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A TECHNIQUE FOR APPLYING THE SYMMETRY METHOD TO SOLVE A PROBLEM OF TORSIONAL VIBRATIONS OF DISKS OF VARIABLE THICKNESS

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The object of this study is a disk of variable thickness. A solution to the problem of natural torsional vibrations of disks of variable thickness was sought. An algorithm to solve the problem for an arbitrary number of different disk profiles has been constructed. The law of change in disk thickness $H(\rho)$, which contains three arbitrary constants α, C, C_1, has been considered. The choice of the disk profile configuration is controlled by changing the values of these three constants.

Exact solutions to the problem in elementary functions are known only in two cases, when $H(\rho)=1/\rho^3$ or $H(\rho)=\rho^{-3}e^{\alpha\rho}$, where ρ is the relative radial coordinate and α is an arbitrary constant. These cases are not sufficient for generalizing conclusions about the behavior of disks during their oscillations.

For the case of a disk that is rigidly fixed along its inner diameter and with a free outer edge, the corresponding relations were derived. They made it possible to calculate natural numbers and study the distribution of angular displacements of the disk. These numerical parameters, along with the frequency indices, are a convenient technique for evaluating the resonant properties of the disk for practice.

A comparison of torsional and radial vibrations of the disk with the chosen law of thickness change was performed. To study torsional vibrations, approximation approaches of thickness change functions were used. It was found that the relative discrepancy between the values of these functions at a certain interval did not exceed 2.2 %. It was found that the differences in the eigenfrequencies of torsional vibrations for the disk of the chosen configuration were significantly smaller than in the case of radial vibrations.

A practical algorithm for applying the method used is presented, which could prove useful for further research based on similar analytical approaches.

The method makes it possible to choose the desired disk configuration for various practical purposes. Owing to this feature, it is possible to provide the required distribution of cyclic stresses, resonant frequencies, and amplitudes for the disk

Keywords: differential equation, disk of variable thickness, torsional vibrations, natural frequencies, method of symmetries

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1. Introduction

The need to study torsional vibrations of disk elements in machines [1–3] is due to their harmful effect on the operational resource, which may decrease due to fatigue damage from the action of variable tangential stresses. In work [1], for example, imbalance in rotary machines caused by operational damage is considered as a similar harmful factor. In [2], the limit torsional vibrations of ship disk energy systems are controlled by an experimental method using optical sensors. From the same point of view, in [3], the operation of the propeller crankshaft of the main engine of the ship is considered.

The most harmful are resonant vibrations and, therefore, the basic current problem in theoretical studies is determining the natural frequencies of disks, the profile of which can vary arbitrarily. The disk configuration is a function of the thickness $H(\rho)$ along the disk radius, which in the form of a variable coefficient is included in the structure of the differential equation of torsional vibrations. To date, solutions to this equation are known for a relatively small number of disk designs. When designing disks, it may be necessary to expand the range of possible profiles for which exact solutions to the given equation have been found. This situation is due to the need to ensure such a distribution of stresses arising during torsional vibrations that the probability of fatigue damage to the disk is minimal. That is, maximum stresses must act in a specified place, for example, at a given distance from the disk mounting location. This is achieved by choosing the appropriate configuration, which warrants an exact solution to the corresponding differential equation of disk vibrations.

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2. Literature review and problem statement

Attempts to find exact solutions to the boundary value problem of torsional vibrations of disks of variable thickness considered in the literature allow us to assert that the available mathematical apparatus is limited mainly to the use of numerical approximate methods. At the same time, our review of the literature [1–10] confirms the assumption of the wide practical application of technical equipment with disks of variable thickness in solving various applied problems. In this case, the disk of variable thickness acts as one of the responsible elements of such equipment.

In work [2], the subject of the study is the scheme for monitoring torsional vibrations of ship power systems. The experimental measurement of the angular velocity of a crankshaft of a homogeneous piston engine shaft using two optical sensors is described, the role of which is performed by two disks of constant thickness, which are placed at opposite ends of this shaft. It is noted that the value of the angular displacements between the ends of the shaft in the region of the angle of rotation is a measure of torsional vibrations. To find the values of the natural frequencies and amplitudes of torsional vibrations of the shaft, the numerical method of finite elements was used. However, in the work, different profiles of disks of variable thickness were not considered in any way. The reported methodology for monitoring torsional vibrations is difficult to use for solving other applied problems that do not concern ship propulsion systems. The lack of consideration in the work of different configurations of disks of variable thickness is probably due to the lack of an analytical apparatus for solving the problem of torsional vibrations of these disks.

Work [3] reports a study on the influence of torsional vibrations on the vibration system of the propeller crankshaft of the main engine of the ship. A method of replacing the moment of inertia of the mass and the stiffness coefficient for torsion of the disks with mechanically equivalent round shafts and round rods of the same diameter is given. To estimate the torques, a multi-stage system of three disks of different diameters of constant thickness is considered. For this purpose, the transfer matrix method is used, but the statement of the boundary value problem, approaches to finding the natural frequencies and vibration forms of the disk system are not given in the work. The lack of approaches to finding the natural frequencies and vibration forms of the disk system is due to technical limitations that exist in the practical use of the method for replacing the moment of inertia of the mass and the stiffness coefficient.

Paper [4] addresses the issue of harmful vibrations of an incorrectly adjusted blade disk of constant thickness, which is very often the cause of cyclic fatigue failure of the disk. As a complex mechanical structure, the bladed disk is one of the key functional components of an aircraft engine. The study shows that the disk with blades is divided into substructures, and based on the finite element method, the vibration response of each substructure is estimated for the structural analysis of the entire structure. In the work, a finite element model of a detuned disk with blades is constructed, for which the natural frequencies are calculated. However, the resulting frequencies relate only to the bending vibration mode of the disk. Torsional vibrations for this disk structure are not considered in the work. The reason for this is probably that the numerical finite element method used by the authors is adapted for a specific complex mechanical disk structure.

In work [5], the object of the study is a homogeneous rotating annular disk, the thickness of which varies according to a hyperbolic law. Two analytical approximation methods are proposed to determine the limiting angular velocities. For this purpose, the two-dimensional theory of plane stresses is considered, and the distributions of stresses and radial displacements are calculated. Possible torsional vibrations of an annular disk of a given profile are not considered in the work. This is due to the fact that, probably, the considered theory of plane stresses is difficult to apply to solving the problem of torsional vibrations of disks of variable thickness with an arbitrarily given configuration.

In contrast to the previous work, in [6] a number of disks with different laws of thickness change are considered: exponential, hyperbolic, parabolic. To determine the natural frequencies and forms of displacements for these disk profiles, an approximate Galerkin method is proposed. An analytical algorithm for finding a formula for calculating the tangential displacements of the disk is described. However, in the final relation, it is necessary to perform a double integration procedure, which is not convenient and possible for cases of arbitrary laws of thickness change. Such an indefinite representation of the solution to the problem does not give a direct answer to the real distribution of frequencies and displacements and cyclic stresses. Thus, the identified difficulties of a mathematical nature significantly complicate the use of the results from the work for devising an analytical apparatus to find solutions to the problem of torsional vibrations of disks with an arbitrary law of thickness change. The fundamental drawback in this case is the use of an approximate method.

The study of the creep deformation behavior of a parabolic profile disk is reported in [7]. To calculate logarithmic deformations and stresses, four differential equations were considered, for the solution of which the numerical method of finite deformations was used. However, the results cannot be used to analyze the tangential vibrations of the considered disk profile. The reason for this is probably the given features of using the numerical method in that it is not adapted to the analytical solution to the problem of tangential vibrations of a parabolic profile disk.

In paper [8], a micro disk with angular acceleration is considered. Based on the Midlin deformation gradient theory and Hamilton's principle, differential equations of motion of this disk and boundary conditions for the radial and tangential components of displacements are derived. At the same time, it is not entirely clear how the results could be used to analyze the tangential vibrations of a disk of variable thickness. The work does not consider the analytical solution to the problem. This is probably due to the fact that Midlin's deformation gradient theory and Hamilton's principle in the formulated statement can only be implemented for the analysis of micro disk oscillations.

Work [9] reports a study on the distribution of stresses and angular velocity for an isotropic disk with a hyperbolic law of thickness change. It was chosen that this disk configuration is characterized by variable density and therefore the theory of transitions was proposed. In this problem, a disk made of a hypothetical material of variable density, which is not related to the subject of the study, is considered. The results obtained in the work, due to the consideration of the isotropic disk configuration and the corresponding mathematical apparatus, cannot be used to find an analytical solution to the problem of torsional vibrations of disks.

The results of the thermoelastic analysis of axisymmetric bending of disks made of a functionally granular material, the properties of which depend on temperature, are given in [10]. To calculate the stress and displacement fields, the Karman theory was proposed, and the numerical method of serial solutions was used. A disk of constant thickness was chosen as the object of the study. The types of vibrations were not clearly separated in the work. Possible plane oscillations of this disk are not covered in the work. The reason for this, apparently, in addition to the use of the conditions of the numerical method, is the authors' consideration of temperature features when studying disks made of such materials.

From our review, it is clear that in some cases the use of disks of constant thickness is not advisable from the point of view of the irrational distribution of cyclic stresses in these disks, which affects their strength. As a result, their replacement with disks of variable thickness is required. As for the disks of variable thickness, which are described in the review, they are limited to configurations of only hyperbolic and parabolic types, which confirms the need to expand the range of profiles for practice. This is an additional justification for the problem statement given below.

So, the problem statement is as follows. Due to the limited number of known exact solutions to the differential equation that describes the torsional vibrations of disks of variable thickness, it is often proposed to use numerical approximate methods. However, questions may arise about the accuracy of the results and the simplicity of numerical calculations. A likely alternative is methods that directly make it possible to obtain an exact analytical solution to the problem for a much larger number of options for changing the thickness of the disks. Moreover, this set of configurations should be determined quite simply by varying the variable coefficients, which in turn adds some flexibility to analysis. Existence of an exact analytical solution to the problem makes it possible to apply standard approaches to determine the natural frequencies, as well as construct the forms of oscillations and stresses.

3. The aim and objectives of the study

The aim of our study is to apply a symmetry method to arbitrarily expand the exact solutions to the problem of torsional vibrations of disks of variable thickness. This, in turn, will make it possible to expand the scope of rational design of profiles of working disks for industrial purposes.

To achieve the goal, the following tasks were set:

– to develop a scheme for applying the symmetry method to find solutions to the differential equation of torsional vibrations of the disk;

– to provide examples of applying the symmetry method to construct new profiles of disks of variable thickness;

– to consider a practical comparison of disk vibrations under radial and torsional variable loads using the symmetry method.

4. The study materials and methods

The object of our study is a disk of variable thickness operating under the mode of torsional (tangential) oscillations. The subject considered is plane torsional oscillations of disks of variable thickness. The mathematical model of the study is defined as a second-order differential equation with variable coefficients.

The hypothesis of the study assumes that from the set of configurations of disks of variable thickness, for which the corresponding differential equation has exact solutions, it is possible to choose the profile that will be able to provide the necessary conditions for the future operation of the selected disk. First of all, this may concern the issue of the optimal distribution of cyclic stresses.

The following assumptions have been accepted. The statement of the problem corresponds to the provisions of the Kirchhoff-Lagrange theory. The disk is a body of rotation and there is a plane of symmetry of the disk, which is perpendicular to the axis of rotation. The thickness of the disk is small compared to its diameter. During free torsional vibrations, the points of the disk located on any radius of the circle remain on this same circle, and all radii are distorted in exactly the same way. Centrifugal effects associated with the rotation of the disk practically do not affect the shapes and natural frequencies of plane vibrations.

To derive an analytical solution to the problem, we searched for the configuration of the disk profile based on approximation approaches and symmetry ideas. That is, the idea of constructing appropriate groups of transformations is applied, owing to which the solving differential equations should remain invariant with respect to these groups.

To solve the frequency equations, the method of successive approximations is employed.

The results of the research correspond to a number of structural materials for which Hooke's law is fulfilled, and Poisson's ratio is 1/3. Density and elastic modulus are constant physical quantities.

5. Results of investigating the problem of torsional vibrations of a disk

5. 1. Scheme of applying the symmetry method to find solutions to the differential equation of torsional vibrations of a disk

The equation of the eigenforms of torsional vibrations of a disk of variable thickness takes the following form [11]:

$$W''+W'\left(\frac{H'}{H}+\frac{3}{\rho}\right)+k^2W=0, \tag{1}$$

where $W=W(\rho)$ – angular displacement;

$\rho=r/a$ – relative radial coordinate;

r – variable radius;

a – disk radius;

$H=H(\rho)$ – disk thickness;

$$k=\omega_{n}a\sqrt{\gamma/gG}, \tag{2}$$

$\omega_c=2\pi f_c$ – circular frequency during torsional vibrations;

γ – specific gravity;

g – acceleration of gravity;

G – disk shear modulus. Dashes define differentiation by ρ.

In the literature, cases for $H=H_0\rho^{-n}$ have been investigated that lead to solutions of $W(\rho)$ in the form of Bessel functions for any values of n, or in the form of elementary functions for n=1 and n=3. Gramel [11] defines the disk profile in the form of $H(\rho)=H_0\rho^{-3}e^{\alpha\rho}$, which also leads to the solution of $W(\rho)$ in elementary (trigonometric) functions.

Disk profiles, which are determined by functions $H(\rho)$, normalized by different values of H_0 and α at different ratios $H(\rho_0)/H(\rho=1)=\eta$, can be selected from the plots, some of which are shown in Fig. 1.

Since the curves $H(\rho)$ at $\rho=0$ are not defined, when constructing real profiles, it is necessary, based on the design requirements, to determine the radius ρ_0 with which the disk design will finally be designed. Considering that in work [12], which reports a study on the radial vibrations of similar disks, $\rho_0=0.2$ was assumed, that value was also adopted in this work. Moreover, this is connected with the planned comparison of natural frequencies for plane and radial vibrations, below in the paper. In the given practical calculations [11], the disk profile with the parameters $H_0=10^4$; $\rho_0=0.2$; $\alpha=1/15$ was chosen.

Next, the vibrations of disks that are rigidly fixed on a round shaft with a radius of ρ_0 and are free along a thin edge at $\rho=1$ are considered. In this case, the boundary conditions for torsional vibrations take the following form:

$$W(\rho_0)=0;\quad W'(\rho=1)=0. \qquad (3)$$

For plots of functions $H(\rho)$, as Fig. 1 demonstrates, limited possibilities for constructing disk profiles are characteristic, which could differ significantly from each other.

The symmetry method is based on the idea of constructing appropriate groups of transformations, owing to which the solving differential equations should remain invariant with respect to these groups. Based on this, it is possible to construct an arbitrary number of symmetries of equation (1), i.e., equations that preserve the form of equation (1).

Based on this, as a result of the corresponding transformations, we obtain:

$$W_n''+\frac{F_n'}{F_n}W_n'+k_n^2W_n=0, \qquad (4)$$

where $F_n=\rho^3H_n$. Depending on the number $n=1,2,3,\ldots$, the following relations hold:

$$\left.\begin{aligned}
&F_1=1/F;W_1=FW';\\
&F_2=F_1V_1^2;\ W_2=W_1/V_1;\ V_1=\int(1/F_1)\,\mathrm{d}\rho+C;\\
&F_3=1/F_2;\ W_3=W_2'F_2;\\
&F_4=F_3V_3^2;\ W_4=W_3/V_3;\ V_3=\int(1/F_3)\,\mathrm{d}\rho+C_1.
\end{aligned}\right\} \qquad (5)$$

The algorithm for constructing symmetries for equation (1) based on relations (5) makes it possible to significantly, in principle to infinity, expand the series of functions $F_n=H_n(\rho)\rho^3$. For these functions, equation (4) for $W_n(\rho)$ will have exact solutions provided that equation (1) for the function $W(\rho)$ also has an exact solution.

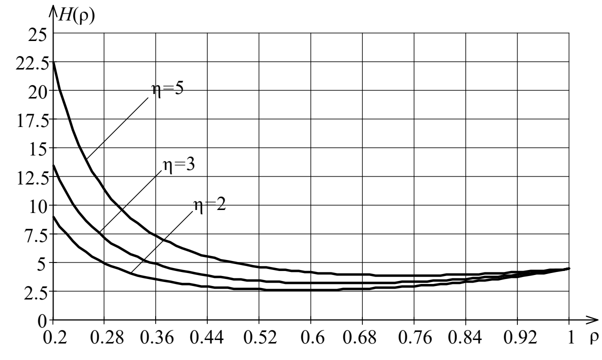

Fig. 1. Dependence of $H(\rho)$ for different values of parameter η

5. 2. Examples of the symmetry method application

The following are examples of the application of this variant of the symmetry method for constructing new profiles of a disk of variable thickness:

1. Let $F=H\rho^3=\text{const}=1$, i.e., $H=1/\rho^3$. According to (1), $W=A\sin k\rho+B\cos k\rho$ is obtained. Further, according to scheme (5), taking into account that $F_1=F=H_1\rho^3$ and $W_1=W$, we find:

$$\left.\begin{aligned}
&F_2=(\rho+C)^2;\ W_2=W_1/(\rho+C)=W/(\rho+C);\\
&F_3=\frac{1}{(\rho+C)^2};\ W_3=(\rho+C)^2W_2'=(\rho+C)W'-W;\\
&F_4=\left[\frac{(\rho+C)^3+C_1}{\rho+C}\right]^2;\ W_4=\frac{(\rho+C)W'-W}{(\rho+C)^3+C_1}.
\end{aligned}\right\} \qquad (6)$$

Despite the complication of functions F_n with increasing $n=1, 2, 3, \ldots$ due to the appearance in their structure of arbitrary constants $C, C_1,\ldots$, the solutions W_n remain fully defined in the form of dependences on the solution W. Some of the graphical dependences $H_n=F_n/\rho^3$ for different C and C_1 are shown in Fig. 2.

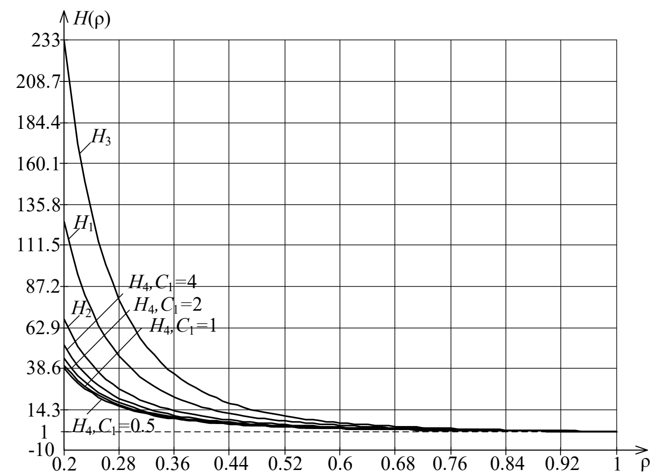

Fig. 2. Dependences of $H_n(\rho)$ for different values of C and C_1

By changing the coefficients C and C_1, it is possible to adjust the ratio of the limiting thicknesses of the disk $\eta=H(\rho_0)/H(\rho=1)$. From the curves shown in Fig. 2, we can conclude about possible technological difficulties in the optimal design of the geometric profiles of the disks, since $\eta>30$.

2. For the case $F_1=H_1\rho^3=e^{2\alpha\rho}$ equation (1) will have a known general solution:

$$W=e^{-\alpha\rho}(A\sin\lambda\rho+B\cos\lambda\rho);\ \lambda=\sqrt{k^2-\alpha^2}. \qquad (7)$$

Based on (5) we obtain:

$$\left.\begin{aligned}
&F_2=e^{-2\alpha\rho};\ W_2=e^{\alpha\rho}(A\sin\lambda\rho+B\cos\lambda\rho);\\
&F_3=\left(e^{\alpha\rho}+Ce^{-\alpha\rho}\right)^2;\ W_3=W_1/\left(e^{2\alpha\rho}+C\right);\\
&F_4=1/\left(e^{\alpha\rho}+Ce^{-\alpha\rho}\right)^2;\ W_4=\left(e^{\alpha\rho}+Ce^{-\alpha\rho}\right)^2W_2';\\
&F_5=\left[\frac{1}{2\alpha}\left(e^{\alpha\rho}-Ce^{-\alpha\rho}\right)+\frac{2C\rho+C_1}{e^{\alpha\rho}+Ce^{-\alpha\rho}}\right]^2;\ W_5=\frac{W_3}{V_3}.
\end{aligned}\right\} \qquad (8)$$

Fig. 3 shows part of the curves $H_n(\rho)=F_n/\rho^3$, constructed according to functions F_n from (8) for some values of the constants α, C, C_1. Fig. 4 additionally defines the curves for the disk configuration $H_5(\rho)=F_5/\rho^3$ for arbitrarily chosen different values of parameters α, C, C_1, taking into account the relationship from (8).

Thus, by changing the values of the coefficients α, C, and C_1, it is possible to obtain profiles with different degrees of concavity or convexity in thickness. This, in turn, significantly expands the possibilities for finding special configurations of a disk of variable thickness.

The graphical dependences $H_5(\rho)$ in Fig. 4 are normalized in such a way that the disk profiles that can be constructed with their help take the form of special configurations. That is, so that the disk thickness decreases with increasing radius from $\rho=\rho_0$ to $\rho=1$.

The symmetry method has made it possible to construct the disk profile function $H_5(\rho)=F_5/\rho^3$ quite simply, for which an exact analytical solution to the problem was obtained. The $H_5(\rho)$ function already contains three independent constants α, C, C_1. This allows for more flexible control over the disk thickness if necessary.

5. 3. Application of the symmetry method in comparing torsional and radial vibrations of a variable-thickness disk

Traditionally, the widespread use of variable-thickness disks in steam and gas turbine engines necessitates the simultaneous study of their torsional and radial vibrations. This primarily concerns the determination of their natural frequencies, in particular for the case of marked plane vibrations. In contrast to equation (1), the differential equation of radial vibrations takes the following form [11]:

$$R''+\left(\frac{H'}{H}+\frac{1}{\rho}\right)R'+R\left(\frac{\nu H'}{\rho H}-\frac{1}{\rho^2}+\lambda^2\right)=0, \qquad (9)$$

$$\lambda^2=\frac{a^2\omega_p^2\gamma\left(1-\nu^2\right)}{gE}, \qquad (10)$$

where:

– $R=R(\rho)$ – radial oscillations;

– $\omega_p=2\pi f_p$ – circular frequency at radial oscillations;

– E – Young's modulus.

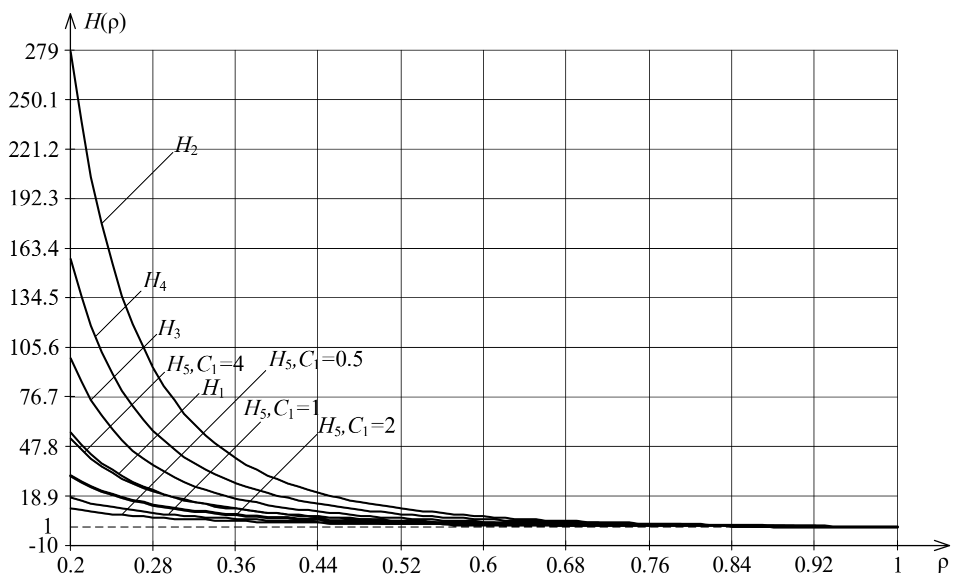

Fig. 3. Dependences of $H_n(\rho)$ for different values of α, C, and C_1

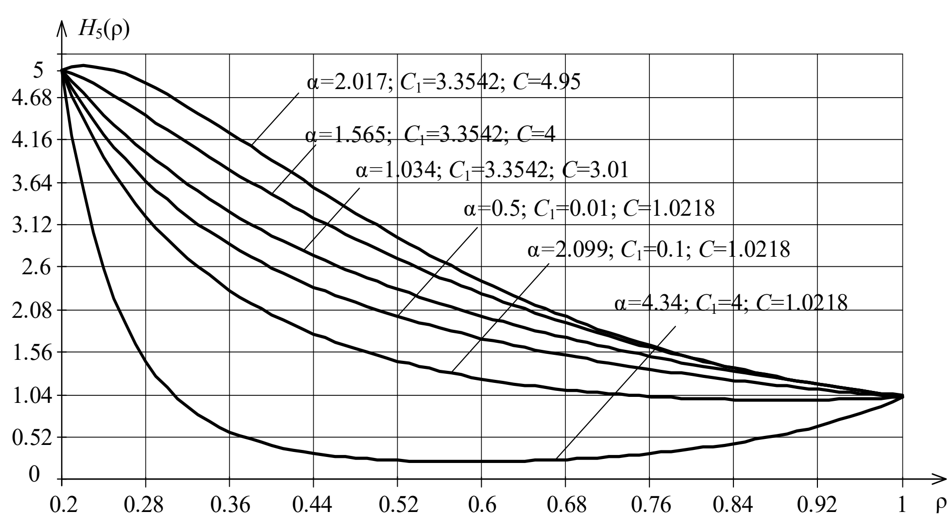

Fig. 4. Dependences of $H_5(\rho)$ for different values of α, C, and C_1

Having different forms, equations (1) and (9) have exact solutions for different non-coincident functions $H(\rho)$. For equation (9), until recently, no solutions have been found at all except those that correspond to the hyperbolic dependence $H=1/\rho^n$, which is practically unsuitable for constructing a real disk profile. In work [12], a number of new exact solutions to equation (9) were obtained for those cases of $H(\rho)$ that can find practical application, such as the Grammel profile.

For further study, the following function [12] was selected from the mentioned series:

$$H = H_0\rho^{2\nu}\left(\rho^{\mu} + C\rho^{-\mu}\right), \quad (11)$$

where $C=1.352$; $\mu=(\nu^2+5/4)^{1/2}=7/6$; $\nu=1/3$. For a given profile H, the solution to equation (9) takes the following form [12]:

$$R = \frac{\left[A(\sin\lambda\rho - \lambda\rho\cos\lambda\rho) + B(\cos\lambda\rho + \lambda\rho\sin\lambda\rho)\right]}{\rho^{3/2+\nu+\mu} + C\rho^{3/2+\nu-\mu}}. \quad (12)$$

For a disk whose thickness $H(\rho)$ corresponds to dependence (11) taking into account the boundary conditions $R(\rho=\rho_0)=0$; $R'(\rho=1)=0$, a frequency equation is constructed. From this equation, the eigenvalues are derived [12]:

$$\lambda_1=2.6241;\ \lambda_2=6.4531778;\ \lambda_3=10.2228. \quad (13)$$

In [12], a solution to the frequency equation for a disk of constant thickness $H(\rho)=\text{const}$ was also obtained:

$\lambda_1=2.2249572;$

$\lambda_2=6.100853;$

$\lambda_3=9.9657101.$ (14)

Determination of the natural frequencies of torsional vibrations of this disk is impossible without solving equation (1) at $H(\rho)$ in the form (11). The exact solution at a given $H(\rho)$ is unknown, therefore, for maximum approximation to it, the symmetry method in its new purpose is used. Above is an algorithm for expanding to infinity the exact solutions to equation (1) for the correspondingly found new expressions for $H(\rho)$. The new functions $H(\rho)$ may have in their structure, as can be seen from the list (8), arbitrary constants α, C, C_1. By a reasonable choice of these constants, changing the form of the functions $F_n=\rho^3H(\rho)$, it is possible to achieve approximation by the selected function F_n of the function $\rho^3H(\rho)$ given by equation (1).

To achieve the set goal, the function $F_2=(e^{\alpha\rho}+Ce^{-\alpha\rho})^2$ was selected from the list (8). In this case of arbitrary constants (C,α), this function, and also W_2, can be replaced by equivalent expressions:

$$F = \left[a\sinh\alpha(1-\rho) + b\cosh\alpha(1-\rho)\right]^2; \quad (15)$$

$$W = \frac{\left[A\sin\sqrt{k^2-\alpha^2}\rho + B\cos\sqrt{k^2-\alpha^2}\rho\right]}{a\sinh\alpha(1-\rho) + b\cosh\alpha(1-\rho)}. \quad (16)$$

Equating F/ρ^3 and $H(\rho)$ according to (15) and (11) at $\rho=\rho_0=0.2$; $\rho_m=1$, three equations are obtained:

$$\frac{\left[a\sinh\alpha(1-\rho) + b\cosh\alpha(1-\rho)\right]^2}{\rho^3} = \rho^{2\nu}\left(\rho^{\mu} + C\rho^{-\mu}\right)^2;$$

$$\left[a\sinh\alpha(1-\rho) + b\cosh\alpha(1-\rho)\right] = \rho^{3/2}\left(\rho^{\mu+\nu} + C\rho^{-\mu+\nu}\right);$$

$$\left[a\sinh\alpha(1-\rho) + b\cosh\alpha(1-\rho)\right] = \rho^{3/2+\nu+\mu} + C\rho^{3/2+\nu-\mu}.$$

Hence, at $\rho_0=0.2$; $\rho_m=1$, $a=-2.477$; $b=2.352$; $\alpha=1.567$ are found. The plots of thicknesses $H(\rho)$ and $H_1(\rho)=F/\rho^3$ on the segment $\rho=0.2\ldots1.0$ are shown in Fig. 5; the quantitative indicators are given in Table 1.

Table 1

Values of functions $H(\rho)$ and $H_1(\rho)$ on the interval $\rho=0.2\div1$

ρ	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$H(\rho)$	0.471	0.633	0.798	0.977	1.178	1.409	1.677	1.989	2.352
$H_1(\rho)$	0.471	0.619	0.783	0.966	1.173	1.409	1.679	1.991	2.352
δ, %	10^{-13}	2.19	1.88	1.086	0.402	$-1.1\cdot10^{-4}$	−0.135	−0.096	0

From analysis of the plots in Fig. 5 and the data in Table 1, it is clear that the functions $H(\rho)$ and $H_1(\rho)$ are almost completely coincident. The average value of the difference between these functions $\delta=(1-H_1(\rho)/H(\rho))\cdot100$ % is quite small and does not exceed 2.2 %; the plots are almost identical, as shown in Fig. 5. It can be argued that $H_1\approx H$ and the approximation technique in this case has achieved its goal.

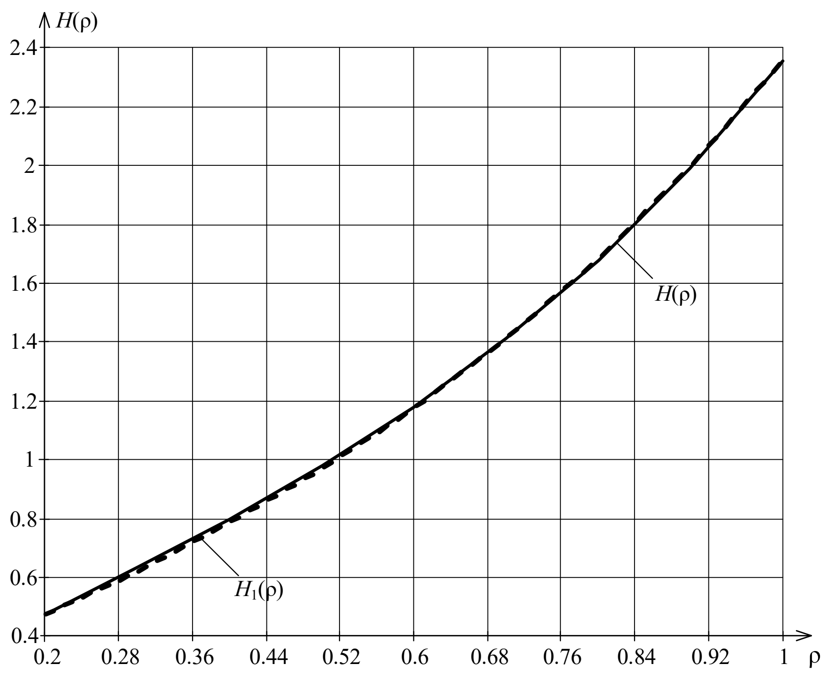

Fig. 5. Graphical interpretation of functions $H(\rho)$ and $H_1(\rho)$

For a disk that is rigidly fixed along the inner radius $\rho=\rho_0$ and free along the outer radius $\rho=1$, the boundary conditions are as follows:

$$W(\rho=\rho_0)=0;\ W'(\rho=1)=0. \quad (17)$$

After entering the solution (16) into these conditions, a system of two equations is obtained in the following form:

$$\left.\begin{aligned}&A\sin q\rho_0 + B\cos q\rho_0 = 0;\\&A\left[q\cos q + \frac{\alpha a}{b}\sin q\right] - B\left[q\sin q - \frac{\alpha a}{b}\cos q\right] = 0.\end{aligned}\right\} \quad (18)$$

By setting the discriminant of this system to zero, we obtain a frequency equation for finding the unknown values of q. After minor transformations, this equation takes the form:

$$tg\left(q - q\rho_0\right) = -\frac{qb}{\alpha a}.$$

Given ρ_0=0.2 and found during approximation a=–2.477; b=2.352; α=1.567, this equation will ultimately take the following form tg0.8q=0.605q. Hence, the following roots are obtained: q_1=5.528; q_2=9.6046; q_3=13.59334; q_4=17.5543. According to the relation $q=(k^2-\alpha^2)^{1/2}$, the eigenvalues k take the following values:

$$k_1=5.7457;\ k_2=9.73158;\ k_3=13.6833;\ k_4=17.62407. \quad (19)$$

One of the roots of the frequency equation is q=0, which means $k=\alpha$. At this value of k, the oscillatory motion of the disk is absent, therefore this root is discarded in the study.

The natural frequencies f_n are calculated after substituting the numbers k_n from sequence (19) into formula (2). If necessary, the forms of torsional vibrations of the disk under consideration can be constructed on the basis of expression (16) after calculating the ratio B/A according to any of the equations in system (18). The constant A is defined as an arbitrary amplitude coefficient.

For a disk of constant thickness, by putting H'=0 into equation (1), its solution in Bessel functions is obtained [13]:

$$W = \frac{1}{\rho}\left[AJ_1\left(k\rho\right) + BY_1\left(k\rho\right)\right].$$

After introducing this solution into the boundary conditions (17), a system of equations is built:

$$\left.\begin{aligned}&AJ_1\left(k\rho_0\right) + BY_1\left(k\rho_0\right) = 0;\\&A\left[kJ_0\left(k\right) - 2J_1\left(k\right)\right] + B\left[kY_0\left(k\right) - 2Y_1\left(k\right)\right] = 0.\end{aligned}\right\}$$

From this system, the frequency equation is obtained:

$$\frac{kJ_0\left(k\right) - 2J_1\left(k\right)}{kY_0\left(k\right) - 2Y_1\left(k\right)} - \frac{J_1\left(k\rho_0\right)}{Y_1\left(k\rho_0\right)} = 0,$$

whose roots at ρ_0=0.2 take the following values:

$$k_1=0.5956;\ k_2=5.8245;\ k_3=9.79844. \quad (20)$$

As mentioned above, when formulating the problem, the relative value of the difference between the natural frequencies of plane vibrations – radial and torsional – can be of practical importance. The value of this difference is determined according to formulas (2) and (10):

$$\frac{f_p}{f_k} = \sqrt{\frac{2}{1-\nu}}\frac{\lambda}{k} = \sqrt{3}\frac{\lambda}{k}.$$

Having a set of eigenvalues for two types of oscillations, it is possible to find the difference of the corresponding frequencies and their mutual location. As can be seen, the difference in the magnitude of the frequencies of torsional oscillations is significantly smaller than in the case of radial oscillations. The same can be said when considering disks of constant thickness, but the degree of difference in quantitative terms will not be the same.

By adding the numbers k_n to the previously found in [12] eigenvalues λ_n for radial oscillations for disks of constant and variable thickness, it becomes possible to determine the magnitude of the differences between these numbers. It is believed that the higher the frequency $f_p(\lambda_n)$ compared to the frequencies $f_c(k_n)$, the more dangerous the resonant (natural) frequencies $f_c(k_n)$. This is explained by the fact that the probability of their primary excitation is higher compared to the resonant frequencies of the radial oscillations of the disk.

6. Discussion of results based on investigating the problem of torsional vibrations of a disk

The differential equation of the eigenforms of torsional vibrations of a disk of variable thickness (1) has known solutions only for the variants when $H=H_0\rho^{-3}$ or $H(\rho)=H_0\rho^{-3}e^{\alpha\rho}$ (H is the disk thickness). These solutions are expressed in elementary functions. However, for configurations of the form $H(\rho)=H_0\rho^{-3}e^{\alpha\rho}$, the possibilities of varying the type of curves that determine the disk profiles are typically limited (Fig. 1).

The main conclusion from our review of the literature is the lack of a general analytical method for solving the problem of torsional vibrations of disks of arbitrary profile. To eliminate this drawback, namely, to expand the number of possible disk profile variants, the method of symmetries is proposed. In this case, it becomes possible to obtain an exact analytical solution to the problem for a much larger number of variants of changing the disk thickness.

Symmetries of the given equation (1) of this problem are found by using special groups of transformations, which in the final result of the transformations make it possible to build an equation that does not differ in form from the given one. That is, if a solution to the problem is found for the thickness change function $H(\rho)$, which is constructed on the basis of transformations by the symmetry method, then a solution to the given equation (1) is automatically found.

Another advantage of the proposed scheme for constructing symmetries for equation (1) based on relations (5) is that it makes it possible to significantly, in principle to infinity, expand the series of functions $H_n(\rho)$.

For example, to construct disk profiles according to scheme (5), functions for two cases F=1 and $F=e^{2\alpha\rho}$ were used. In these cases, the equations for W will be equations with constant coefficients, the solutions to which are known. According to scheme (5), for these individual values of F, exact solutions will be known for F_1, F_2,.... Dividing the value of F_i by ρ^3, we shall obtain possible disk profiles, which is shown for these cases in Fig. 2, 3. From this and other possible sets of profiles, one can always choose the one required for design.

This approach to finding a solution to the problem, in contrast to numerical approximate methods, is also more convenient for researchers. This is explained by the fact that due to the presence of arbitrary coefficients C, C_1, C_2, ... in the relations of system (5), it is possible to quite simply adjust the ratio of the limiting thicknesses of the disk and change the disk profile itself, as shown in Fig. 4.

This extremely useful feature of the symmetry method makes it possible to resolve the task of rational design of disk elements for structures by optimally choosing their configurations. Moreover, this choice makes it possible to take into account both the distribution of the disk thickness along the radius and to estimate the values of the natural frequencies and their corresponding vibration forms.

In confirmation of this, two practical examples are given, in which an extended set of functions in (6) and (8) is obtained based on the functions $H=1/\rho^3$ and $H_1=e^{2\alpha\rho}/\rho^3$. In this case, for example, the solutions W_n from system (8) remain fully defined in the form of dependences on the solution W from (7). The choice of the disk profile configuration can be controlled by changing the values of three arbitrary constants α, C, C_1.

A comparison of the torsional and radial vibrations of a disk of variable thickness, described by a single function (11), is carried out. For the torsional vibration mode, the solution to equation (1) for the thickness change function (11) is found by combining the symmetry method and approximation approaches of the initial function $H(\rho)$ and the given function $H_1(\rho)$. From the analysis of data in Table 1, it follows that the average difference of these functions does not exceed 2.2 % in the relative value of the parameter δ.

A characteristic feature of this disk profile (11) is the observance of the same curvature $\eta=H(\rho_0)/H(\rho=1)=5$. Provided that the disk is rigidly fixed along the inner radius $\rho=\rho_0$ and is free along the outer radius $\rho=1$, the natural frequencies of oscillations (13) and (19) are found. A comparison of these results shows that the magnitude of the difference in the frequencies of torsional oscillations is much smaller than the values for the case of radial oscillations.

The reported algorithm for obtaining an exact analytical solution to the problem of torsional oscillations of disks provides the opportunity to vary the configuration of a disk of variable thickness almost arbitrarily. Given this, it is possible to provide, for example, the necessary distribution of stresses or amplitudes along the radius. This technique is convenient when, for structural reasons, it is necessary to move the zone of action of maximum stresses away or closer to the place of fastening the disk.

The limitation of the symmetry method in the given form of the system of relations (5) is that it can be applied only to equations of the type (1) or (4). In other cases of II-order equations, they must first be brought to form (4) by appropriate transformations. This is the disadvantage, or rather, the inconvenience in the application of the method and the algorithm as a whole.

Further development of research on applying the symmetry method involves finding opportunities for its use in applied problems for IV-order differential equations.

7. Conclusions

1. A scheme for constructing symmetries for the equation of eigenforms of torsional vibrations of a disk of variable thickness is presented. The resulting relations make it possible to significantly expand the series of thickness functions $H_n(\rho)$ due to the fact that they contain a series of arbitrary constants C_i. The method of symmetries allowed us to obtain an exact analytical solution to the problem of torsional vibrations of disks of variable thickness in a wide range of changes in their configurations.

2. Two practical examples of the application of the symmetry method are given, in which the known solutions to the problem for disks are used as a basis, where the thickness varies according to the laws $H=1/\rho^3$ and $H_1=e^{2\alpha}/\rho$, respectively. Due to the provisions of the symmetry method, a number of new configurations of the disk of torsional vibrations are found. In particular, various configurations of the disk $H_5(\rho)$ are constructed, for which an exact analytical solution to the problem on eigenvalues is obtained. The resulting profiles have the same curvature $\eta=H(\rho_0)/H(\rho=1)=5$ and are characterized by the fact that their appearance is regulated by choosing the values of three arbitrary constants α, C and C_1. It is characteristic that by changing the values of constants α, C and C_1, it is quite easy to change the geometry of the disk. For example, when $\alpha\approx 2$, then with increasing values of the constants C and C_1, one can observe the transition of the curve $H_5(\rho)$ to a set of arbitrary shapes – from concave to convex. That is, by varying the values of these constants, it is possible to adjust the curves that describe the law of change in the thickness of the disk and its angular displacements.

3. A comparison of two problems of torsional and radial vibrations of a disk of variable thickness, which is rigidly fixed along the inner contour and free along the outer edge, has been carried out. For the problem of torsional vibrations, in order to obtain an analytical solution, approximation approaches of the functions $H(\rho)$ and $H_1(\rho)$ were used, and the relative discrepancy of the values of these functions on a certain interval was found to not exceed 2.2 %. Frequency equations were built, and the natural frequencies of vibrations λ_i were determined (for ordinal numbers $i=1, 2, 3$). A comparison of the frequencies for torsional and radial vibrations indicates that in practice the resonant frequencies of torsional vibrations are more dangerous. This is explained by the fact that they lie much lower than the resonant frequencies of radial vibrations.

Conflicts of interest

The authors declare that they have no conflicts of interest in relation to the current study, including financial, personal, authorship, or any other, that could affect the study, as well as the results reported in this paper.

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Data availability

All data are available, either in numerical or graphical form, in the main text of the manuscript.

Use of artificial intelligence

The authors confirm that they did not use artificial intelligence technologies when creating the current work.

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