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NUMERICAL OPTIMIZATION OF CONTROL STRATEGIES FOR COUPLED VIBRATIONAL SYSTEMS

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The study examines the numerical solution of vibration control problems in a coupled system consisting of two interacting objects. The problem is solved under the assumption that the left boundary of the distributed system is fixed, while an object with lumped parameters is attached to the right boundary, where a boundary control action is also applied to the distributed system. Special attention is given to obtaining a numerical solution to the problem. The solution is approached using two methods: the gradient projection method and, due to the linearity of the boundary problem concerning phase coordinates and control inputs, the method of successive approximations. By introducing an additional variable, the one-dimensional wave equation is approximated using the method of lines, transforming it into a system of ordinary differential equations of the 2n-th order. The resulting variational problem for the system with lumped parameters is then numerically solved based on Pontryagin's maximum principle. The approximately optimal controls, obtained using the gradient projection method with a specially chosen step size, form a minimizing sequence of controls. Based on the numerical results, functional convergence is established. The method of successive approximations provides an optimal control solution as early as the second iteration, regardless of the initial control. This demonstrates the method's efficiency and reliability for solving linear optimal control problems. The developed numerical techniques can be applied to optimize the dynamic behavior of complex mechanical structures, enhance system stability, and improve operational efficiency in various engineering applications

Keywords: Pontryagin's maximum principle, wave equation, method of straight lines, functional convergence

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1. Introduction

The study of coupled systems with distributed and lumped parameters remains a highly relevant scientific topic due to its broad applicability in engineering, mechanics, and industrial systems. In recent years, significant advancements have been made in the theory of systems with distributed parameters, enriching the field with new analytical and computational methods. These developments have contributed to more effective approaches in modeling and controlling dynamic processes in coupled structures, enabling improved performance and stability in various applications. Integrating modern mathematical techniques and computational tools has further enhanced the precision and efficiency of control strategies for such systems.

The control of vibrations in coupled systems with distributed and lumped parameters is crucial in numerous practical applications. Oscillatory processes occur in various industrial and engineering domains, necessitating effective control mechanisms to enhance performance, reduce wear, and prevent structural failures. Examples include ship roll stabilization [1], where unwanted oscillations must be minimized to ensure navigational safety, and crane boom damping [2], which requires precise control to prevent excessive swaying. Other applications include vibratory conveyors used in material handling [3], active vibration protection systems in aerospace engineering, and shock absorption mechanisms in structural design [4]. These diverse applications highlight the necessity for further research into vibration control strategies tailored to systems with both distributed and lumped parameters [5].

The mathematical modeling of such systems typically relies on methods of mathematical physics, where governing equations are often expressed as partial differential equations (PDEs) subject to specific boundary and initial conditions. While exact analytical solutions exist for certain simplified cases, most practical problems necessitate the use of approximate and numerical methods to determine feasible control strategies. In particular, the wave equation serves as a fundamental model for describing the propagation of vibrations in elastic media, mechanical systems, and structural elements. The choice of boundary conditions significantly affects the system's behavior and the available control methodologies. Fixed boundaries restrict movement, while free boundaries allow oscillations that must be managed effectively.

Given the complexities associated with coupled systems exhibiting both distributed and lumped parameters, the study of their controllability and optimal control remains a critical scientific and engineering challenge. Addressing these issues contributes not only to advancing theoretical knowledge in mathematical modeling and control theory but also to developing innovative and effective vibration control techniques applicable to a wide range of technical systems. Therefore, research devoted to the development of advanced control methods for such coupled vibratory systems is of significant relevance.

2. Literature review and problem statement

One of the relevant applied problems in vibration control is the problem of damping vibrations in coupled systems. Due to numerous applications and its important theoretical significance, the control of vibratory processes with various boundary conditions has long attracted the attention of researchers, and there is extensive literature dedicated to this topic [5].

The paper [6] presents a comprehensive study on applied engineering solutions in industries such as automotive, maritime, aviation, and power generation. It is shown that innovative practical solutions have been developed for controlling nonlinear oscillations induced by limited energy sources. However, unresolved issues remain concerning the fundamental theoretical aspects of nonlinear oscillations, which are not deeply explored in this work. The reason for this may be the focus on industrial applications rather than on mathematical modeling and dynamic system analysis. A way to overcome this gap is to integrate advanced mathematical approaches into practical engineering solutions, ensuring a more comprehensive understanding of oscillatory behavior.

The study [7] presents methods that expand the understanding of optimization problems and their application to dynamic systems with oscillations. It is shown that optimization algorithms used in telecommunications networks or time series processing can be adapted for vibration control in mechanical systems. However, unresolved issues remain regarding the practical implementation of these optimization methods for real vibration control tasks. The reason for this may be the lack of specific examples and detailed discussions on their adaptation to mechanical systems. A possible way to overcome this limitation is to conduct case studies that demonstrate the real-world applicability of these algorithms in vibration suppression tasks.

The research in [8] provides a valuable resource on classical control methods, including the use of transfer functions and state-space representations for controlling oscillatory processes. It is shown that these approaches offer efficient strategies for vibration control, including the design of controllers to minimize unwanted oscillations. However, unresolved issues exist regarding the adaptation of these methods to complex, multi-component, or nonlinear systems. The reason for this may be that traditional methods require significant computational resources or may be inadequate for highly dynamic environments. A way to address this challenge is the development of hybrid control strategies that combine classical and modern optimization techniques for improved performance in complex vibratory systems.

The paper [9] explores the use of Dynamic Light Scattering (DLS) for monitoring and characterizing the aggregation dynamics of nanoparticles in colloidal systems. It is shown that DLS is a valuable tool for analyzing particle size distribution and aggregation, which can have implications for vibrational process optimization. However, unresolved issues remain regarding the comparison of DLS with alternative analytical methods, which could provide a more complete understanding of particle behavior under different conditions. The reason for this may be the methodological limitations of DLS in handling complex colloidal systems. A potential way to address this issue is the integration of complementary analysis techniques alongside DLS for a more robust assessment of vibrational process stability.

The study [10] investigates the oscillatory behavior of third-order differential equations, which is essential for optimizing oscillatory processes in various dynamic systems. It is shown that understanding relationships between solutions and their derivatives enables the development of new criteria for analyzing oscillatory behavior. However, unresolved issues exist in applying these theoretical results to practical engineering applications. The reason for this may be the lack of a detailed analysis of potential limitations and challenges in real vibration systems. A way to overcome these difficulties is to develop experimental validation studies that compare theoretical predictions with empirical data.

The numerical study in [11] presents computational experiments that confirm the theoretical results obtained in [10]. It is shown that numerical simulations provide valuable insights into system behavior. However, unresolved issues remain concerning the accuracy and sensitivity of the method to variations in initial data. The reason for this may be the absence of a thorough error analysis. A way to address this limitation is to implement rigorous error estimation techniques and sensitivity analyses to enhance the reliability of numerical results in practical applications.

The paper [12] examines the use of a reliable position controller for managing an elastic joint to suppress vibrations. It is shown that the Resonance Ratio Control (RRC) method effectively regulates system oscillations by setting a fixed ratio between resonance and anti-resonance frequencies. However, unresolved issues arise regarding the adaptability of this method to dynamic systems with variable characteristics. The reason for this may be the fixed nature of the frequency ratio, which may not be suitable for changing external conditions or system parameters. A way to overcome this limitation is to develop adaptive control techniques that can adjust control parameters in real time based on system dynamics.

All this suggests that it is advisable to conduct a study on the development of advanced vibration control strategies that integrate modern optimization techniques, adaptive control methodologies, and experimental validation approaches. Addressing the unresolved challenges identified in the literature will contribute to the advancement of effective vibration suppression solutions in coupled dynamic systems.

3. The aim and objectives of the study

The study aims to develop effective numerical method for solving optimal control problems in vibrating systems, where the parameters are distributed along two boundaries. This will make it possible to enhance control strategies for vibrations in coupled systems, ensuring improved stability and performance.

To achieve this aim, the following objectives are accomplished:

– to formulate the optimal control problem for vibrations of coupled systems and finding numerical solutions to the control problems;

– to develop algorithms for finding a sequence of controls that minimize the functional;

– to justify the convergence of the functional based on the conducted calculations.

4. Materials and methods

The object of the study is the dynamic behavior and control of vibrations in coupled systems with distributed and lumped parameters. The main hypothesis of the study is that the application of advanced numerical optimization techniques enables effective control of vibrations in coupled systems, ensuring stability and improved performance. The study is based on several assumptions: the system's equations of motion are accurately described by known mathematical models, the control inputs can be continuously adjusted to optimize system behavior, the applied numerical methods provide sufficient accuracy and convergence for solving the control problem, and the system parameters remain within a predefined range during operation. To simplify the analysis, certain approximations were adopted: the system dynamics were linearized to facilitate the application of analytical and numerical methods, external disturbances and noise were neglected in the primary model formulation, ideal boundary conditions were assumed without material degradation or nonlinearity, and spatial variables were discretized to reduce computational complexity while maintaining sufficient accuracy.

The research employed a range of numerical methods to solve optimal control problems for vibrations in coupled systems, addressing both spatial and temporal parameters of the model [13]. Optimization approaches and variational principles were utilized to determine optimal control inputs and regulate the system's dynamic behavior throughout its operation. The study primarily relied on the gradient projection method and the method of successive approximations, both of which are well-established in scientific practice due to their efficiency and convergence properties in solving complex problems.

The gradient projection method was applied within the variational framework to minimize a functional dependence on both system states and control inputs. This functional, representing the quality or effectiveness of control, guided the optimization process. The method involved iterative calculations of the functional's gradient, allowing control adjustments in the direction of the negative gradient to incrementally improve the system's behavior.

The method of successive approximations was used for solving linear boundary value problems that model system dynamics. This iterative approach refined control inputs at each step based on the system's current state. The optimization procedure incorporated the Hamiltonian function, which was maximized concerning the control variable, enabling the computation of optimal control inputs at each stage.

To enhance numerical accuracy, spatial variable discretization was employed, allowing for more precise modeling of the system's dynamic behavior. Additionally, temporal integration was performed using the Runge-Kutta method, a highly accurate and stable numerical technique well-suited for differential equations with distributed parameters. This integration method ensured computational precision, which is essential for obtaining reliable control solutions [14].

For practical implementation, specialized software was developed to automate computations, including discretization, gradient calculation, and iterative control adjustments. The software effectively integrated all methodological steps, facilitating efficient and precise optimization processes.

The research methodology was validated through numerical experiments, ensuring that the proposed approach could be effectively applied under various conditions. The structured combination of numerical techniques, including optimization algorithms, spatial discretization, and temporal integration, provided a robust framework for addressing vibratory control problems in coupled systems.

5. Results of the development of numerical methods for vibration control

5. 1. Formulation of the optimal control problem and deriving the computational formula

In the area $Q=\{0<x<1.0\leq t\leq T\}$ let's consider the vibrations of the system described by the following boundary value problem [4]:

$$u_{tt}(x,t)=a^2u_{xx}(x,t),(x,t)\in Q, \quad (1)$$

$$u(x,0)=\phi(x),u_t(x,0)=g(x),\ 0\leq x\leq 1, \quad (2)$$

$$u(0,t)=0,u(1,t)=y(t),\ 0<t\leq T, \quad (3)$$

$$\ddot{y}(t)+(a\mu)^2\,y(t)=p(t)+bu_x(1,t),\ 0<t\leq T, \quad (4)$$

$$y(0)=y^0,\ \dot{y}(0)=y^1, \quad (5)$$

where $\phi(x)\in C^2[0,1]$ is the initial state of an object with distributed parameters, $g(x)\in C^1[0,1]$ is the initial velocity of movement of an object with distributed parameters. The external force $p(t)\in C[0,T]$ is selected as a control. It is assumed that $0\leq p(t)\leq p_{\max}$, where $p_{\max}$ is the value specified based on technological conditions.

From the alignment of initial and boundary conditions, it is possible to obtain that:

$$\begin{cases}\phi(0)=0,\phi''(0)=0,g(0)=0,\\ \phi(1)=y^0,g(1)=y^1,\\ p(0)=a^2\phi''(1)-b\phi'(1)+(a\mu)^2\phi(1).\end{cases} \quad (6)$$

The first condition in (3) indicates that the left end of the distributed system is fixed, while the second condition in (3) means that a lumped system is attached to the right end of this system, and there, an external force $p(t)$ is applied to the distributed system.

It is required to control the external force $p(t)$ in such a way that at time $t=T$, the system is in a state that differs little from its rest state. Such a deviation is taken as the functional:

$$F=\int_0^1\left[u^2(x,T)+u_t^2(x,T)\right]\mathrm{d}x. \quad (7)$$

The control $p(t)$ that satisfies the above-mentioned conditions is called a permissible control. In particular, if $inf\,F=0$, then it is possible to completely dampen the vibrations of the coupled system by time T.

The results that determine the general solution of the boundary value problem (1)–(5) depend on how the coefficients α, β, and μ are related. It can be obtained by separately solving the problem with zero initial conditions $y^0=0$, $y^1=0$ and the problem with homogeneous boundary conditions. In this case, the general solution of the boundary value problem (1)–(5) is the sum of the solutions of the considered boundary problems [5].

Let's find the solution y(t) to the Cauchy problem (4), (5):

$$y(t)=y^0\cos(a\mu t)+\frac{1}{a\mu}y^1\sin(a\mu t)+$$

$$+\frac{1}{a\mu}\int_0^t\left(p(\tau)+bu_x(1,\tau)\right)\sin\left(a\mu(t-\tau)\right)\mathrm{d}\tau. \quad (8)$$

Considering the condition $u(1,t)=y(t)$, it is possible to write the boundary condition that the function $u(x,t)$ must satisfy at the point $x=1$:

$$u(1,t)=y^0\cos(a\mu t)+\frac{1}{a\mu}y^1\sin(a\mu t)+$$
$$+\frac{1}{a\mu}\int_0^t\big(p(\tau)+bu_x(1,\tau)\big)\sin\big(a\mu(t-\tau)\big)\mathrm{d}\tau. \quad (9)$$

If the variable y(t) is excluded from the conditions (3)–(5), then (3)–(5) can be represented in the following form:

$$u(0,t)=0,\, u_{tt}(1,t)+(a\mu)^2u(1,t)=p(t)+bu_x(1,t), \quad (10)$$

$$u(1,0)=y^0,\, u_t(1,0)=y^1. \quad (11)$$

It should be noted that depending on the sign of the product $b^2–4a^4\mu^2$, in [16], using the continuation method, formulas for the general solution of the boundary value problem (1)–(5) were derived, and control laws were determined.

From the results obtained in this paper, as a special case, the control laws for the vibration of a string with first-kind boundary conditions can be derived, i.e., for the control problem described by the boundary value problem (1)–(3), where the control function is $y(t)$.

In this paper, two methods are used for the numerical solution of the considered problem – the gradient projection method and the method of successive approximations. The boundary value problem describing the control process is replaced by ordinary differential equations using the direct method, and the resulting variational problem is solved numerically based on Pontryagin's maximum principle.

Let $u_t(x,t)=\upsilon(x,t)$ be denoted. Then, the boundary value problem (1)–(5) and the functional (7) can be written in the following form:

$$u_t(x,t)=\upsilon(x,t), \quad (12)$$

$$\upsilon_t(x,t)=a^2u_{xx}(x,t),\ 0<x<1.0<t\le T, \quad (13)$$

$$u(x,0)=\phi(x),\ \upsilon(x,0)=g(x),\ 0\le x\le 1, \quad (14)$$

$$u(0,t)=0,\ \upsilon_t(1,t)+(a\mu)^2u(1,t)=p(t)+bu_x(1,t), \quad (15)$$

$$0<t\le T,$$

$$u(1,0)=y^0,\ \upsilon(1,0)=y^1, \quad (16)$$

$$F=\int_0^1\left[u^2(x,T)+\upsilon^2(x,T)\right]\mathrm{d}x. \quad (17)$$

Therefore, the problem (1)–(7) reduces to determining the function $p(t)$ from the condition of minimizing the functional (17) subject to the constraints (12)–(16).

Using the direct method, it is possible to construct a finite-dimensional approximation of the boundary value problem (12)–(16). Let $x_i=ih$, $i=0,1,\ldots,n$ is the grid of lines with a step $h=1/n$ on the segment [0,1]. Let $u_i(t)=u(x_i,t)$, $\upsilon_i(t)=v(x_i,t)$, $i=1,2,\ldots,n$ be denoted. Then, taking into account the conditions $u_0(t)=0$, the problem (12)–(16) at the grid nodes can be approximated by a system of ordinary differential equations:

$$\begin{cases}\dot{u}_i=\upsilon_i, i=1,2,\ldots,n,\\ \dot{\upsilon}_1=\dfrac{a^2}{h^2}\left[-2u_1+u_2\right],\\ \dot{\upsilon}_i=\dfrac{a^2}{h^2}\left[u_{i-1}-2u_i+u_{i+1}\right], i=2,3,\ldots,n-1,\\ \dot{\upsilon}_n=\dfrac{1}{h}\left[-bu_{n-1}+\left(b-h(a\mu)^2\right)u_n\right]+p(t),\end{cases} \quad (18)$$

with the following initial conditions:

$$\begin{cases}u_i(0)=\phi(x_i), \upsilon_i(0)=g(x_i), i=1,2,\ldots,n,\\ u_n(0)=y^0, \upsilon_n(0)=y^1.\end{cases} \quad (19)$$

Thus, the original problem (12)–(17) reduces to determining the external force $p(t)$ from the condition of minimizing the approximating sum:

$$F=h\sum_{i=1}^{n}\left[u_i^2(T)+\upsilon_i^2(T)\right]. \quad (20)$$

To bring the system (18) into the canonical Cauchy form, let's introduce an $2n$-dimensional vector with components $(u_1, u_2,\ldots,u_n, u_{n+1}, u_{n+2},\ldots,u_{2n})$, where $u_{n+i}=\upsilon_i$, $i=1, 2, \ldots, n$, and rewrite the system of equations (18), (19) as follows:

$$\begin{cases}\dot{u}_i=u_{n+i}, i=1,2,\ldots,n,\\ \dot{u}_{n+1}=\dfrac{a^2}{h^2}\left[-2u_1+u_2\right],\\ \dot{u}_{n+i}=\dfrac{a^2}{h^2}\left[u_{i-1}-2u_i+u_{i+1}\right], i=2,3,\ldots,n-1,\\ \dot{u}_{2n}=\dfrac{1}{h}\left[-bu_{n-1}+\left(b-h(a\mu)^2\right)u_n\right]+p(t),\end{cases} \quad (21)$$

with the following initial conditions:

$$\begin{cases}u_i(0)=\phi(x_i), u_{n+i}(x_i)=g(x_i), i=1,2,\ldots,n-1,\\ u_n(0)=y^0, u_{2n}(0)=y^1.\end{cases} \quad (22)$$

The functional (20) takes the form:

$$F=h\sum_{i=1}^{2n}u_i^2(T). \quad (23)$$

Thus, the problem is reduced to determining the function $p(t)$ from the condition of minimizing (23) subject to the constraints (21), (22). Let's construct the Hamiltonian function for the problem (21)–(23):

$$H=\sum_{i=1}^{n}\psi_i u_{n+i}+\psi_{n+1}\frac{a^2}{h^2}\left(-2u_1+u_2\right)+$$
$$+\frac{a^2}{h^2}\sum_{i=2}^{n-1}\psi_{n+i}\left(u_{i-1}-2u_i+u_{i+1}\right)+$$
$$+\psi_{2n}\frac{1}{h}\left[\left(-bu_{n-1}+\left(b-h(a\mu)^2\right)u_n\right)+hp(t)\right]. \quad (24)$$

For the convenience of constructing the conjugate system of functions H, let's first represent them in the following form:

$$H=\sum_{i=1}^{n}u_{n+i}\psi_i+\frac{a^2}{h^2}u_1\left(-2\psi_1-\psi_2\right)+$$
$$+\frac{a^2}{h^2}\sum_{i=2}^{n-2}u_i\left(\psi_{n+i-1}-2\psi_{n+i}+\psi_{n+i+1}\right)+$$
$$+\frac{a^2}{h^2}u_{n-1}\left(\psi_{2n-2}-2\psi_{2n-1}-\frac{hb}{a^2}\psi_{2n}\right)+$$
$$+\frac{a^2}{h^2}u_n\left(\psi_{2n-1}+\frac{h}{a^2}\left(b-h(h\mu)^2\right)\right)+\psi_{2n}\cdot p(t). \quad (25)$$

Then, the system of conjugate equations, according to the standard formulation of the maximum principle, can be written in the following form:

$$\begin{cases} \dot{\psi}_1 = -\frac{a^2}{h^2}\left[-2\psi_{n+1}+\psi_{n+2}\right], \\ \dot{\psi}_i = -\frac{a^2}{h^2}\left[\psi_{n+i-1}-2\psi_{n+i}+\psi_{n+i+1}\right], i=2,3,\ldots,n-2, \\ \dot{\psi}_{n-1} = -\frac{a^2}{h^2}\left[\psi_{2n-2}-2\psi_{2n-1}-\frac{hb}{a^2}\psi_{2n}\right], \\ \dot{\psi}_n = -\frac{a^2}{h^2}\left[\psi_{2n-1}+\frac{h}{a^2}\left(b-h(a\mu)^2\psi_{2n}\right)\right], \\ \dot{\psi}_{n+i} = -\psi_i, i=1,2,\ldots,n. \end{cases} \quad (26)$$

The boundary conditions for $\psi_i(T)$ and $F'=-dH/dp$ are as follows:

$$\psi_i(T) = -2hu_i(T), i=1,2,\ldots,2n, \quad (27)$$

$$F'(p) = -\psi_{2n}(t). \quad (28)$$

To compute the gradient for a given control $p(t)$, it is first necessary to solve the system of equations (21) with the initial conditions (22), which allows determining the state function values $u_i(T)$, where $i=1,2,\ldots,n$ at the final time T. Then, the obtained values $u_i(T)$ are substituted into expression (27), after which the adjoint system of equations (26) is integrated with the boundary conditions (27) defined over the final time interval. Once the adjoint system is solved, the gradient $F'(p)$ can be computed using formula (28), which enables determining the direction for optimizing the control parameter $p(t)$.

5. 2. The development of an algorithm for determining the minimizing sequence

When using the gradient projection method, a sequence of control functions is constructed, starting from some admissible control $p^*(t)$. The main work when transitioning from one control $p^*(t)$ to the next $p^{k+1}(t)$ is related to calculating the gradient of the functional using the formula (28).

The algorithm for solving the problem consists of the following steps:

Step 1. An admissible control $p^k(t)$ is selected. Based on the selected $p^k(t)$ using the Runge-Kutta method (or the Euler method, if stability conditions allow), the system of equations (21), (22) is integrated with the "forward direction". The values of $u_i(t)$, $i=1, 2,\ldots, 2n$ are computed and stored. It should be noted that for linear systems, it is not necessary to store all the values of the functions $u_i(t)$, $i=1, 2,\ldots, 2n$ in the computer's memory. Only their final values need to be remembered, and then the conjugate system (26) can be integrated simultaneously with the system (21) from $t=T$ to $t=0$. However, this approach increases the computational load and, the computation process, especially for nonlinear systems, often becomes unstable.

Step 2. The values of the approximating sum (23) are computed.

Step 3. The values of $\psi_i(T)$, $i=1, 2, \ldots, 2n$ are computed using the formulas (27).

Step 4. In the "backward direction" of time, the system (26), (27) is integrated, and the gradient of the functional (23) is computed using the formulas (28).

Step 5. Taking into account $p^{k+1}(0)=a^2\phi'(1)-b\phi(1)$, the new control $p^{k+1}(t)$ for the values $t_j=j\Delta t, j=1,2,\ldots,m, m\Delta t=T$ is computed using the formula:

$$p^{k+1}(t) = \begin{cases} 0, \text{ if } p^k(t)+\delta p^k(t)<0, \\ p_{\max}, \text{ if } p^k(t)+\delta p^k(t)>p_{\max}, \\ p^k(t)+\delta p^k(t), \text{ if } 0\le p^k(t)+\delta p^k(t)\le p_{\max}. \end{cases} \quad (29)$$

Here:

$$\delta p^k(t) = \alpha\cdot\frac{\psi_{2n}^k(t)}{\left|\max\psi_{2n}^k(t)\right|}, k=0,1,\ldots, \quad (30)$$

where Δt is the discretization step in time, k is the iteration number, and the parameter $\alpha>0$ is chosen using one of the methods described in [3].

Step 6. A step is taken with the new control $p^{k+1}(t)$, returning to Step 2.

5. 3. Obtaining a numerical solution to the problem

To obtain a numerical solution to the problem (21)–(23) using the described algorithm, computer codes were developed. The code allows for the automatic numerical solution of a fairly wide range of optimal control problems. To verify the correctness of the entire computational process, intermediate computational results were checked using an unconditional jump operator. This included analyzing the behavior of the Cauchy problem (21), (22) with fixed admissible control, evaluating the sum (23) immediately after integrating the system of equations (21), (22), and examining the behavior of the adjoint problem (26), (27). Additionally, the boundary conditions (27) for system (26) were computed, along with the gradient values $F'(p)$ using formula (28).

The systems of equations (21), (22) and the adjoint system (26), (27) were integrated using a constant step size $\Delta t=0.01$, and results were output at step size $\Delta t=0.05$. The interval [0, 1] was divided into ten equal parts with a step size $h=0.1$. Clearly, for the chosen values of Δt and h, the stability condition $\Delta t^2\le 0.5h^2$ holds, meaning that the system of equations (21), (22) and the system (26), (27) could also be integrated using Euler's method.

The problem was solved with the following parameter and function values:

$$T=0.2, p_{\max}=4, \alpha=0.2, a=1, b=6, \mu=3,$$

$$\phi(x)=x^3(1-x), g(x)=x(1-x).$$

As the initial approximation, the function $p^0(t)$ was taken as:

$$p^0(t) = p_{\max}\cdot\frac{t}{T}. \quad (31)$$

The value of the functional (23) for the initial approximation (31) was found to be 0.123, which is computed after integrating the system of equations (21), (22). It is important to note that condition (6) in the boundary problem (1)–(5) does not allow for a wide variation of the functions and initial data. It is easy to see that for the selected parameter values and functions, the following conditions are satisfied:

$$\begin{cases} b^2-4a^2(a\mu)^2=0, \\ \phi(0)=g(0)=0, \\ \phi'(0)=\phi''(0)=0, \\ \phi(1)=y^0=0, \\ g(1)=y^1=0, \\ p(0)=a^2\phi''(1)-b\phi'(1)+(a\mu)^2\phi(1)=0. \end{cases} \quad (32)$$

Taking these conditions into account, the solution of the boundary problem (1)–(5) for $0 \le t \le 2$ has the following form:

$$u(x,t)=\frac{1}{2}\left[\Phi(x+t)+\Phi(x-t)\right]+$$
$$+\frac{1}{2}\int_{x-t}^{x+t}G(z)\mathrm{d}z+\int_{0}^{\theta}(\theta-s)\exp\left(3(\theta-s)\right)\bar{p}(s)\mathrm{d}s-$$
$$-\int_{0}^{\eta}(\eta-s)\exp\left(3(\eta-s)\right)\bar{p}(s)ds, \qquad (33)$$

where $\theta=\theta(x,t)=t-1+x, \eta=\eta(x,t)=t-1-x$, and $\bar{p}(t)$, used in solving the problem with zero initial conditions, has the form of:

$$\bar{p}(t)=\begin{cases}p(t), t>0,\\ 0, t\le 0.\end{cases}$$

In formula (33), the functions $\Phi(z)$ and $G(z)$ are extensions of the functions $\phi(z)$ and $g(z)$, respectively. These extensions for $1 \le z \le 2$ are determined as follows:

$$\Phi(z)=(z-1)\exp\left(3(z-1)\right)+$$
$$+\int_{1}^{z}(z-\varsigma)\exp\left(3(z-\xi)\right)f(\zeta)\mathrm{d}\zeta, \qquad (34)$$

$$G(z)=\int_{1}^{z}\left[1+3(z-\xi)\right]\exp\left(3(z-\xi)\right)\rho(\zeta)\mathrm{d}\zeta, \qquad (35)$$

where $2 \le z \le 3$:

$$\rho(z)=3z^{3}-\frac{29}{2}z^{2}+2z+\frac{3}{2}. \qquad (36)$$

The extensions of the functions $\phi(z)$ and $\psi(z)$ for $2 \le z \le 3$ are determined using similar formulas, while for $-2 \le z \le 0$, they are given by specific formulas $\Phi(z)=-\Phi(-z)$, $G(z)=-G(-z)$ [5].

As seen from the given formulas (33)–(36), their practical implementation involves significant computational difficulties. To overcome this challenge in the numerical solution of the boundary value problem (12)–(16), the method of straight lines is applied. Table 1 shows the convergence of the functional for the controls given in equation (31).

Table 1

Convergence of the functional for the given controls

No. of iteration	F	$\max \psi_{2n}(t)$
0	1.2299 E-01	3.8597 E-01
1	1.1871 E-01	3.7337 E-0.1
2	1.1462 E-01	3.6082 E-01
3	1.1071 E-01	3.4835 E-01
4	1.0699 E-01	3.3599 E-01
5	1.0348 E-01	3.2384 E-01
6	1.0015 E-01	3.1178 E-01
7	9.6985 E-02	2.9979 E-01
10	8.8550 E-02	2.6475 E-01
14	7.9548 E-02	2.1995 E-01
26	6.8265 E-02	1.3540 E-01
27	6.8245 E-02	1.3495 E-01
28	6.8245 E-02	1.3494 E-01

In further iterations, the functional value did not change. The data presented in the table clearly show that the approximated optimal control exhibits convergence of the functional. It should be noted that convergence of the functional always occurs if the solution of the approximating system (21) converges to the solution of the original boundary value problem [1].

The sequence of controls obtained in some intermediate iterations is shown in Fig. 1. From the analysis of this figure and the table data, it follows that the gradient projection method in problem (21), (22) provides a minimizing sequence of controls.

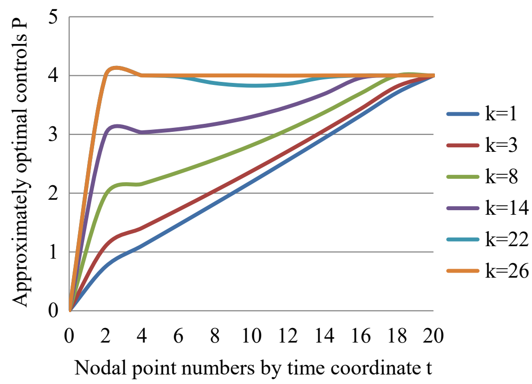

Fig. 1. Sequence of intermediate controls

It should be noted that when using the method of successive approximations, at each step, the control variable is selected according to the maximum principle based on the condition of maximizing the Hamiltonian function, i.e., during iterations, according to the formula:

$$p^{k+1}(t)=\frac{p_{\max}}{2}\left(1+\mathrm{sign}\left(\psi_{2n}^{k}(t)\right)\right), k=0,1,\ldots. \qquad (37)$$

The question of the convergence conditions for this process of successive approximations remains open. Even in the case of convergence, it is generally unknown whether the obtained control is truly optimal, as the maximum principle provides only a necessary condition for optimality. Nevertheless, for certain variational problems where the existence and uniqueness of the solution are clear from physical considerations, the method of successive approximations can be used to find optimal controls.

Due to the linearity of the system of equations (21) with respect to the phase variables and control, the method of successive approximations in the problem (21)–(23), regardless of the initial control, yields the optimal control as early as the second iteration:

$$p^{*}(t)=\begin{cases}p_{\max}, \text{ if } t>0,\\ 0, \text{ if } t=0.\end{cases} \qquad (38)$$

It is easy to see that as the number of iterations increases, the sequence of controls constructed using formulas (29), (30) converges to the function $p^{*}(t)$.

6. Discussion of experimental results relative to vibration control strategies in coupled systems with distributed and lumped parameters

Based on the data provided in Table 1, the functional's convergence of the control sequence is confirmed. The graphs shown in Fig. 1 demonstrate that, despite the incorrectness of problem (1)–(5) with the quadratic functional (7), the gradient projection method (31), (32) with a special step selection produces a converging sequence of controls and requires only a small number of iterations. Calculations show that the use of the method of successive approximations allows a solution to be obtained as early as the second approximation.

A key advantage of the proposed method compared to similar studies [5, 7, 8] is its ability to eliminate the need for complex mathematical transformations and computationally intensive operations. This simplifies the problem-solving process, reduces computational load, and still maintains high accuracy and reliability of the results.

Within the scope of this work, the direct method is applied, which has proven effective both in solving boundary problems of mathematical physics and in addressing optimal control problems for systems with distributed parameters. This approach significantly reduces the labor intensity of numerical calculations while maintaining high accuracy in the obtained solutions. Moreover, due to its versatility, the direct method is particularly well-suited for optimal control problems in dynamic systems, whose processes are described by nonlinear boundary conditions.

Unlike the approaches presented in [9, 10], which rely on iterative numerical schemes requiring multiple recalculations per step, the proposed method provides a more computationally efficient framework. While methods in [11] emphasize spectral decomposition techniques for vibration suppression, they often suffer from sensitivity to parameter variations, making them less robust for practical applications. Additionally, the control strategies described in [12] incorporate adaptive elements, but their implementation demands high computational power, making real-time applications challenging. In contrast, the developed approach ensures stability and efficiency even with limited computational resources, making it well-suited for engineering applications where real-time control is essential.

It should be noted that the results of research on vibration control in coupled systems consisting of two objects hold significant theoretical importance. However, the practical application of the formulas derived in these works is associated with substantial computational challenges when constructing and solving boundary value problems and control variables. The present study aims to develop practical applications of vibration control methods in coupled systems and to refine the conditions of controllability. The results obtained may find applications in various fields of engineering, including transportation mechanics, robotics, construction, and vibration isolation, contributing to the further advancement of vibration suppression technologies in complex technical systems.

One of the significant limitations of the conducted research is the lack of a detailed and rigorous analysis of the convergence of the approximate solution to the Cauchy problem (21), (22), which is related to the original boundary value problem (1)–(5). This aspect represents an important issue, as the convergence of the solutions to the approximation problem (21)–(23) directly affects the accuracy and reliability of the results obtained. Future research could focus on deriving a priori estimates for linear and non-homogeneous systems of ordinary differential equations, which would allow for a deeper understanding of the behavior of the solution when approximating boundary problems. Moreover, the issue of the convergence of approximate solutions in the context of optimal control, due to the ill-posed nature of the considered optimal control problem, remains unresolved and requires further investigation. Verifying the correctness and accuracy of the proposed method, especially in practical applications, is essential for evaluating its real effectiveness and suitability for solving complex optimal control problems.

One of the most promising directions for future research is the extension of the proposed method to more complex systems involving more than two interconnected objects. Such an expansion will open new horizons for applying this approach, enabling the solution of larger-scale and more complex problems involving interactions between different elements. In particular, studying multi-component systems with interconnected elements provides a unique opportunity to significantly improve the accuracy and efficiency of the method, especially in the context of optimal control.

The introduction of additional objects and more complex interconnections between them implies that the methods used to solve problems within a single object must be adapted to the new, more intricate structure. This will require the development of new algorithms that can effectively account for numerous variables and their interactions. Given the multi-component nature of such systems, new approaches must be flexible and efficient in considering all interacting elements, minimizing errors arising from model simplifications.

An additional benefit of expanding the scope of the method is the potential for more accurate modeling of real systems, where elements often interact with one another in varying degrees of complexity. Therefore, further development of optimal control methods for such complex systems will not only contribute theoretically but also have practical value, enabling the solution of problems that are traditionally too complex for existing methods.

7. Conclusions

1. In the course of the research, the optimal control problem for a system with distributed and lumped parameters was formulated, taking into account controllability and optimality criteria. Constraints on admissible controls and system states were defined, and the necessary optimality conditions were derived. It was shown that, in this problem, the optimal control formally represents a piecewise constant function, successively taking the specified boundary values.

2. An algorithm was developed to find the minimizing sequence of controls that ensures the optimal result in a system with distributed and lumped parameters. The algorithm is based on an iterative method, which involves the successive refinement of controls, considering the specified optimality criterion. For constrained problems where a function of the final state is minimized, the gradient projection method is more efficient than the method of successive approximations.

3. Numerical methods for solving vibration control problems in a coupled system consisting of two interacting objects have been implemented. These methods are based on the discretization of the original model for spatial coordinates and the application of the gradient method, ensuring the convergence of the minimizing control sequence in terms of the functional, as well as the method of successive approximations. Computational schemes have been developed to efficiently determine an approximately optimal control solution. Based on numerical experiments, it has been established that when using the method of successive approximations, if the initial control $p^0(t)$ is relay-type, then all subsequent $u^k(t)$ controls also remain relay-type, and the optimal control is found as early as the second iteration.

Conflict of interest

The authors declare that they have no conflict of interest in relation to this research, whether financial, personal, authorship or otherwise, that could affect the research and its results presented in this paper.

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Data availability

Data will be made available on reasonable request.

Use of artificial intelligence

The authors have used artificial intelligence technologies within acceptable limits to provide their own verified data, which is described in the research methodology section.

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