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DEVELOPMENT AND IMPLEMENTATION OF A NUMERICAL APPROACH FOR OPTIMAL CONTROL OF OSCILLATIONS IN COUPLED SYSTEMS WITH DISTRIBUTED AND LUMPED PARAMETERS

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This study focuses on numerical solutions to control problems in oscillatory systems consisting of two distinct objects. The primary issue addressed is the effective modeling and control of oscillations in these systems, particularly through the interaction of two objects. The research yields significant results by demonstrating a method to transform the complex boundary value problem into a more manageable system of ordinary differential equations using the method of straight lines. The findings reveal the influence of boundary conditions on the dynamics of an object characterized by distributed parameters. The results' unique features include applying Pontryagin's maximum principle to solve the associated variational problem, effectively integrating the behavior of both objects in the system. The numerical approach adopted in this research simplifies the problem and enhances the precision of the solutions obtained. Moreover, the study examines the convergence of numerical methods, improving their applicability to practical scenarios. The computational results demonstrate the convergence of the functional and show that the gradient projection method provides a convergent sequence in the control space, even for ill-posed optimal control problems. The conditions under which these results are most applicable include scenarios where boundary effects play a critical role in system dynamics, offering a robust framework for further investigation and application in real-world systems. This work significantly contributes to the understanding of oscillatory systems and provides a foundation for future research in optimal control strategies, thereby advancing the field of dynamic system control

Keywords: wave equations, oscillatory processes, method of successive approximations, controlled boundary effects

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1. Introduction

Studying control systems involving oscillatory processes and mixed parameter types (distributed and lumped) remains highly significant in contemporary research and industry. Systems modeled by wave equations and second-order differential equations characterize various fields of applied science, including engineering, materials science, and energy production. Specifically, oscillatory systems, where differential equations with boundary effects govern dynamics, hold critical importance for their capacity to simulate real-world applications, such as vibrational control in mechanical structures, electromagnetic wave propagation, and energy transfer in infrastructure [1, 2]. These systems provide the foundational models for designing and managing manufacturing, robotics, oil field development processes, and aeromechanics [3].

Moreover, integrating controlled boundary effects in oscillatory systems presents both practical and theoretical challenges that are far from fully resolved. Although foundational theories have been established in [4], the continuous development of numerical methods and control principles has revealed areas where current models lack robustness, particularly in achieving precision control of oscillations in varying boundary conditions [5]. These deficiencies are evident in applications requiring customized oscillation patterns or the precise timing of oscillation phases, such as wave damping, stabilization, and resonance avoidance in structural engineering and environmental processes. The power of oscillatory processes with various boundary conditions has long interested researchers. It is extensively covered in the literature [6, 7].

Additionally, industries like oil and gas encounter control problems that are heavily dependent on managing oscillatory processes governed by a mix of partial and ordinary differential equations [4]. For instance, optimizing the extraction rate and direction in fields with oscillatory boundary conditions (gas-oil-water interfaces) necessitates sophisticated boundary regulation, which is essential for maximizing resource extraction efficiency. Here, the application of control theories proves invaluable, though its potential for this class of oscillatory systems remains underutilized and not fully explored. However, for some problems involving parabolic equations, necessary conditions for optimality in Pontryagin's maximum principle have been obtained, and numerical methods for their solution have been developed [8]. Advances in this area could lead to significant efficiency improvements in resource development and sustainability.

Therefore, research on controlling oscillatory systems with distributed and lumped parameters, focusing on optimal boundary management and dynamic model enhancement, remains relevant and essential for advancing theoretical control models and practical applications.

2. Literature review and problem statement

The specific aspects of nonlinear oscillations are vast and varied, impacting systems in ways that are sometimes counterintuitive. In multiphase systems – like those involving liquids, granular media, or elastic bodies – oscillations can induce diverse motion forms, including translational displacement, vortex flows, and nonlinear waves such as shock waves [9]. These oscillations maintain oscillatory behavior and can prompt monotonic motions, stabilize equilibrium states, and produce resonance effects. Such phenomena occur due to energy transformations from oscillatory movements to other forms under nonlinear dynamic interactions. Small disturbances, which may seem insignificant, can decisively influence the behavior of multiphase systems under nonlinear resonance [10].

Nonlinear resonances, broadly defined, allow for more extensive capture regions in multiphase systems than traditional linear systems. This property makes oscillatory applications highly desirable in technology. For example, wave impacts can significantly amplify forces in gravitational separation processes (e.g., ore beneficiation or liquid purification) [11]. Here, resonant forces can exceed gravitational forces, providing a powerful tool for creating or enhancing technological processes. Oscillatory effects are also beneficial in selective media separation processes – such as purifying fuels or separating rare metals with similar specific weights – where current technical solutions remain ineffective [12].

The study [13] examines a mathematical model of a flexible elastic system characterized by distributed and discrete parameters. The model is represented by a partial differential equation with non-standard boundary conditions, whose complexity precludes the derivation of exact analytical solutions. To resolve this challenge, the asymptotic method of small parameters is combined with the numerical method for normal fundamental solution systems. These approaches facilitate the analysis of vibrations and provide a framework for determining the complex eigenvalues of the boundary value problem. The conditions under which vibration processes with varying characteristics occur are identified, and the relationship between vibration frequencies and the physical parameters of the hybrid system is explored. This paper primarily relies on the asymptotic method of small parameters and numerical approaches, which may limit its applicability to systems with more complex or highly nonlinear behaviors. Additionally, the study does not fully address the computational efficiency or potential challenges in implementing the proposed methods for real-world systems with varying boundary conditions.

The study [14] examines a two-link manipulator with flexible components, simulating endpoint vibration signals. Results indicate that residual vibrations are highly sensitive to deceleration time; the RMS is minimal when the inverse of deceleration time matches the first natural frequency and maximal when it matches half of the first natural frequency. However, the paper primarily focuses on specific trapezoidal velocity profiles and predefined starting and stopping positions, which may limit its generalizability to more complex or dynamic operational scenarios. Additionally, while the study highlights the sensitivity of vibrations to deceleration time, it does not explore alternative control strategies or feedback mechanisms to actively minimize residual vibrations.

The study [15] aims to assess the feasibility of estimating the parameters of various devices within seismic control systems and evaluating the maximum responses of key quantities. The observability condition was used to determine optimal sensor layouts and establish modeling assumptions for joint parameter estimation while excluding unsuitable configurations. This paper relies heavily on the observability condition and specific modeling assumptions, which may limit its applicability to more complex or less structured seismic systems. Additionally, the influence of partially unmeasured inputs is only investigated through a single case study, leaving uncertainty about its generalizability to diverse real-world scenarios.

The main goal of the study [16] is to examine how amplitude changes in mass transfer under reduced gravity-driven flow. The effect of a heat source or sink on periodic, mixed convective gravity-driven flow along an inclined plate has been assessed. The study explores the oscillatory behavior of heat transfer and mass transmission at three different inclined angles. Gravity is assumed to act under maximum temperature differences. The governing dimensional model is converted into non-dimensional equations using appropriate scaling variables. These non-dimensional equations are then simplified into real and imaginary parts to make the calculations smoother. The finite difference method is applied to transform the non-dimensional equations into an algebraic system, which is solved using the Gaussian elimination scheme to calculate the velocity, temperature distribution, and mass transfer rates. This paper mainly focuses on idealized conditions such as maximum temperature differences and assumes constant gravity, which may not fully represent real-world, variable gravitational environments. Additionally, the study does not address potential experimental limitations or the applicability of the model to complex, nonlinear systems that could exhibit more diverse behaviors than those considered in the simplified approach.

The paper [17] examines optimal control for a distributed oscillatory system modeled by a quasilinear hyperbolic differential equation in higher-order derivatives. Control quality is assessed via an integral functional with a small parameter. By using asymptotic expansions, the original boundary-value problem is approximated with a countable set of ordinary differential equations. A truncated version of this system, paired with an adapted quality functional, enables the practical formulation of the control problem. However, the truncated system may not fully capture the dynamics of the original countable system, potentially impacting the control quality in complex scenarios. Moreover, expanding the original system asymptotically introduces challenges in accurately approximating boundary interactions, particularly for higher-order derivative terms.

Recent research has advanced the understanding of controlled oscillatory processes, but several unresolved issues remain that limit practical applications. The study [18] investigates the oscillatory properties of a new class of third-order differential equations, focusing on the recursive relationships between solutions and their derivatives. The study provides generalized criteria for the oscillation of all solutions, broadening the scope of applications compared to prior research. Importantly, these results do not impose additional restrictive conditions, and examples demonstrate their practical significance. Although the findings cover a wide range of applications, identifying the full extent of systems where the criteria can be reliably applied remains an open question.

The paper [19] discusses optimal control problems characterized by oscillations (chattering controls) and concentrations (impulsive controls). In these problems, the control signal can concentrate at points where the state signal experiences discontinuities. The authors utilize methods from functional analysis, specifically anisotropic parametrized measures, to provide a clear framework for understanding the integral cost associated with these control signals. They also explore integrating these concepts with the Lasserre hierarchy of semidefinite programming relaxations, which are numerical methods for solving optimization problems. Nevertheless, the methods may need further exploration to determine their applicability to nonlinear control systems beyond the current scope.

It should be noted that in the problems of control for oscillations of coupled systems, as formulated in the aforementioned studies under various settings, only theoretical results have been obtained. All this allows us to assert that it is expedient to conduct a study on developing and implementing numerical methods for solving the problem.

3. The aim and objectives of the study

The study aims to develop numerical methods for solving optimal control problems in oscillatory systems with distributed and lumped parameters along two boundaries. This will make it possible to optimize control strategies for complex coupled systems by applying the gradient projection method and the method of successive approximations.

To achieve this aim, the following objectives are accomplished:

– to formulate the problem of optimal control of oscillations in coupled systems under the assumption that objects with concentrated parameters are attached to both boundaries of distributed systems;

– to reduce the original control problem to solving a variational problem associated with ordinary first-order differential equations and obtaining calculation formulas;

– based on the numerical solution of the problem, to develop an algorithm, to establish convergence in terms of the functional.

4. Materials and methods

This research focuses on developing numerical solutions for optimal control problems in oscillatory processes involving coupled objects.

The proposed numerical methods, including the gradient projection method and the method of successive approximations, combined with the discretization of spatial variables and time integration using the Runge-Kutta method, provide an effective framework for solving optimal control problems in oscillatory processes involving coupled systems. By iteratively refining control strategies based on functional gradients, these approaches are hypothesized to yield accurate and convergent solutions, thereby optimizing system performance and validating their applicability through numerical experiments.

The gradient projection method is based on formulating the optimal control problem as a variational problem, aiming to minimize a functional that depends on both the control and the system state. The process involves calculating the functional gradient concerning the control and optimizing the control in the direction of the anti-gradient through an iterative approach. This allows progressively refining the control at each step until the optimal solution is reached. On the other hand, the method of successive approximations is applied when dealing with linear boundary value problems that describe the system dynamics. This iterative method involves adjusting the control at each iteration based on the current system state and maximizing the Hamiltonian function concerning the control variable to determine the optimal control for each step.

An algorithm is devised to solve the optimal control problem for oscillatory processes. Initially, a system of differential equations is derived using the method of lines, which discretizes spatial variables and enables a time-dependent analysis of the system dynamics. The derived system of equations is then integrated over time using the Runge-Kutta method. This numerical integration facilitates predicting the system's future state over a specified time interval. Following the time integration, the functional gradient concerning the control is computed, which is crucial for adjusting the control parameters toward achieving an optimal solution. The control is refined iteratively based on the evaluated gradient, leading to progressively improved solutions that minimize the functional and optimize the system control.

To facilitate this research, a computer program is developed and implemented to execute the outlined algorithm, enabling efficient computation and iteration through the control strategies. The adequacy of the proposed models and methods is validated through numerical experiments, demonstrating their effectiveness in solving the optimal control problems of oscillatory processes.

5. Results of research on numerical method development for optimal control of oscillations

5. 1. Statement of the optimal control problem of oscillations in coupled systems

Let $Q=\{0<x<1,\ 0<t\le T\}$ and oscillations of a system be described by a boundary value problem [20]:

$$u_{tt}(x,t)=u_{xx}(x,t),(x,t)\in Q, \quad (1)$$

$$u(x,0)=\phi(x),\ u_t(x,0)=g(x),\ 0\le x\le 1,\ 0\le x\le 1, \quad (2)$$

$$\alpha_1 u_x(0,t)-\beta_1 u(0,t)=z_1(t),$$

$$\alpha_2 u_x(1,t)+\beta_2 u(1,t)=z_2(t),\ 0\le t\le T, \quad (3)$$

$$\ddot{z}_1(t)+\lambda_1^2 z_1(t)=b_1 u_x(0,t)+p_1(t),\ 0\le t\le T, \quad (4)$$

$$z_1(0)=z_1^0,\ \dot{z}_1(0)=z_1^1, \quad (5)$$

$$\ddot{z}_2(t)+\lambda_2^2 z_2(t)=b_2 u_x(1,t)+p_2(t),\ 0\le t\le T, \quad (6)$$

$$z_2(0)=z_2^0,\ \dot{z}_2(0)=z_2^1. \quad (7)$$

Excluding variables $z_1(t)$ and $z_2(t)$ from conditions (3) to (7), we can represent them in the following form:

$$\alpha_1 u_{xtt}(0,t)-\beta_1 u_{tt}(0,t)+$$
$$+\left(\lambda_1^2\alpha_1-b_1\right)u_x(0,t)-\lambda_1^2\beta_1 u(0,t)=p_1(t),$$
$$\alpha_1 u_x(0,0)-\beta_1 u(0,0)=z_1^0,$$
$$\alpha_1 u_{xt}(0,0)-\beta_1 u_t(0,0)=z_1^1, \quad (8)$$
$$\alpha_2 u_{xtt}(1,t)+\beta_2 u_{tt}(1,t)+$$
$$+\left(\lambda_2^2\alpha_2-b_2\right)u_x(1,t)+\lambda_2^2\beta_2 u(1,t)=p_2(t),$$
$$\alpha_2 u_x(1,0)+\beta_2 u(1,0)=z_2^0,\ \alpha_2 u_{xt}(1,0)+\beta_2 u_t(1,0)=z_2^1, \quad (9)$$

where $\phi(x)\in C^3[0,1]$ is the initial state of the object with distributed parameters, and the function $g(x)\in C^2[0,1]$ is the initial velocity distribution of the displacement point of the distributed object. The functions $p_1(t)$, $p_2(t)\in C[0,T]$ are controls selected to satisfy the compatibility conditions:

$$\alpha_1\phi'''(0)-\beta_1\phi''(0)+$$
$$+\left(\lambda_1^2\alpha_1-b_1\right)\phi'(0)-\lambda_1^2\beta_1\phi(0)=p_1(0), \quad (10)$$
$$\alpha_2\phi'''(1)+\beta_2\phi''(1)+$$
$$+\left(\lambda_2^2\alpha_2-b_2\right)\phi'(1)+\lambda_2^2\beta_2\phi(1)=p_2(0). \quad (11)$$

It is required to stay within the specified range $0\le p_i^0\le$ $\le p_i(t)\le p_i^1$ where p_i^0, p_i^1, $i=1,2$ control the functions $p_i(t)$ in such a way that at any given time $t=T$ the system is in a state little different from its rest state. The functional accepted as such deviation is:

$$F=\int_0^1\left[u^2(x,T)+u_t^2(x,T)\right]\mathrm{d}x. \quad (12)$$

The control functions $p_i(t)$ that satisfy the above conditions we call admissible controls, and the boundary problem (1)–(7) – the direct boundary problem. In this case, we deal with the problem of the optimal damping of oscillations in interconnected systems. In particular, if INF $F=0$, then it is possible to achieve complete damping of oscillations in interconnected systems by the moment T.

In [7], it is proven that if the conditions:

$$\alpha_1\phi'(0)-\beta_1\phi(0)=z_1^0,\ \alpha_1\psi'(0)-\beta_1\psi(0)=z_1^1, \quad (13)$$

$$\alpha_2\phi'(1)+\beta_2\phi(1)=z_2^0,\ \alpha_2\psi'(1)+\beta_2\psi(1)=z_2^1, \quad (14)$$

are met, then any functions $p_1(t)$, $p_2(t)$, satisfying the aforementioned conditions (8), (9) uniquely determine the solution $u(x,t)\in C^3[Q]$ to the boundary problem (1), (2), (8), (9), which at $0\le t\le 1$ can be represented as a sum of "direct" and "inverse" waves.

Setting the coefficients b_i, $i=1,2$ to zero for the system (1)–(7) implies that the oscillation mode of the distributed-parameter object does not affect the lumped-parameter object. Solutions to control problems for such a system with different combinations of coefficients α_i, β_i, $i=1,2$ are obtained in [6].

Let us formulate the aforementioned optimal control problem, the boundary problems of which contain first-type boundary conditions, i.e. $\alpha_1=\alpha_2=0$ and $-\beta_1=\beta_2=1$. Then, on both boundaries, we obtain first-type boundary conditions:

$$u(0,t)=z_1(t),\ u(1,t)=z_2(t),\ 0\le t\le T. \quad (15)$$

From conditions (8) and (9), taking into account the conditions $\alpha_1=\alpha_2=0$ and $-\beta_1=\beta_2=1$ we have:

$$u_{tt}(0,t)-b_1u_x(0,t)+\lambda_1^2u(0,t)=p_1(t),$$
$$u(0,0)=z_1^0,\ u_t(0,0)=z_1^1, \quad (16)$$
$$u_{tt}(1,t)-b_2u_x(1,t)+\lambda_2^2u(1,t)=p_2(t),$$
$$u(1,0)=z_2^0,\ u_t(1,0)=z_2^1. \quad (17)$$

Thus, the boundary value problem (1)–(7) takes the form:

$$u_{tt}(x,t)=u_{xx}(x,t),\ (x,t)\in Q, \quad (18)$$
$$u(x,0)=\phi(x),\ u_t(x,0)=g(x),\ 0\le x\le 1, \quad (19)$$
$$u_{tt}(0,t)-b_1u_x(0,t)+\lambda_1^2u(0,t)=p_1(t),$$
$$u(0,0)=z_1^0,\ u_t(0,0)=z_1^1, \quad (20)$$
$$u_{tt}(1,t)-b_2u_x(1,t)+\lambda_2^2u(1,t)=p_2(t),$$
$$u(1,0)=z_2^0,\ u_t(1,0)=z_2^1. \quad (21)$$

The compatibility conditions are expressed as:

$$\phi''(0)-b_1\phi'(0)+\lambda_1^2\phi(0)=p_1(0),$$
$$\phi''(1)-b_2\phi(1)+\lambda_2^2\phi(1)=p_2(0). \quad (22)$$

It is required to find admissible controls $p_1(t)$, $p_2(t)$ such that the solution $u(x,t)$ of the boundary value problem (18)–(21) under the conditions (22) minimizes the functional (12).

In [5], it is proven that if conditions:

$$\phi(0)=z_1^0,\ g(0)=z_1^1,\ \phi(1)=z_2^0,\ g(1)=z_2^1, \quad (23)$$

and (22) are met, then any functions $p_1(t)$, $p_2(t)$, satisfying conditions (16), (17) uniquely determine the solution $u(x,t)\in C^2[Q]$ of the boundary value problem (18)–(23).

5. 2. Reduction of the original problem to solving a variational problem

Finite-dimensional approximation in solving control problems for oscillations typically relies either on the Fourier method or on the method of straight lines. When using the Fourier method, it is important to note that the eigenfunctions of the corresponding boundary value problems form complete orthonormal systems in some functional space. This allows the use of finite Fourier series to construct systems of ordinary differential equations that describe finite-dimensional models of the original control problems. However, in the case of the controlled processes considered here, this approximation method is ineffective. This is because the system of eigenfunctions of the boundary value problem (18)–(21) is not orthonormal, and there is no proof that it is complete in any functional space.

To apply the method of lines, let us divide the interval [0,1] into n equal parts by points $x_i=ih$, $i=0,1,\ldots,n$, $nh=1$. At each point x_i, the values of the functions $u(x_i,t)$ and $u_t(x_i,t)=v(x_i,t)$ will be denoted by $u_i(t)$ and $v_i(t)$, respectively. The second-order derivative $u_{xx}(x_i,t)$ concerning the variable x will be replaced by numerical differentiation formulas by the expression:

$$u_{xx}(x_i,t)\approx\frac{1}{h^2}\left[u_{i+1}(t)-2u_i(t)+u_{i-1}(t)\right],\ i=1,2,\ldots,n-1.$$

The first-order derivative $u_x(x,t)$ at the boundary points $x=0$ and $x=1$ will be replaced by the expressions:

$$u_x(0,t)\approx\frac{1}{2h}\left[-3u_0(t)+4u_1(t)-u_2(t)\right],$$

$$u_x(1,t)\approx\frac{1}{2h}\left[u_{n-2}(t)-4u_{n-1}(t)+3u_n(t)\right].$$

With such an approximation of derivatives, an equal order of error is ensured when replacing the derivatives in the equations and boundary conditions with finite differences. Consequently, we obtain a system of ordinary differential equations corresponding to the boundary conditions and equations of the boundary value problem (18)–(21) instead of the original boundary value problem:

$$\begin{aligned}
&\dot{u}_i(t)=\upsilon_i(t),\ i=0,1,\ldots,n,\\
&\dot{\upsilon}_0(t)=\frac{b_1}{2h}\left[\left(-\frac{2h\lambda_1^2}{b_1}\right)u_0(t)+4u_1(t)-u_2(t)\right]+p_1(t),\\
&\dot{\upsilon}_i(t)=\frac{1}{h^2}\left[u_{i=1}(t)-2u_i(t)+u_{i+1}(t)\right],\ i=1,2,\ldots,n-1,\\
&\dot{\upsilon}_n(t)=\frac{b_2}{2h}\begin{bmatrix}u_{n-2}(t)-4u_{n-1}(t)+\\+\left(-\frac{2h\lambda_2^2}{b_1}\right)u_n(t)\end{bmatrix}+p_2(t),
\end{aligned}\tag{24}$$

with the following initial conditions:

$$u_i(0)=\phi(x_i),\ \upsilon_i(0)=g(x_i),\ i=1,2,\ldots,n-1,\tag{25}$$

$$u_0(0)=z_1^0,\ u_n(0)=z_2^0,\ \upsilon_0(0)=z_1^1,\ \upsilon_n(0)=z_2^1.\tag{26}$$

Therefore, the problem is reduced to determining the functions $p_1(t)$, $p_2(t)$ by minimizing the sum:

$$F=h\sum_{i=1}\left[u_i^2(T)+\upsilon_i^2(T)\right],\tag{27}$$

under conditions (24)–(26).

To transform the system of equations (24) into the canonical Cauchy form, we introduce $2n+2$ – dimensional vector with components $(u_0, u_1,\ldots, u_n, u_{n+1}, u_{n+2},\ldots, u_{2n}, u_{2n+1})$ where $u_{n+1+i}=v_i$, $i=0,1,\ldots,n$ and notations:

$$c_1=-\frac{2h\lambda_1^2}{b_1}-3,\ c_2=-\frac{2h\lambda_2^2}{b_2}+3.\tag{28}$$

In this case, the system of equations (24) can be written as:

$$\begin{aligned}
&\dot{u}_i=u_{n+1+i},\ i=0,1,\ldots,n,\\
&\dot{u}_{n+1}=\frac{b_1}{2h}\left[c_1u_0+4u_1-u_2\right]+p_1(t),\\
&\dot{u}_{n+1+i}=\frac{1}{h^2}\left[u_{i=1}-2u_i+u_{i+1}\right],\ i=1,2,\ldots,n-1,\\
&\dot{u}_{2n+1}=\frac{b_2}{2h}\left[u_{n-2}-4u_{n-1}-c_2u_n\right]+p_2(t),
\end{aligned}\tag{29}$$

with initial conditions:

$$\begin{aligned}
&u_i(0)=\phi(x_i),\ i=1,2,\ldots,n-1,\ u_0(0)=z_1^0,\ u_n(0)=z_2^0,\\
&u_{n+1+i}(0)=g(x_i),\ i=1,2,\ldots,n-1,\\
&u_{n+1}(0)=z_1^{01},\ u_{2n+1}(0)=z_2^1,
\end{aligned}\tag{30}$$

Functional (29) is written as:

$$F=h\sum_{i=1}^{2n+1}u_i^2(T).\tag{31}$$

Thus, considering the conditions (22), the optimal control problem ultimately reduces to determining the functions $p_1(t)$ and $p_2(t)$ from the minimizing condition (31) under the constraints (31), (32).

We construct the Hamiltonian function for the problem (29)–(31) as follows:

$$\begin{aligned}
H=&\sum_{i=0}^{n}\psi_iu_{n+1+i}+\frac{b_1}{2h}\psi_{n+1}\left[c_1u_0+4u_1-u_2\right]+\\
&+\psi_{n+1}p_1(t)+\frac{1}{h}\sum_{i=1}^{n-1}\psi_{n+1+i}\left[u_{i-1}-2u_i+u_{i+1}\right]+\\
&+\frac{b_2}{2h}\psi_{2n+1}\left[u_{n-2}-4u_{n-1}+c_2u_n\right]+\psi_{2n+1}p_2(t).
\end{aligned}$$

Then, the system of adjoint differential equations takes the form:

$$\begin{aligned}
&\dot{\psi}_0=-\frac{\partial H}{\partial u_0}=-\frac{1}{h^2}\left[\frac{hb_1c_1}{2}\psi_{n+1}+\psi_{n+2}\right],\\
&\dot{\psi}_1=-\frac{\partial H}{\partial u_{01}}=-\frac{1}{h^2}\left[2hb_1\psi_{n+1}-2\psi_{n+2}+\psi_{n+3}\right],\\
&\dot{\psi}_2=-\frac{\partial H}{\partial u_0}=-\frac{1}{h^2}\left[-\frac{hb_1}{2}\psi_{n+1}+\psi_{n+2}-2\psi_{n+3}+\psi_{n+4}\right],\\
&\dot{\psi}_i=-\frac{\partial H}{\partial u_i}=-\frac{1}{h^2}\left[\psi_{n+i}-2\psi_{n+i+1}+\psi_{n+i+2}\right],\ i=3,4,\ldots,n-3,\\
&\dot{\psi}_{n-2}=-\frac{\partial H}{\partial u_{n-2}}=-\frac{1}{h^2}\begin{bmatrix}\psi_{2n-2}-2\psi_{2n-1}+\\+\psi_{2n}+\frac{hb_2}{2}\psi_{2n+1}\end{bmatrix},\\
&\dot{\psi}_{n-1}=-\frac{\partial H}{\partial u_{n-1}}=-\frac{1}{h^2}\left[\psi_{2n-1}-2\psi_{2n}+2hb_2\psi_{2n+1}\right],\\
&\dot{\psi}_n=-\frac{\partial H}{\partial u_n}=-\frac{1}{h^2}\left[\psi_{2n}+\frac{hb_2c_2}{2}\psi_{2n+1}\right],\\
&\dot{\psi}_{n+1+i}=\frac{\partial H}{\partial u_{n+1+i}}=-\psi_i,\ i=1,2,\ldots,n.
\end{aligned}\tag{32}$$

The boundary conditions for (32) and the approximating analog of the finite-dimensional function (31) take the form:

$$\psi_i(T)=2hu_i(T),\ i=1,2,\ldots,2n+1,\ \psi_0(T)=0,\tag{33}$$

$$F'(p)=\left(\psi_{n+1}(t),\psi_{2n+1}(t)\right).\tag{34}$$

The first component of the pair (34) is the "partial derivative" of the function (31) concerning the variable $p_1(t)$, and the second component – concerning the variable $p_2(t)$.

5. 3. Development of a numerical solution algorithm for the problem

For the numerical solution of the optimal control problem, the gradient projection method in the control space is applied, starting from some admissible control $p^k(t)=(p_1^k(t),p_2^k(t))$, consisting of pairs $(p_1^k(t),p_2^k(t))$. When transitioning from controls $p^k(t)$ to a new control $p^{k+1}(t)$, the main task is computing the functional gradient according to formula (34) [5].

The algorithm for solving the problem consists of the following steps:

1. Select some admissible controls $p^k(t)$ (their selection can be based on any physical considerations).

2. Given the specified control $p^k(t)$, the system of equations (29), (30) is integrated with the "forward direction" using the Runge-Kutta method (or possibly the Euler method if stability conditions are met), and the values of $u_i(T)$, $i=0,1,\ldots,2n+1$ within the time interval $0\leq t\leq T$ are determined.

3. The values of the approximating sum (31) are computed.

4. Formulas (33) calculate the values $\psi_i(T)$, $i=0, 1, \ldots, 2n+1$.

5. In the "backward direction" of time, the system (32), (33) is integrated, and the values of the functions $\psi_{n+1}(t)$, $\psi_{2n+1}(t)$ within the time interval are determined.

6. Taking into account the compatibility conditions (23) at $t>0$, the new control $p^{k+1}(t)$ is computed using the formulas:

$$p_i^{k+1}(t)=\begin{cases} p_i^0, \text{ if } p_i^k(t)-\delta p_i^k(t)<p_i^0, \\ p_i^1, \text{ if } p_i^k(t)-\delta p_i^k(t)>p_i^1, \\ p_i^k(t)-\delta p_i^k(t), \text{ if } p_i^0\leq p_i^k(t)-\delta p_i^k(t)\leq p_i^1, \end{cases} \quad (35)$$

and at $t=0$ using the formulas $p_i^k(0)=p_i(0)$, $i=1,2$. Here:

$$\delta p_1^k(t)=\mu_1\cdot\frac{\psi_{n+1}^k(t)}{\left|\max\psi_{n+1}^k(t)\right|},$$

$$\delta p_2^k(t)=\mu_2\cdot\frac{\psi_{2n+1}^k(t)}{\left|\max\psi_{2n+1}^k(t)\right|},\ k=0,1,\ldots, \quad (36)$$

where k is the number of iterations, and parameters $\mu_i>0$ are chosen by one of the methods described in [5].

7. The obtained results are printed.

8. A step is taken with the new control $p^{k+1}(t)$, returning to step 2.

Computer programs have been developed to implement this task following the outlined algorithm. To ensure the correctness of the computational process, as the program is being developed, the results of some intermediate calculations have been verified, including the behavior of the direct problem (29), (30) with controls:

$$p_1^0(t)=\begin{cases}1.5, \text{ if } t=0,\\ 2, \text{ if } t\geq 0.01,\end{cases}\quad p_2^0(t)=\begin{cases}2, \text{ if } t=0,\\ 1.5, \text{ if } t\geq 0.01,\end{cases} \quad (37)$$

the value of the sum (31), boundary conditions for the adjoint system (32), (33), the values of the functions $\psi_{n+1}(t)$ and $\psi_{2n+1}(t)$, which are the "partial" derivatives of the function (12), respectively, concerning the variable $p_i(t)$, $i=1, 2$. The systems (29), (30) and (32), (33) are integrated with a constant step size $\Delta t=0.01$, and the results are displayed in the step $\Delta t=0.05$. The interval $[0, 1]$ is divided into ten equal parts in increments $h=0.1$. At the initial iteration, functions (38) are adopted. Note that for the selected values Δt and h, conditions $2\Delta t^2<h^2$ of stability of difference schemes are met, therefore, the direct and conjugate systems can be integrated using the Euler method.

The problem is solved with the following parameter values:

$$T=0.2,\ p_i^0=1,\ p_i^1=2,\ \nu_i=0.1,\ b_i=\lambda_i^2=1,\ i=1,2.$$

According to (22) and (23) in calculations, a solution to the equation $\phi''(0)-\phi'(x)+\phi(x)=p_i(0)$, satisfying the condition $\phi(0)=1$, $\phi'(0)=0$, was chosen as a function $\phi(x)$. It is easy to check that if $p_1(0)=1.5$, then this solution has the form:

$$\phi(x)=\frac{1}{2}e^{\frac{x}{2}}\left[-\cos\left(\frac{\sqrt{3}}{2}x\right)+\frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}x\right)\right]+1.5. \quad (38)$$

Then, taking into account the equalities:

$$\phi'(x)=\frac{1}{\sqrt{3}}e^{\frac{x}{2}}\left[\sin\left(\frac{\sqrt{3}}{2}x\right)\right], \quad (39)$$

$$\phi''(x)=\frac{1}{2}e^{\frac{x}{2}}\left[\cos\left(\frac{\sqrt{3}}{2}x\right)+\frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}x\right)\right], \quad (40)$$

from the condition $\phi''(0)-\phi'(0)+\phi(0)=1.5$, under which problem (18)–(21) has a unique solution, it follows that $p_1(0)=1.5$. By direct calculation, we find that if $p_2(0)=2$, then $\phi''(1)-\phi'(1)+$ $+\phi(1)=1.5$, i.e. $p_2(0)=2$.

Note that depending on the choice of valid values $p_i(0)$, in general, the solution:

$$\phi(x)=e^{\frac{x}{2}}\left[c_1\cos\left(\frac{\sqrt{3}}{2}x\right)+c_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right]+p_i(0), \quad (41)$$

of the equation $\phi''(x)-\phi'(x)+\phi(x)=p_i(0)$, constants c_1 and c_2 take different values. The function $g(x)$ is given as $g(x)=1-x$.

We have:

$$\phi(0)=z_1^0=1,\ g(0)=z_1^1=1,$$

$$\phi(1)=z_2^0=1.32848,\ g(1)=z_2^1=0.$$

Table 1 shows the results of calculations with the above data with initial controls (37). More than thirty iterations were made. The rate of convergence in the functional was as follows.

Table 1

The rate of convergence in the functional

Number of iterations	$F(p)$	$\lvert\max\psi_{n+1}(t)\rvert$	$\lvert\max\psi_{2n+1}(t)\rvert$
0	2.070237	0.490997	0.223982
5	2.045609	0.475560	0.205008
10	2.024625	0.460251|	0.194800
15	2.012774	0.450815	0.191255
20	2.008364	0.447349	0.189479
25	2.007621	0.446819	0.188420
30	2.007572	0.446819	0.187728

Fig. 1 shows the sequence of controls $p_1^k(t)$ for some intermediate iterations. From an examination of these graphs, it is clear that with an increase in the number of iterations, the sequence of functions $p_1^k(t)$ approaches their lower limit at $t>0$. The optimal result $p_1(t)$ was obtained after 25 iterations.

Fig. 2 shows the sequence of controls $p_2^k(t)$, obtained for 6 intermediate iterations. As the number of iterations increases, the sequence of controls $p_2^k(t)$, approaches its lower limit at $t>0$. The optimal $p_2(t)$, obtained after 40 iterations is borderline. The value of the functional turned out to be 2.007114.

Fig. 3 presents solutions to the conjugate boundary value problem (32)–(34) with functions $\psi_{n+1}(t)$, $\psi_{2n+1}(t)$ corresponding to optimal controls:

$$p_1^*(t)=\begin{cases}1.5, \text{ if } t=0,\\ 1, \text{ if } t\geq 0.01,\end{cases}\quad p_2^*(t)=\begin{cases}2, \text{ if } t=0,\\ 1, \text{ if } t\geq 0.01.\end{cases}$$

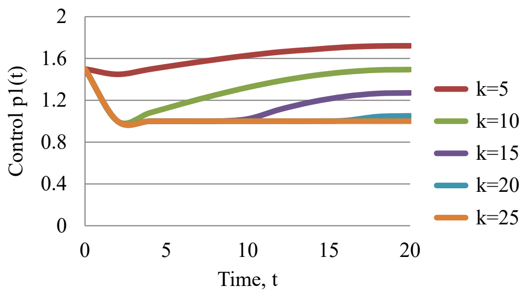

Fig. 1. Sequence of controls $p_1^k(t)$ during iterations

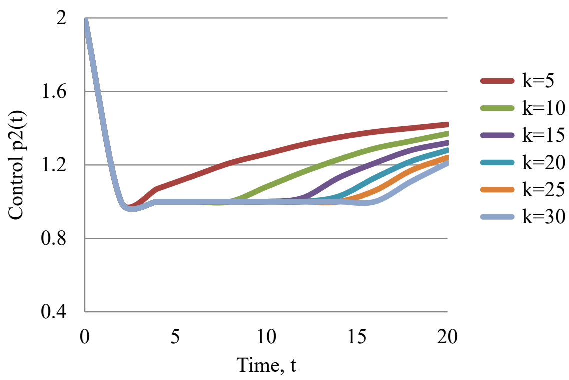

Fig. 2. Sequence of controls $p_2^k(t)$ during iterations

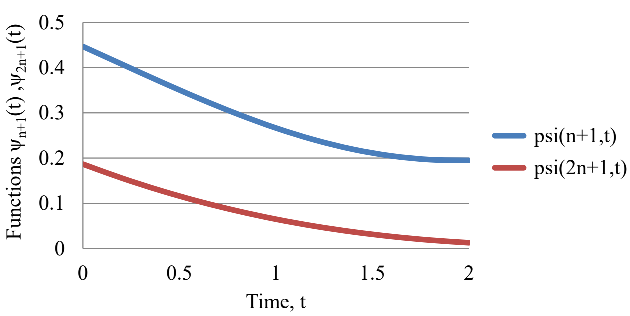

Fig. 3. Behavior of functions $\psi_{n+1}(t)$, $\psi_{2n+1}(t)$ in time

Thus, the gradient projection method for the problem considered gives significantly more accurate results, although it requires many iterations. It should also be noted that the gradient projection method did not show a tendency to become "loose" and, for any initial control, gave a convergent minimizing sequence of controls.

6. Discussion of the results, assessing the quality of control achieved for minimizing oscillatory behavior

This section examines the algorithm developed to solve the optimal control problem for oscillations in coupled systems. The study formulated a control problem for a system composed of objects with distributed and lumped parameters, enabling the specification of clear conditions and criteria for determining the optimal control in the given formulation. In solving this problem, significant results were obtained, such as the convergence of solutions in terms of the functional and the convergence of the minimizing sequence of control actions. These results are supported by the data presented in Table 1 and Fig. 1–3, which visually illustrate solution behavior and confirm the effectiveness of the chosen method.

One key advantage of this research, compared to similar studies like [5–7], is its ability to avoid complex mathematical and numerical computations. Instead, the method of straight lines is applied, which has proven to be one of the most effective approaches for solving boundary value problems in mathematical physics. Using the method of straight lines greatly simplifies numerical calculations while maintaining high solution accuracy. Additionally, this method is well-suited for solving optimal control problems with distributed parameters, allowing for an in-depth approach to the critical issues discussed in Section 3.

However, certain limitations arise when applying this approach to practical control problems for oscillatory systems with coupled objects. First, nonlinear boundary conditions often need to be considered, which significantly complicates the problem formulation and requires additional computational resources and methods to accommodate various conditions and constraints. Second, it is necessary to account for different restrictions on the system's state and control actions, which further complicates the process of finding an optimal solution.

One limitation of the study is the lack of a convergence analysis for the Cauchy problem solution that approximates the original boundary value problem as presented in [5, 6]. This aspect could be explored in future research by using a priori estimates for solutions to systems of linear and inhomogeneous ordinary differential equations, which would allow for a more accurate assessment of solution behavior when approximating boundary value problems. Additionally, convergence of approximate solutions in terms of control has not yet been demonstrated, leaving open questions regarding the correctness and accuracy of the obtained approximations for practical applications.

Future research could extend this study to oscillations in multiple coupled objects, which would enhance the applicability of the developed method and improve its accuracy for solving more complex optimal control problems in systems with a larger number of interacting elements.

7. Conclusions

1. The problem of optimal control of oscillations in coupled systems with distributed and lumped parameters is considered under the assumption that objects with lumped parameters are attached to both boundaries of the distributed systems, where boundary control actions are also applied to the distributed-parameter object. The system is controlled through these boundary actions applied to the lumped-parameter objects, which enables modification of the oscillatory behavior of the entire system. The primary objective is to select control actions that minimize a functional characterizing the quality of the oscillatory process.

2. By applying the method of straight lines – one of the most effective methods for solving boundary value problems in mathematical physics and optimal control problems with distributed parameters – the original optimal control problem is reduced to a variational problem associated with first-order ordinary differential equations, which constitutes a key step. The issues involving ordinary differential equations are solved using the gradient projection method based on Pontryagin's maximum principle.

3. An algorithm was developed to solve the optimal control problem for oscillations in coupled systems with distributed and lumped parameters, with the primary objective of finding an optimal control that minimizes a specified functional

characterizing the damping of oscillations in the coupled systems. The developed algorithm for solving the optimal control problem is supported by numerical data, illustrating the reduction in oscillatory behavior, and showcasing how the functional values converge to an optimal solution across iterations. Such data highlight the practical applicability and efficiency of the algorithm in damping oscillations in coupled systems.

Conflict of interest

The authors declare that they have no conflict of interest in relation to this research, whether financial, personal, authorship or otherwise, that could affect the research and its results presented in this paper.

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Data availability

Data will be made available on reasonable request.

Use of artificial intelligence

The authors have used artificial intelligence technologies within acceptable limits to provide their own verified data, which is described in the research methodology section.

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