structure was designed and tested experimentally. However, the results show certain errors when modeling for a homogeneous environment.

Work [8] investigated the influence of control parameters on dimensionless speed, temperature, skin friction and local heat exchange rate for two thermal boundary conditions: Newtonian heating and convection. The thermophysical properties of the liquid remain constant throughout the study for a constant temperature of the plate surface. Geometric mapping makes it possible to analyze the behavior of heat flow and temperature distribution in relation to the influence of dimensionless parameters. Studies confirm the influence of boundary conditions on the rate of heat transfer, with Newtonian heating increasing the rate and convective heating causing it to decrease. This is due to the heating at the boundary in Newtonian heating, which improves the transfer of thermal energy, unlike convective heating. As a result, the heat is dissipated due to the moving fluid, which limits the transfer rate. The thermophysical properties of the liquid are assumed to be constant, which does not reflect the real conditions in the heat treatment process. The thermophysical parameters of the liquid may depend on temperature and other factors, and failure to take this into account in the model may lead to the emergence of significant errors in the research results.

Thermal modeling of electronic devices is one of the most important tools for assessing their reliability under various operating modes. In [9], a thermal model of electronic devices is presented, which is based on experimental temperature measurement data obtained by an infrared camera. These data are input to the constructed mathematical model, which is based on the finite difference method and some known physical dependences. The model built was verified by comparing simulation data with experimental data. It can be used to study the thermal behavior of the device under various operating conditions. The temperature distribution is determined experimentally, which introduces an error into the developed mathematical model based on the finite difference method. As a result, the results contain significant errors.

Paper [10] proposed dynamic compact thermal models for predicting the temperature of the body of portable devices, in particular, a smartphone and a laptop, based on the convolution method. The models allow rapid determination of case temperature by accounting for the step response of each heat source but are limited to two devices and are experimental. The model contributes to the improvement of thermal design and the determination of temperature control strategies in the early stages of development. The model built is experimental and does not make it possible to determine the temperature regimes for more than two portable electronic devices.

The solution for the steady-state reaction of thick cylinders subjected to pressure and external heat flow on the inner surface is reported in [11]. The influence of the temperature gradient on the deformations of the environment is not taken into account, which significantly worsens the accuracy of the model.

A functional defect causes an increase in temperature and thermal stresses in thermoelectric materials, which reduces the reliability of devices. In [12], thermoelectric-elastic fields around an elliptical defect in a two-dimensional thermoelectric plate were investigated using the complex variable method. The results show that the temperature at the tip of the defect increases with its size and can exceed the material's melting point, and stresses can exceed the yield strength. This is important for material failure analysis.

A thermal analysis of cylinders of different thickness, made of functionally graded materials, which are under the influence of heterogeneous heat flows concentrated on the inner and outer layers was performed in [13, 14]. The studies do not make it possible to analyze the thermal state of the cylinders for local thermal disturbance.

Functionally graded materials with a continuous change in properties are useful for thermal protection and biomedical applications. In the case of a thin coating on the substrate, the usual mesh discretization is ineffective. The devised method [15] of approximate transfer uses the concept of finite differences to transfer boundary conditions from the coating to the substrate. This makes it possible to numerically consider only the substrate with convection conditions using a hybrid finite element method. The method has been tested for different types of coatings and can be used to build models of thermal conductivity in electronic devices containing separate assemblies and their elements with semi-through inclusions. Using the concept of finite differences to transfer boundary conditions can limit the accuracy of numerical calculations, especially for complex systems with continuously changing properties.

In work [16], the authors simplify the nonlinear three-dimensional problem of thermal conductivity, reducing it to the Laplace equation using an intermediate function. A generalized ternary function is proposed, and a general solution to the Laplace equation is derived. Three specific problems are analyzed: it is shown that the heat flux of the nonlinear problem coincides with the results for the linear problem, while the temperature field differs. At the flat border of the defect, the heat flow has a singularity, and its intensity is proportional to the root of the fourth power from the width of the defect. The disadvantage of this approach is that the simplification of the nonlinear problem to the Laplace equation can lead to a loss of accuracy in determining the temperature field, since the nonlinearity inherent in the original problem is not always an adequately reproduced linear model.

Existing methods have been improved and new approaches have been devised to construct mathematical models that make it possible to analyze heat exchange in piecewise homogeneous media [17]. Planar and spatial models of heat exchange are given, in which the differential equations contain coefficients dependent on the thermophysical properties of the phases and the geometric structure. Approaches for determining analytical and analytical-numerical solutions to boundary value problems of thermal conductivity are reported in [18]. Heat exchange processes occurring in homogeneous and layered structures with foreign inclusions of canonical form were analyzed in [19].

Linear and nonlinear mathematical models have been built for determining the temperature field and analyzing temperature regimes in isotropic environments with local thermal heating [20]. Analytical solutions were derived and algorithms for the numerical implementation of the temperature distribution by spatial coordinates were developed. The results make it possible to analyze heat exchange processes and increase the thermal resistance of structures.

In the cited works [17–20], models that take into account local heating, the heterogeneity of environments, and the thermal sensitivity of their structural materials have remained insufficiently researched. The use of classical analytical and numerical methods does not make it possible to effectively take into account the given factors for individual elements and nodes of electronic device designs. Therefore, a technique for constructing mathematical models of thermal conductivity, in which these factors are taken into account, is given.

Our review of the literature reveals that there is not a strict, logical, theoretically grounded technique for building linear and nonlinear mathematical models of thermal conductivity for heterogeneous thermosensitive media. The components of modern digital devices are small in size and contain significant thermal power. Therefore, in research, it is important to take into account the heterogeneity of these elements, the thermal sensitivity of structural materials, and the locality of thermal heating.

3. The aim and objectives of the study

The aim of our study is to construct linear and nonlinear mathematical models of thermal conductivity for isotropic media that contain foreign semi-transparent inclusions and are subject to local external heating. As a result, there is an opportunity to increase the accuracy of determining temperature fields, which would further affect the effectiveness of methods for designing modern electronic devices.

To achieve this goal, the following tasks must be solved:

– to build a linear mathematical model for determining the temperature field in a medium with a foreign semi-transparent inclusion heated by a locally concentrated heat flow;

– to construct a non-linear mathematical model for determining the temperature field in a thermosensitive (the thermophysical parameters of the material depend on the temperature) environment with a foreign half-through inclusion heated by a locally concentrated heat flow.

4. The study materials and methods

The object of research is the heat conduction process for isotropic media with foreign semi-transparent inclusions, which are heated by a locally concentrated heat flow.

Our hypothesis: the study was carried out within the framework of the classical theory of thermal conductivity.

The following assumptions and simplifications were accepted in the research process: the media are not anisotropic, that is, the values of thermophysical parameters are constant in spatial directions. The process of heat conduction is stationary since the change in the temperature field is determined only by spatial coordinates.

An isotropic layer containing a foreign half-through cylindrical inclusion with a radius R assigned to the cylindrical coordinate system ($Or\varphi z$) is given. On the boundary surface of the layer L_+ in the region of inclusion $\{(r, \varphi, h)$: $0 \le r \le R$, $0 \le \varphi \le 2\pi\}$ the given structure is heated by a concentrated heat flux, the surface density of which is q_0=const, and on the other surface of the layer $L_-=\{(r, \varphi, -l)$: $0 \le r < \mu$, $0 \le \varphi \le 2\pi\}$ there is a convective heat exchange with the environment with temperature t_c=const according to Newton's law. On inclusion surfaces $K_R=\{(R, \varphi, z)$: $0 \le \varphi \le 2\pi$, $0 \le z \le h\}$, $K_0=\{(r, \varphi, 0)$: $0 \le \varphi \le 2\pi$, $0 \le r \le R\}$ there is an ideal thermal contact $t_0(R, z)=t_1(R, z)$, $\lambda_0 \dfrac{\partial t_0(r,z)}{\partial r} = \lambda_1 \dfrac{\partial t_1(r,z)}{\partial r}$ for $r=R$ and $t_0(r, 0)=t_1(r, 0)$, $\lambda_0 \dfrac{\partial t_0(r,z)}{\partial z} = \lambda_1 \dfrac{\partial t_1(r,z)}{\partial z}$ for $z=0$ (0 for inclusion, 1 for layer) (Fig. 1).

The behavior of the temperature $t(r, z)$ as a function of the spatial coordinates r and z in the given inhomogeneous structure was obtained by solving the generalized heat conduction equation [17, 19]:

$$\frac{1}{r} div\left[r\lambda(r,z) grad\ \theta(r,z)\right]=0, \tag{1}$$

under boundary conditions:

$$\lambda_1 \frac{\partial\theta(r,z)}{\partial z}\bigg|_{z=-l} = \alpha\,\theta(r,z)\big|_{z=-l},$$

$$\lambda_0 \frac{\partial\theta(r,z)}{\partial z}\bigg|_{z=h} = q_0 S_-(R-r), \tag{2}$$

where $q(r, z)=t(t, z)-t_c$; $\lambda(r, z)$ – thermal conductivity coefficient of the heterogeneous layer; α – coefficient of heat transfer from the boundary surface of the layer; L_-; $\lambda_0(t)$, $\lambda_1(t)$ – thermal conductivity coefficients of inclusion and layer materials, respectively; $S_\pm(z)$ are asymmetric unit functions:

$$S_\pm(\zeta)=\begin{cases}1, & \zeta>0,\\ 0.5\mp 0.5, & \zeta=0,\\ 0, & \zeta<0.\end{cases}$$

In the case of intensive thermal load for certain temperature intervals, the thermophysical parameters of structural materials become dependent on temperature. As a result, the environment becomes thermally sensitive, and on this basis, nonlinear boundary value problems arise. Therefore, an isotropic thermosensitive layer containing a half-through inclusion of the canonical form is given. The surface of the layer L_- is considered thermally insulated. On the inclusion surfaces K_R, K_0, the conditions of ideal thermal contact $t_0(R, z)=t_1(R, z)$, $\lambda_0(t)\dfrac{\partial t_0(r,z)}{\partial r} = \lambda_1(t)\dfrac{\partial t_1(r,z)}{\partial r}$ for $r=R$ and $t_0(r, 0)=t_1(r, 0)$, $\lambda_0(t)\dfrac{\partial t_0(r,z)}{\partial z} = \lambda_1(t)\dfrac{\partial t_1(r,z)}{\partial z}$ for $z=0$ are set.

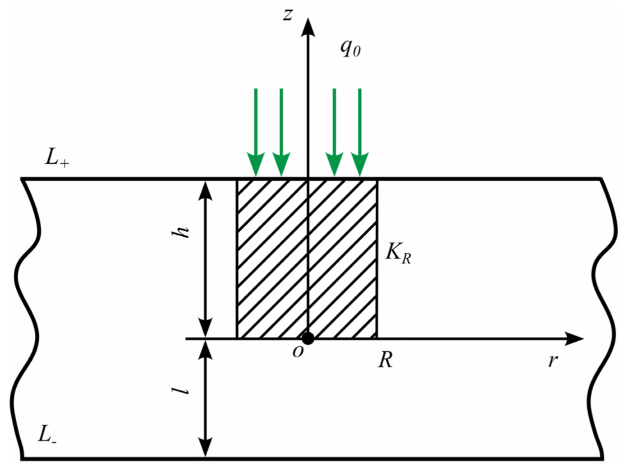


Fig. 1. Cross-section of an isotropic layer with a half-through inclusion in the plane φ=0, which is heated by a heat flux

Taking into account the thermal sensitivity of the materials of the environment, the temperature distribution $t(r, z)$ in the cylindrical coordinate system in the given structure is determined by solving the nonlinear heat conduction equation [17, 19]:

$$\frac{1}{r} div\left[r\lambda(r,z,t) grad\ t(r,z)\right]=0, \tag{3}$$

under boundary conditions:

$$t(r,z)\big|_{r\to\infty}=0,\quad \frac{\partial t(r,z)}{\partial r}\bigg|_{r\to\infty}=0,\quad \frac{\partial t(r,z)}{\partial z}\bigg|_{z=-l}=0,$$

$$\lambda_0(t)\frac{\partial t(r,z)}{\partial z}\bigg|_{z=h} = q_0 S_-(R-r), \tag{4}$$

where $\lambda(r,z,t)$ is the thermal conductivity coefficient of the heterogeneous thermosensitive layer; $\lambda_0(t)$, $\lambda_1(t)$ are thermal conductivity coefficients of inclusion and layer materials, respectively.

5. Results of the construction of mathematical models of thermal conductivity for media with semi-through inclusions

5. 1. Linear mathematical model of thermal conductivity for a heterogeneous layer with heat flow

The coefficient of thermal conductivity of a layer with a foreign inclusion is shown in the form:

$$\lambda(r,z)=\lambda_1+(\lambda_0-\lambda_1)S_-(R-r)S_-(z), \tag{5}$$

The following function has been introduced:

$$T(r,z)=\lambda(r,z)\theta(r,z), \tag{6}$$

and it was differentiated by the variables r and z taking into account the expression for the thermal conductivity coefficient $\lambda(r, z)$ (5). As a result, the following ratio was obtained:

$$\lambda(r,z)\frac{\partial\theta(r,z)}{\partial r}=\frac{\partial T(r,z)}{\partial r}+$$
$$+(\lambda_0-\lambda_1)\theta(r,z)\big|_{r=R}\delta_+(r-R)S_-(z),$$

$$\lambda(r,z)\frac{\partial\theta(r,z)}{\partial z}= \\ =\frac{\partial T(r,z)}{\partial z}-(\lambda_0-\lambda_1)\theta(r,z)\big|_{z=0}S_-(R-r)\delta_-(z). \qquad (7)$$

Here $\delta_\pm(\zeta)=\dfrac{dS_\pm(\zeta)}{d\zeta}$ are Dirac's asymmetric delta functions.

As a result of substituting expressions (7) into ratio (1), a second-order differential equation with partial derivatives with discontinuous and singular coefficients is built:

$$\Delta T+(\lambda_0-\lambda_1)F(r,z)=0, \qquad (8)$$

where $\Delta=\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial}{\partial r}\right)+\dfrac{\partial^2}{\partial z^2}$ is the Laplace operator in the cylindrical coordinate system;

$$F(r,z)=\frac{R}{r}\theta(r,z)\big|_{r=R}\delta'_+(r-R)S_-(z)- \\ -\theta(r,z)\big|_{z=0}S_-(R-r)\delta'_-(z).$$

As a result, the desired temperature field in the given system is determined by equation (8) under boundary conditions (2).

The unknown functions $\theta(R,z)$, $\theta(r,0)$ are approximated by linear constant functions for the variables z and r:

$$\theta(R,z)=\theta_1+\sum_{i=1}^{n-1}(\theta_{i+1}-\theta_i)S_-(z-z_i),$$

$$\theta(r,0)=\theta_1+\sum_{j=1}^{k-1}(\theta_{j+1}-\theta_j)S_-(r-r_j), \qquad (9)$$

where $z_i\in(0;h)$; $z_1\le z_2\le\ldots\le z_{n-1}$; $r_j\in(0;R)$; $r_1\le r_2\le\ldots\le r_{k-1}$; n, k are the numbers of interval division $(0;h)$,$(0;R)$; $\theta_i\left(i=\overline{1,n}\right)$, $\theta_j\left(j=\overline{1,k}\right)$ are unknown approximate temperature values $\theta(R,z)$, $\theta(r,0)$.

After substituting expressions (9) into equation (8), a second-order differential equation with partial derivatives with a discontinuous and singular right-hand side is constructed:

$$\Delta T=(\lambda_0-\lambda_1)F_1(r,z). \qquad (10)$$

Here:

$$F_1(r,z)=\theta(r,0)S_-(R-r)\delta'_-(z)-\frac{R}{r}\theta(R,z)\delta'_+(r-R)S_-(z).$$

Henkel's integral transformation along the r coordinate is applied to equation (10) and boundary conditions (2) taking into account ratio (6).

As a result, an ordinary differential equation of the second order with constant coefficients and a discontinuous and singular right-hand side is derived:

$$\frac{d^2\bar{T}}{dz^2}-\xi^2\bar{T}=(\lambda_0-\lambda_1)\left[A(\xi)\frac{\delta^{'}_-(z)}{\xi}-A(\xi,z)S_-(z)\right], \qquad (11)$$

under boundary conditions:

$$\left.\frac{d\bar{T}(\xi,z)}{dz}\right|_{z=h}=\frac{q_0R}{\xi}J_1(R\xi),$$

$$\left.\frac{d\bar{T}(\xi,z)}{dz}\right|_{z=-l}=\frac{\alpha}{\lambda_1}\bar{T}(\xi,z)\big|_{z=-l}, \qquad (12)$$

where:

$$A(\xi)=R\theta_kJ_1(R\xi)-\sum_{j=1}^{k-1}r_j(\theta_{j+1}-\theta_j)J_1(r_j\xi),$$

$$A(\xi,z)=R\xi J_1(R\xi)\left[\theta_1+\sum_{i=1}^{n-1}(\theta_{i+1}-\theta_i)S_-(z-z_i)\right],$$

$\bar{T}((\xi,z)=\int_0^\infty rJ_0(r\xi)T(r,z)\mathrm{d}r$ – the transform of the function $T(r, z)$;

$J_v(x)=\sum_{n=0}^{\infty}(-1)^n\dfrac{(x/2)^{v+2n}}{n!(v+n)!}$ – Bessel function of the first kind of the J-th order;

z is a parameter of Henkel's integral transformation.

The general solution to equation (11) is determined by the method of variation of constants:

$$\bar{T}(\xi,z)=c_1e^{\xi z}+c_2e^{-\xi z}+(\lambda_0-\lambda_1)A_3(\xi,z),$$

where:

$$A_3(\xi,z)=A(\xi)\frac{ch\xi z}{\xi}S_-(z)- \\ -\frac{R}{\xi}J_1(R\xi)\left[\theta_1A_1(\xi,z)+A_2(\xi,z)\right],$$

$$A_1(z)=(ch\xi z-1)S_-(z),$$

$$A_2(\xi,z)=\sum_{i=1}^{n-1}(\theta_{i+1}-\theta_i)(ch\xi(z-z_i)-1)S_-(z-z_i).$$

Boundary conditions (12) were used, and as a result, the solution to problem (11), (12) was derived in the form:

$$\bar{T}(\xi,z)=\frac{\lambda_0-\lambda_1}{\xi}\times \\ \times\left\{AB_1(\xi,z)-RJ_1(R\xi)\left[B_2(\xi,z)+B_3(\xi,z)\right]\right\}+ \\ +\frac{q_0RE(\xi,z)}{\xi^2E(\xi)}J_1(R\xi). \qquad (13)$$

Here:

$$B_1(\xi,z)=ch\xi zS_-(z)-P(z),$$

$$B_2(\xi,z)=\theta_1\left[(ch\xi z-1)S_-(z)-P(\xi,z)\right],$$

$$P(\xi,z)=sh\xi h\frac{E(\xi,z)}{E},$$

$$B_3(\xi,z)=\sum_{i=1}^{n-1}(\theta_{i+1}-\theta_i)\begin{bmatrix}(ch\xi(z-z_i)-1)\times \\ \times S_-(z-z_i)-D_i(\xi)\end{bmatrix},$$

$$D_i(\xi)=sh\xi(h-z_i)\frac{E(\xi,z)}{E(\xi)},$$

$$E(\xi,z)=(\lambda_1\xi+\alpha)e^{\xi(z+l)}+(\lambda_1\xi-\alpha)e^{-\xi(z+l)},$$

$$E(\xi)=(\lambda_1\xi+\alpha)e^{\xi(h+l)}-(\lambda_1\xi-\alpha)e^{-\xi(h+l)}.$$

The inverse Henkel integral transformation was applied to ratio (13) and the following result was obtained:

$$T(r,z)=\int_0^\infty \xi J_0(r\xi)\overline{T}(\xi,z)\mathrm{d}\xi. \qquad (14)$$

Unknown approximation values $\theta_i\left(i=\overline{1,n}\right)$ and $\theta_j\left(j=\overline{1,k}\right)$ of temperatures $\theta(R,z)$ and $\theta(r,0)$ are determined by solving the system of $n+k$ linear algebraic equations constructed after certain mathematical transformations from expression (14).

As a result, the desired temperature field in a layer with a half-through cylindrical inclusion is expressed by formula (14), from which the temperature value at an arbitrary point of the "layer-inclusion" structure is derived.

5. 2. Nonlinear mathematical model of thermal conductivity for a heat-sensitive inhomogeneous layer with heat flux

The coefficient of thermal conductivity for a heat-sensitive layer with a foreign inclusion is given in the form:

$$\lambda(r,z,t)=\lambda_1(t)+\left[\lambda_0(t)-\lambda_1(t)\right]S_-(R-r)S_-(z). \qquad (15)$$

A linearizing function has been introduced:

$$\vartheta(r,z)=\int_0^{t(r,z)}\lambda_1(\zeta)\mathrm{d}\zeta+ \left\{\begin{matrix}\int_{t(R,z)}^{t(r,z)}\left[\lambda_0(\zeta)-\lambda_1(\zeta)\right]\mathrm{d}\zeta- \\ -\int_{t(R,0)}^{t(r,0)}\left[\lambda_0(\zeta)-\lambda_1(\zeta)\right]\mathrm{d}\zeta\end{matrix}\right\}S_-(R-r)S_-(z), \qquad (16)$$

after differentiation of which by the variables r and z, the relation is obtained:

$$\lambda(r,z,t)\frac{\partial t(r,z)}{\partial r}=\frac{\partial\vartheta(r,z)}{\partial r}+ \left\{\left[\lambda_0(t)-\lambda_1(t)\right]\frac{\partial t(r,z)}{\partial r}\right\}\Bigg|_{z=0}S_-(R-r)S_-(z),$$

$$\lambda(r,z,t)\frac{\partial t(r,z)}{\partial z}=\frac{\partial\vartheta(r,z)}{\partial z}+ \left\{\left[\lambda_0(t)-\lambda_1(t)\right]\frac{\partial t(r,z)}{\partial z}\right\}\Bigg|_{r=R}S_-(R-r)S_-(z). \qquad (17)$$

Taking into account expressions (17), the original equation (3) is transformed into the following form:

$$\Delta\vartheta+\frac{1}{r}\frac{\partial}{\partial r}\left\{\begin{matrix}r\left[\lambda_0(t)-\lambda_1(t)\right]\times \\ \times\frac{\partial t(r,z)}{\partial r}S_-(R-r)\end{matrix}\right\}\Bigg|_{z=0}S_-(z)+ \frac{\partial}{\partial z}\left\{\left[\lambda_0(t)-\lambda_1(t)\right]\frac{\partial t(r,z)}{\partial z}S_-(z)\right\}\Bigg|_{r=R}S_-(R-r)=0. \qquad (18)$$

The linearizing function (16) made it possible to transform the boundary conditions (4), which took the following form:

$$\vartheta(r,z)\big|_{r\to\infty}=0,\quad \frac{\partial\vartheta(r,z)}{\partial r}\bigg|_{r\to\infty}=0,\quad \frac{\partial\vartheta(r,z)}{\partial z}\bigg|_{z=-l}=0, \qquad (19)$$

$$\frac{\partial\vartheta(r,z)}{\partial z}\bigg|_{z=h}= =\left\{q_0-\left[\lambda_0(t)-\lambda_1(t)\right]\frac{\partial t(r,z)}{\partial z}\right\}\Bigg|_{\substack{r=R\\ z=h}}S_-(R-r). \qquad (20)$$

As a result, the nonlinear boundary value problem (3), (4) is reduced to a partially linearized differential equation with partial derivatives of the second order with discontinuous coefficients (18), linearized boundary conditions (19), and partially linearized boundary condition (20).

The unknown functions $t(R,z)$ and $t(r,0)$ are approximated by piecewise constant functions in the spatial coordinates z and r:

$$t(R,z)=t_1+\sum_{i=1}^{n-1}(t_{i+1}-t_i)S_-(z-z_i),$$

$$t(r,0)=t_1+\sum_{j=1}^{k-1}(t_{j+1}-t_j)S_-(r-r_j), \qquad (21)$$

where $z_i\in(0;h)$; $z_1\le z_2\le\cdots\le z_{n-1}$; $r_j\in(0;R)$; $r_1\le r_2\le\cdots\le r_{k-1}$; n, k are the numbers of dividing the intervals $(0;h)$, $(0;R)$; $t_i\left(i=\overline{1,n}\right)$, $t_j\left(j=\overline{1,k}\right)$ are unknown approximate temperature values $t(R,z)$, $t(r,0)$.

The relation (21) was differentiated by the variables z and r, respectively, and as a result, we obtained:

$$\frac{\partial t(R,z)}{\partial z}=\sum_{i=1}^{n-1}(t_{i+1}-t_i)\delta_-(z-z_i),$$

$$\frac{\partial t(r,0)}{\partial r}=\sum_{j=1}^{k-1}(t_{j+1}-t_j)\delta_-(r-r_j). \qquad (22)$$

A second-order linear differential equation with partial derivatives and a discontinuous and singular right-hand side with respect to the linearizing function (r,z) is built as a result of substituting expressions (22) into relations (18) and (20):

$$\Delta\vartheta= =-\frac{1}{r}\sum_{j=1}^{k-1}r_j(t_{j+1}-t_j)\begin{bmatrix}\lambda_0(t_{i+1})- \\ -\lambda_1(t_{i+1})\end{bmatrix}S_-(z)\delta'_-(r-r_j)- -\sum_{i=1}^{n-1}(t_{i+1}-t_i)\left[\lambda_0(t_{i+1})-\lambda_1(t_{i+1})\right]S_-(R-r)\delta'_-(z-z_i), \qquad (23)$$

under a linear boundary condition:

$$\frac{\partial\vartheta(z)}{\partial z}\bigg|_{z=h}=q_0S_-(R-r). \qquad (24)$$

Henkel's integral transformation along the r coordinate was applied to equation (23) and boundary conditions (19), (24), and a second-order ordinary differential equation with constant coefficients and a singular right-hand side was derived:

$$\frac{d^2\overline{\vartheta}}{dz^2}-\xi^2\overline{\vartheta}=-\frac{R}{\xi}J_1(R\xi)\times \times\sum_{i=1}^{n-1}A_i(\xi)\delta'_-(z-z_i)-\xi A(\xi)S_-(z), \qquad (25)$$

under boundary conditions:

$$\left.\frac{d\bar{\vartheta}(\xi,z)}{dz}\right|_{z=-l}=0,\quad \left.\frac{d\bar{\vartheta}(\xi,z)}{dz}\right|_{z=h}=\frac{R}{\xi}q_0J_1(R\xi), \tag{26}$$

where $\bar{\vartheta}(\xi,z)=\int_0^\infty r\vartheta(r,z)J_0(r\xi)\mathrm{d}r$ is the $J(r,z)$ function transformant;

$$A_i(\xi)=(t_{i+1}-t_i)\left[\lambda_0(t_{i+1})-\lambda_1(t_{i+1})\right],$$

$$A(\xi)=\sum_{j=1}^{k-1}r_jJ_1(r_j\xi)(t_{j+1}-t_j)\left[\lambda_0(t_{j+1})-\lambda_1(t_{j+1})\right].$$

The general solution to equation (25) is derived as:

$$\bar{\vartheta}(\xi,z)=C_1e^{\xi z}+C_2e^{-\xi z}-$$
$$-\frac{1}{\xi}\left[RJ_1(R\xi)\sum_{i=1}^{n-1}\begin{matrix}A_i(\xi)ch\xi(z-z_i)S_-(z-z_i)+\\+A(\xi)(ch\xi z-1)S_-(z)\end{matrix}\right],$$

and using boundary conditions (26) the constants of integration c_1 and c_2 are found. As a result, the solution to problem (25), (26) is obtained:

$$\bar{\vartheta}(\xi,z)=\frac{1}{\xi}\left[\begin{matrix}\frac{Rq_0}{\xi}J_1(R\xi)P(\xi,z)+A(\xi)B(\xi,z)+\\+RJ_1(R\xi)\sum_{i=1}^{n-1}A_i(\xi)B_i(\xi,z)\end{matrix}\right]. \tag{27}$$

Here:

$$B(\xi,z)=P(\xi,z)sh\xi h-(ch\xi z-1)S_-(z);$$

$$B_i(\xi,z)=P(\xi,z)sh\xi(h-z_i)-ch\xi(z-z_i)S_-(z-z_i);$$

$$P(\xi,z)=\frac{ch\xi(z+l)}{sh\xi(h+l)}.$$

The inverse Henkel integral transformation was applied to relation (27) and the expression for the linearizing function J(r, z) was determined in the following form:

$$\vartheta(r,z)=\int_0^\infty \xi J_0(r\xi)\bar{\vartheta}(\xi,z)\mathrm{d}\xi. \tag{28}$$

As a result of substituting the temperature dependence of the coefficient of thermal conductivity of the layer materials and including it in relations (16) and (28), certain mathematical transformations are performed. As a result, a system of nonlinear algebraic equations is constructed. It is used to determine the unknown approximate values $t_i\left(i=\overline{1,n}\right)$, $t_j\left(j=\overline{1,k}\right)$ of temperature $t(R, z)$ and $t(r, 0)$, respectively.

The desired temperature field $t(r, z)$ for the given structure is determined using the resulting nonlinear algebraic equation. To this end, relations (16) and (28) are used as a result of substituting in them a specific expression of the temperature dependence of the coefficient of thermal conductivity of structural materials.

A partial example. The dependence of the coefficient of thermal conductivity on the temperature of the structural materials of the structure is given in the form of a ratio:

$$\lambda=\lambda_m^0(1-k_mt), \tag{29}$$

where λ_m^0, k_m are the reference and temperature coefficients of thermal conductivity of materials for inclusion (m=0) and layer (m=1).

Using relation (16) for the linearizing function and expression (29), the formulas for determining the temperature $t(r, z)$ in the inclusion region $\{(r,\varphi,z): r\le R, 0\le\varphi\le 2\pi, z\le h\}$ are derived:

$$t(r,z)=\frac{1}{k_0}\left(1-\sqrt{1-k_0\left(\frac{2\vartheta(r,z)}{\lambda_0^0}+\vartheta+\vartheta(r)+\vartheta(z)\right)}\right), \tag{30}$$

and in area outside the inclusion $\left\{\begin{matrix}(r,\varphi,z): r>R,\\ 0\le\varphi\le 2\pi,\\ -l\le z\le h\end{matrix}\right\}$:

$$t(r,z)=\frac{1}{k_1}\left(1-\sqrt{1-\frac{2k_1\vartheta(r,z)}{\lambda_1^0}}\right), \tag{31}$$

where:

$$\vartheta=t_n(\lambda j-t_nk\lambda),\quad \vartheta(r)=t(r,0)\left[t(r,0)k\lambda-\lambda j\right],$$

$$\vartheta(z)=t(R,z)\left[t(R,z)k\lambda-\lambda j\right],$$

$$\lambda j=2\left(\frac{\lambda_1^0}{\lambda_0^0}-1\right),\quad k\lambda=\frac{\lambda_1^0}{\lambda_0^0}k_1-k_0.$$

Formulas (30) and (31) describe the temperature field in the thermosensitive "layer-inclusion" structure.

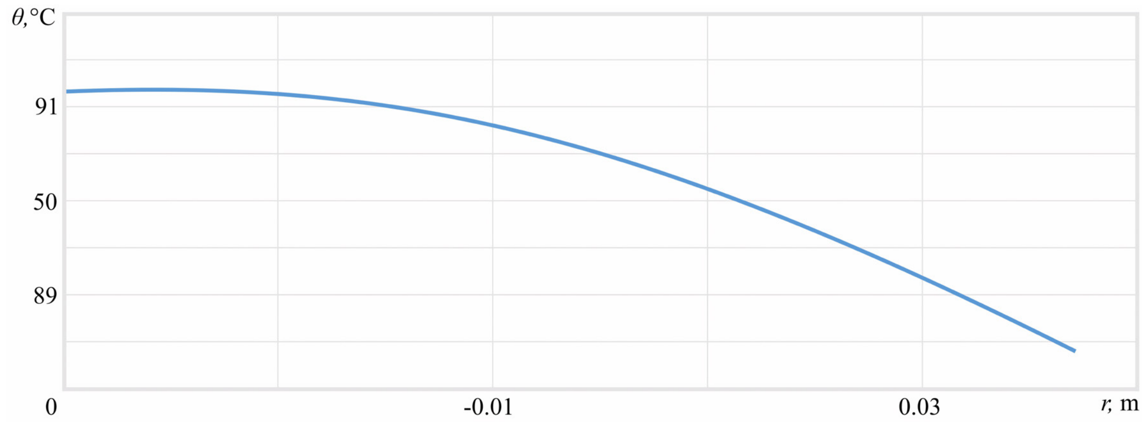
The main material of the layer is silicon, and the foreign inclusion material is silver. In the temperature range [20 °C; 1230 °C] the temperature dependence of the coefficient of thermal conductivity for the given materials was established by the interpolation technique in the form:

$$\lambda_1(t)=67.9\frac{\mathrm{W}}{\mathrm{degree}\cdot\mathrm{m}}\left(1-0.0005\frac{1}{\mathrm{degree}}t\right),$$

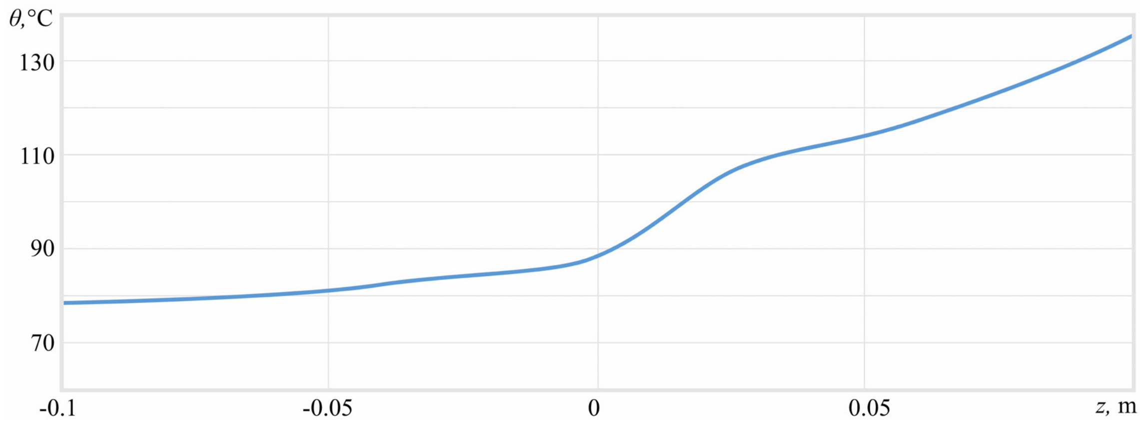
$$\lambda_0(t)=422.54\frac{\mathrm{W}}{\mathrm{degree}\cdot\mathrm{m}}\left(1-0.00031\frac{1}{\mathrm{degree}}t\right), \tag{32}$$

which is a partial case of relation (29).

According to (14), numerical calculations of the temperature distribution θ(r,z) in the layer were performed for the following initial data: λ_1=67.9 W/(m·degree) for silicon and λ_0=419 W/(m·degree) for silver according to temperature t=27 °C; q_0=200 W/m^2; l=0.1 m; h=0.075 m; R=0.05 m; α=17.64 W/(m^2·degree). The temperature change θ(r, z) is illustrated depending on the spatial coordinates r (z=0) (Fig. 2, *a*) and for z (r=0.05) (Fig. 2, *b*). It can be seen from the behavior of the curves that the temperature as a function of spatial coordinates is smooth and monotonic. It reaches maximum values at the boundary surface of the plate L_+, where the heat flow is concentrated. The number of partitions n and m of the intervals (0;h) and (0;R) is chosen to be nine. As a result, the numerical experiment was performed with an accuracy of 10^{-6}.



a



b

Fig. 2. Dependence of temperature $\theta(r,z)$ on spatial coordinates: *a* – dependence of temperature $\theta(r,z)$ on radial r coordinate; *b* – dependence of temperature $\theta(r,z)$ on the axial z coordinate

The results obtained for the selected materials based on the linear dependence of the coefficient of thermal conductivity on temperature differ from the results obtained for a constant coefficient of thermal conductivity by 7 %. Their slight difference is explained by the fact that the values of the temperature coefficient of thermal conductivity for the considered materials, as shown by ratio (32), are small.

6. Discussion of results related to investigating the constructed mathematical models of heat conduction in media with semi-through inclusions

The boundary value problems of thermal conductivity have been stated in accordance with a real physical process that is studied in heterogeneous media. Given this, differential equations of heat conduction and boundary conditions clearly describe mathematical models of the stationary process of heat conduction, which correspond to a certain physical model. The shape of the curves in Fig. 2, which are built on the basis of numerical results obtained from the analytical-numerical solution to the boundary value problem, indicates their correspondence to the physical process. This is confirmed by the non-discontinuity of temperature as a function of spatial coordinates and the fulfillment of boundary conditions at the edges of the environment.

In our studies, the theory of generalized functions was applied to effectively describe the thermophysical parameters of the environment with a foreign half-through inclusion and local temperature disturbances. As a result, differential equations with partial derivatives contain discontinuous and singular coefficients. To linearize the nonlinear boundary value problem (2), (3), the linearizing function (16) was introduced, and as a result, the partially linearized boundary value problem (18) to (20) was obtained. Approximation of temperature as a function of spatial coordinates was carried out on inclusion surfaces, which made it possible to apply Henkel's integral transformation. As a result, analytical and numerical solutions (14), (33) to boundary value problems were derived. The temperature distribution is determined using relations (14), (30), (31), and is geometrically shown in Fig. 2.

It is worth noting that in our studies, an approach for linearization of boundary value problems of thermal conductivity for heat-sensitive media was considered using an analytical-numerical method. Unlike work [1], in which a homogeneous environment is analyzed, and the use of the Kirchhoff transformation made it possible to linearize the boundary value problem, in our studies a new linearizing function has been proposed. Its application to a nonlinear problem made it possible to effectively obtain an analytical-numerical solution, which, in turn, leads to a minimum error in the results, in contrast to the use of numerical methods, which was not achieved in works [3, 4]. The use of generalized functions provided an effective description of the thermophysical parameters of media with foreign semi-transparent inclusions, which made it possible to solve the differential equations of thermal conductivity with partial derivatives of the second order, which contain discontinuous and singular coefficients. This approach was not used in [2].

The current studies relate only to the stationary process of heat conduction and were performed for media with foreign semi-through inclusions. In the future, it is possible to build on these studies for layered media with foreign half-through inclusions, as well as for non-stationary heat conduction processes and anisotropic layered media with the same inclusions.

Since in the architecture of modern electronic devices, separate nodes and their elements are concentrated in the form of structures with foreign semi-through inclusions, there is a need to construct mathematical models of the heat conduction process. These models could be both linear and nonlinear for isotropic layered media containing foreign half-through inclusions. As a result, the given mathematical models of heat conduction are simplified, but they make it possible to construct more complex mathematical models of the heat conduction process for composite media based on them.

Based on our analytical and numerical solutions to linear and nonlinear boundary value problems of heat transfer, it is proposed to develop computational algorithms and software tools for their numerical implementation. This will make it possible to conduct research for a number of materials used in the process of designing digital electronic devices, regarding the influence of their thermal sensitivity on the temperature distribution.

It is proposed to take into account the presence of half-through foreign inclusions in isotropic media, as well as the thermal sensitivity of structural materials for the analysis of thermal regimes, which significantly complicates the solution to the corresponding linear and nonlinear boundary value problems of thermal conductivity. The sought solutions to these problems more adequately describe the temperature behavior as a function of spatial coordinates.

Our study was carried out for a stationary heat conduction process, therefore the models built are limited and make it possible to determine the temperature change only by spatial coordinates. Heat conduction problems contain boundary conditions of the first, second, and third kind at the boundary surfaces of media, which is a disadvantage, although this does not reduce the generality of research.

Further studies may focus on the construction of mathematical models for determining temperature fields in layered media with foreign semi-transparent inclusions for the non-stationary process of heat conduction and more complex boundary conditions, such as thermal radiation.

7. Conclusions

1. A linear mathematical model for determining the temperature field and, subsequently, for analyzing thermal regimes in the structures of electronic devices with a foreign half-through inclusion when heated by a locally concentrated heat flow, has been constructed. An analytical-numerical solution to the boundary-value problem was derived and, on its basis, the temperature behavior as a function of spatial coordinates was determined graphically.

2. A nonlinear mathematical model has been built for determining the temperature field and, subsequently, for analyzing thermal regimes in thermosensitive structures of electronic devices with foreign half-through inclusion when heated by a locally concentrated heat flow. A linearizing function was introduced, using which a nonlinear boundary value problem was partially linearized. The piecewise linear approximation of the temperature on the inclusion surfaces has made it possible to fully linearize this problem. On this basis, an analytical-numerical solution to the initial boundary value problem was obtained for the linear dependence of the coefficient of thermal conductivity of the structure materials, which determines the temperature distribution in the structure. The results obtained for the selected materials based on the linear dependence of the coefficient of thermal conductivity on temperature differ from the results obtained for a constant coefficient of thermal conductivity by 7 %.

Conflicts of interest

The authors declare that they have no conflicts of interest in relation to the current study, including financial, personal, authorship, or any other, that could affect the study, as well as the results reported in this paper.

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Data availability

All data are available, either in numerical or graphical form, in the main text of the manuscript.

Use of artificial intelligence

The authors confirm that they did not use artificial intelligence technologies when creating the current work.

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