This study investigates heat exchange processes for thermally sensitive media with local near-surface and internal heating. As a result of the thermal load, significant temperature gradients arise. To establish temperature regimes for effective operation of electronic devices, linear and nonlinear mathematical models for determining the temperature field have been constructed, which could allow further analysis of temperature regimes.

Based on the stated linear and nonlinear boundary value problems of thermal conductivity, their analytical and numerical solutions have been derived. Using these solutions, numerical calculations of the temperature distribution in spatial coordinates for given geometric and thermophysical parameters have been performed. Reliability of the results has been confirmed by experimental findings and the determined numerical values of temperature distribution in the medium.

For an effective description of local heating, the theory of generalized functions was used. A technique for linearizing nonlinear mathematical models has been introduced. As a result, linear second-order differential equations with partial derivatives and a singular right-hand side have been derived.

The numerical results reflect the temperature distribution in the medium in spatial coordinates for the given geometric and thermophysical parameters. The number of divisions of the interval (0; x*) was chosen to be 9, which made it possible to derive numerical values of temperature with an accuracy of 10^{-6}. The obtained numerical values of temperature for silicon under a linear temperature dependence of the thermal conductivity coefficient differ from the results obtained for its constant value by 2%

Keywords: temperature field, thermal conductivity of material, thermal stability of structures, heat-sensitive material, thermally-active zone

UDC 536.24

DOI: 10.15587/2706-5448.2025.341525

CONSTRUCTION OF MATHEMATICAL MODELS OF HEAT EXCHANGE FOR DIGITAL DEVICES WITH LOCAL NEAR-SURFACE AND INTERNAL HEATING

Vasyl Havrysh
Doctor of Technical Sciences, Professor*

Svitlana Yatsyshyn
Corresponding author
PhD
Department of Software Engineering
Ukrainian National Forestry University
Henerala Chuprynky str., 103, Lviv, Ukraine, 79057
E-mail: yatsyshyn@nltu.edu.ua

Viktoria Maikher
PhD*

Oksana Hrytsai
PhD*

Fedir Honchar
PhD
Department of General Physics**
*Department of Software**
**Lviv Polytechnic National University
S. Bandery str., 12, Lviv, Ukraine, 79013

Received 25.07.2025
Received in revised form 24.09.2025
Accepted 06.10.2025
Published 31.10.2025

How to Cite: *Havrysh, V., Yatsyshyn, S., Maikher, V., Hrytsai, O., Honchar, F. (2025). Construction of mathematical models of heat exchange for digital devices with local near-surface and internal heating. Eastern-European Journal of Enterprise Technologies, 5 (5 (137)), 31–40. https://doi.org/10.15587/2706-5448.2025.341525*

1. Introduction

Construction of heat transfer mathematical models for state-of-the-art electronic devices is an important area of research that has attracted significant attention from both the scientific community and industry.

The temperature field in microelectronic devices is a critical factor that determines their efficiency and reliability. An increase in temperature can reduce performance, increase power consumption, and shorten the device's service life. Heat generation is related to the passage of electric current through components, which leads to their heating and the formation of a temperature field.

With the increasing complexity and miniaturization of electronic devices, effective heat dissipation is essential to enable optimal performance and reliability. The relative influence of temperature is the highest (55%) compared to humidity (19%), vibration (20%), and dust (6%). Various studies have focused on investigating new materials, structures, and models to increase thermal conductivity and improve heat control in electronic systems.

Experimental methods for determining the temperature field, such as thermal impedance microscopy, infrared microscopy, or thermometry, make it possible to obtain data on the temperature distribution in devices. However, each of these methods has its limitations and is used under specific conditions. Computer simulations based on mathematical models of thermal processes make it possible to predict temperature regimes of device operation and define optimal design parameters.

Important factors affecting the temperature regime are the location of components and their density on board. Uneven heat dissipation, in particular for elements in the center of the board, can lead to local overheating. Various technologies are used to ensure effective cooling: heat dissipation materials, fans, thermal pipes, thermal pastes, etc. In addition, the use of temperature sensors make it possible to control heating and timely reduce the risks of overheating.

The relevance of studying the temperature field is increasing due to the significant thermal loads on modern devices. The existence of significant temperature gradients in local sectors of the medium can cause overheating, leading to damage to components or device failure. Therefore, construction of mathematical models of heat transfer is an important task for predicting thermal regimes and enabling stable operation of electronic devices without the need for expensive experiments.

2. Literature review and problem statement

It is shown in [1] that the improvement of thermal interface materials is an important direction for improving heat dissipation and increasing the performance of electronic devices. However, the issues regarding the effectiveness of such materials under difficult heating conditions arising from local heating by heat sources remain unresolved.

In [2], the importance of optimizing thermal processes in porous materials is emphasized; however, the method used does not make it possible to take into account local thermal disturbances, which are critically important as a result of heating of electronic devices. This creates limitations in the application of such models for actual devices.

A mathematical model of heat transfer in porous materials with temperature-dependent thermophysical parameters is reported in [3], which is relevant for structures of complex architecture, in particular digital electronic devices. However, the model does not take into account local thermal disturbances characteristic of such devices, which limits its accuracy.

Analytical solutions [4] describe the distribution of temperature, displacements, and stresses in simply supported laminated plates under thermomechanical loading, taking into account the temperature dependence of materials. However, they do not reflect local temperature loads, which limits their application to real operating conditions.

A thermal conductivity model for electronic devices is reported in [5]. However, the influence of near-surface temperature disturbances in the environment and the thermal sensitivity of structural materials on thermal resistance has not been investigated, which limits its application.

Studies on nanofluid-based microchannel radiators and thermoelectric generators emphasize the importance of modeling and optimizing the temperature control system. This is necessary for effective heat dissipation in electronic devices. Works [6, 7] reflect effective approaches to modeling thermal processes in such devices; however, due to the linearization of nonlinear heat conduction problems through the thermal sensitivity of their structural materials, there is a need for significant computational resources to ensure accuracy.

In work [8], experimental studies on the behavior of the temperature field in nanomaterials were thoroughly performed. However, significant errors in numerical temperature values were obtained based on measurements.

The reconstruction of the temperature field is important for thermal regulation of electronic equipment. In [9], a deep learning method is proposed, which combines UNet and multilayer perceptron (MLP) to transform the problem of temperature field reconstruction into a regression problem. UNet is responsible for the reproduction of the general temperature field, while MLP makes it possible to predict zones with large temperature gradients. The results of numerical experiments are obtained with an error value that is less than 1°K, but the use of this method requires a large amount of data for training, which complicates its application under actual conditions.

In [10], the thermomechanical loads of columns under longitudinal thermal heating with different boundary conditions were investigated. The temperature distribution was determined by the differential quadrature method (DQM); the deflection analysis was performed based on the Euler-Bernoulli theory. The results were confirmed by FEM and literature data. The main drawback is the simplification of the model, which does not take into account significant temperature gradients arising from critical thermal loads.

In [11], the results of studies are reported, which emphasize the importance of thermal interface materials for improving heat dissipation in electronic devices. It is shown that such materials play a significant role in facilitating heat transfer between device components, which contributes to increasing their performance. However, the tasks associated with the need to design materials with improved characteristics that could ensure the effective operation of devices under difficult conditions of local heating remain unfulfilled. The reason is the complexity of experimental verifications and the high cost of designing such materials.

In study [12], the concept of double Cattaneo-Christov diffusion in entropy-optimized nanofluids with a variable thermal conductivity coefficient is considered. It is shown that the use of such models make it possible to take into account multiple diffusion mechanisms; however, the consideration of complex thermal properties of liquids and their influence on heat transfer remains insufficient. This is due to the difficulties of modeling the behavior of magnetized liquids under actual conditions.

The authors of [13] built an analytical model for regenerative cooling systems using elastic caloric rubber. The importance of using such systems to optimize heat transfer is indicated, but the model is limited in terms of analyzing thermal processes for environments with local both external and internal heating. This is due to the limitations of the modeling technologies used, which do not make it possible to take into account transient heating regimes.

An option to overcome these difficulties is to construct complex mathematical models of thermal conductivity, which take into account the heterogeneity of media, the locality of thermal heating, and the variability of thermal properties of materials. This approach was partially applied in [12, 13], but it is not possible to take into account important aspects of complex thermal processes associated with the thermal sensitivity of structural materials.

In [14], dynamic compact thermal models for predicting the case temperature of portable devices, such as smartphones and laptops, using the convolution method are reported. The models make it possible to quickly determine the case temperature taking into account the response of each heat source, which contributes to the optimization of thermal design and the choice of a temperature control strategy at the early stages of development. However, their application is limited to two types of devices and is not covered by a wide range of portable equipment.

A solution is given in [15] for the steady state reaction of thick cylinders subjected to pressure and external heat flux on the inner surface. However, the influence of the temperature gradient on the deformation of the medium is not taken into account, which significantly worsens the accuracy of the model.

A nonlinear mathematical model for determining the temperature field in a thermosensitive layered plate with foreign inclusions has been built in [16]. The use of this model does not make it possible to analyze temperature regimes in a flat thermosensitive medium with local internal and external heating.

In [17], a nonlinear model for determining the temperature field in a thermosensitive layer with a through-going foreign inclusion was constructed. A linearizing function was introduced, which allowed the nonlinear axisymmetric boundary value problem to be reduced to a quasi-linear problem. The layer is heated by a heat flux concentrated in a circle at the edge of the medium. The model does not provide for local temperature disturbances.

A linear mathematical model for determining the temperature field in a segmentally homogeneous layer with a thermally active inclusion was built in [18]. The model does not provide for the analysis of temperature regimes for the case of a thermosensitive medium and local internal heating as the heat sources are uniformly concentrated in the volume of the cylindrical inclusion.

In [19], a linear mathematical model was constructed for the analysis of temperature regimes in elements of electronic devices of a segmentally homogeneous structure with through-going foreign elements. The model does not provide for the temperature dependence of the thermophysical parameters of structural materials and external local heating.

Our review of the literature reveals a problem related to the lack of theoretically justified approaches for linearization of heat conduction problems in thermally sensitive media. Current mathematical models do not sufficiently reflect heat transfer between structural elements of electronic devices, taking into account local thermal heating, which limits their effectiveness for multifunctional structures. The main reasons are the complexity of describing thermal processes in such media, as well as the high cost of experiments and the difficulty of creating conditions for proper validation of models.

All this indicates the feasibility of conducting research on the construction of linear and nonlinear mathematical models of heat transfer for isotropic flat media with thermally active heating zones. Such models could be used to predict the thermal modes of operation of modern electronic devices, which would contribute to increasing their efficiency and reliability.

3. The aim and objectives of the study

The purpose of our study is to build linear and nonlinear mathematical models for determining temperature fields in isotropic flat media with thermally active heating zones. As a result, it will be possible to increase their accuracy of determination and more effectively analyze temperature regimes, which will further affect the effectiveness of design methods for modern electronic devices.

To achieve this goal, it is necessary to solve the following problems:

– to construct a linear mathematical model of heat transfer in an isotropic flat media with local near-surface heating;

– to build a nonlinear mathematical model of heat transfer in an isotropic thermosensitive (thermophysical parameters of the material depend on temperature) flat media with local near-surface heating;

– to construct a linear mathematical model of heat transfer in an isotropic flat media with local internal heating;

– to build a nonlinear mathematical model of heat transfer in an isotropic thermosensitive flat media with local internal heating.

4. The study materials and methods

The object of our study is the process of heat transfer in isotropic flat media with local near-surface and internal heating.

Research hypothesis: if the temperature fields in a flat medium are caused by local near-surface and internal thermal heating, then they could be described by analytical-numerical solutions to linear and nonlinear boundary value problems of heat conduction. Differential equations with partial derivatives of the second order of these problems contain right-hand sides with the Dirac delta function, which make it possible to describe the local concentration of heating.

It is assumed in the process of the study that the flat medium is isotropic, that is, the values of thermophysical parameters are constant in spatial directions. The constructed linear and nonlinear mathematical models of heat transfer are simplified since the change in the temperature field, and the analysis of temperature regimes are determined only by spatial coordinates.

Experimental temperature measurements were performed using the SKF TKTL 21 infrared thermometer. A cube-shaped plate with a volume of 0.008 m^3 was taken for the experiment. The thermometer was set to an emissivity value for silicon of $\varepsilon \approx 0.8–0.9$. Before measurement, the sample was placed in the working environment for 10–15 s to achieve a sensor temperature value equal to the ambient temperature. After that, the sample was bound to a Cartesian rectangular coordinate system. Specific measurement points were determined along the ordinate axis on the sample surface. The measurement was performed with a device placed at a distance of 30 cm and at an angle of 90° to the sample surface. Temperature measurements were performed under a "Scan" mode at the indicated points.

To build linear and nonlinear mathematical models of heat transfer in an isotropic flat medium with thermally active heating zones, the theory of generalized functions was used. This approach made it possible to effectively describe local near-surface and internal heating. This led to the solution of boundary value problems of heat conduction, which contain partial differential equations with discontinuous and singular right-hand sides. To solve nonlinear boundary value problems of heat transfer due to the thermal sensitivity of the medium material, a linearization method is given. This method involves initially using the Kirchhoff transformation (11), which made it possible to linearize nonlinear differential equations (3), (6), as well as partially boundary conditions (4), and derive linear differential equations (13), (24), as well as quasi-linear boundary condition (15). For the final linearization (15), a segment-constant approximation of temperature (16) as a function of the spatial coordinate x on the boundary surface of the plate $x = l$ is introduced.

An isotropic plate with a thickness of 2δ with thermally insulated front surfaces $|z| = \delta$, referred to the Cartesian rectangular coordinate system ($Oxyz$), is considered. In the near-surface region $\Omega_0 = \{(x,y,z):|x| \le H, l, |z| \le \delta\}$ of the plate, uniformly distributed internal heat sources with specific power $q_0 = \text{const}$ are concentrated. On the boundary surface of the layer $L_+ = \{(x,l,z):|x| < \infty, |z| \le \delta\}$, convective heat exchange with the environment with a constant temperature $t_c = \text{const}$ occurs according to Newton's law. On its other surface $L_- = \{(x, -l, z):|x| < \infty, |z| \le \delta\}$ boundary conditions of the second kind are given (Fig. 1).

In the above structure, the temperature distribution $t(x, y)$ in spatial coordinates x and y is determined by solving the heat conduction equation

$$\Delta\theta(x,y)=-\frac{q_0}{\lambda}S_-\left(H-|x|\right)\delta(y-l), \quad (1)$$

under boundary conditions

$$\theta(x,y)|_{|x|\to\infty}=0,\ \left.\frac{\partial\theta(x,y)}{\partial x}\right|_{|x|\to\infty}=0,$$

$$\lambda\left.\frac{\partial\theta(x,y)}{\partial y}\right|_{y=l}=\alpha\,\theta(x,y)\big|_{y=l},\ \left.\frac{\partial\theta(x,y)}{\partial y}\right|_{y=-l}=0, \quad (2)$$

where λ is the thermal conductivity coefficient of the plate; α is the heat transfer coefficient from the surface L_+; $\theta(x, y)=t(x, y)-t_c$; Δ is the Laplace operator in the Cartesian rectangular coordinate system; $S\_(\zeta)$ is the asymmetric unit function; $\delta(\zeta)=dS(\zeta)/d\zeta$ – Dirac delta function; $S(\zeta)$ is the symmetric unit function;

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2};\ S_-(\zeta)=\begin{cases}1, & \zeta\geq 0,\\ 0, & \zeta<0;\end{cases}$$

$$S(\zeta)=\begin{cases}1, & \zeta>0,\\ 0.5, & \zeta=0,\\ 0, & \zeta<0.\end{cases}$$

A thermosensitive plate is considered (Fig. 1) that is isotropic with respect to thermophysical parameters (thermophysical parameters depend on temperature).

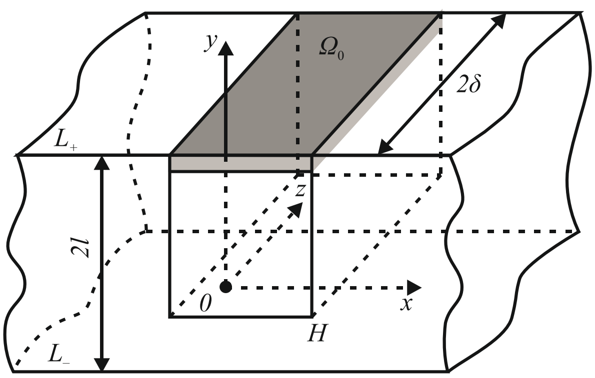

Fig. 1. Isotropic plate under the influence of near-surface heating

In the given structure, the temperature distribution $t(x, y)$ in the spatial coordinates x and y is determined by solving the nonlinear heat conduction equation

$$div\left[\lambda(t)\,grad\,t(x,y)\right]=-q_0S_-\left(H-|x|\right)\delta(y-l), \quad (3)$$

under boundary conditions

$$t(x,y)|_{|x|\to\infty}=0,\ \left.\frac{\partial t(x,y)}{\partial x}\right|_{|x|\to\infty}=0,$$

$$\left.\frac{\partial t(x,y)}{\partial y}\right|_{y=-l}=0,\ \lambda(t)\left.\frac{\partial t(x,y)}{\partial y}\right|_{y=l}=\alpha\left(t|_{y=l}-t_c\right), \quad (4)$$

where $\lambda(t)$ is the thermal conductivity coefficient of the thermally sensitive plate.

An isotropic plate with a thickness of 2δ with respect to thermophysical parameters is considered, referred to the Cartesian rectangular coordinate system ($Oxyz$). The front surfaces of this plate are thermally insulated $|z|=\delta$. In the volume of a thin rectangular parallelepiped $\Omega_0=\{(x, y, z):|x|\leq H, 0, |z|\leq\delta\}$ of the isotropic plate, uniformly distributed internal heat sources with specific power q_0 are concentrated. On the boundary surface of the layer $L_+=\{(x, l, z): |x|<\infty, |z|\leq\delta\}$, convective heat exchange with the environment with a constant temperature t_c occurs according to Newton's law. On its other surface $L_-=\{(x, -l, z): |x|<\infty, |z|\leq\delta\}$, boundary conditions of the second kind are given (Fig. 2).

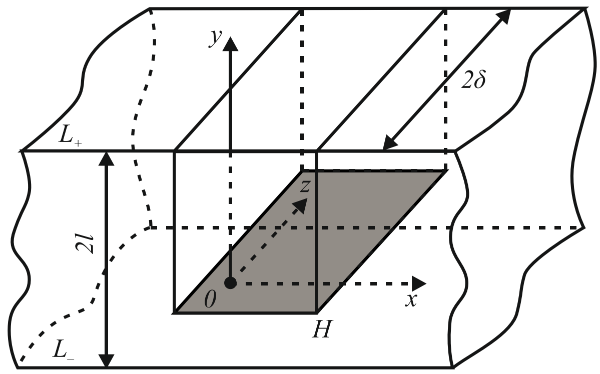

Fig. 2. Isotropic plate under the influence of internal heating

In the above structure, the temperature distribution $t(x, y)$ in spatial coordinates x and y is determined by solving the heat conduction equation

$$\Delta\theta(x,y)=-\frac{q_0}{\lambda}S_-\left(H-|x|\right)\delta(y), \quad (5)$$

under boundary conditions (2).

A thermally sensitive plate is considered, isotropic with respect to thermophysical parameters (Fig. 2).

In the given structure, it is necessary to determine the temperature distribution $t(x, y)$ in the spatial coordinates x and y, which is obtained by solving the nonlinear heat conduction equation

$$div\left[\lambda(t)\,grad\,t(x,y)\right]=-q_0S_-\left(H-|x|\right)\delta(y), \quad (6)$$

under boundary conditions (4).

5. Results of investigating mathematical models of heat transfer in isotropic flat media with local heating

5. 1. Linear mathematical model of heat transfer in an isotropic plate with near-surface heating

The integral Fourier transform along the x coordinate was applied to equation (1) and boundary conditions (2); a second-order inhomogeneous ordinary differential equation with constant coefficients and a singular right-hand side was obtained

$$\frac{d^2\bar{\theta}}{dy^2}-\xi^2\bar{\theta}=-\sqrt{\frac{2}{\pi}}\frac{q_0}{\lambda\xi}\sin H\xi\,\delta(y-l), \quad (7)$$

under boundary conditions

$$\left.\frac{d\bar{\theta}(y)}{dy}\right|_{y=l}=\frac{\alpha}{\lambda}\bar{\theta}(y)\Big|_{y=l},\ \left.\frac{d\bar{\theta}(y)}{dy}\right|_{y=-l}=0, \qquad (8)$$

where $\bar{\theta}(y)$ is the transformant of function $\theta(x, y)$;

$$\bar{\theta}(y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\xi x}\theta(x,y)\mathrm{d}x;$$

ξ is the parameter of the integral Fourier transform, $i^2=-1$.

The general solution to equation (7) is defined as

$$\bar{\theta}(y)=c_1e^{\xi y}+c_2e^{-\xi y}-$$
$$-\sqrt{\frac{2}{\pi}}\frac{q_0}{\lambda\xi^2}\sin H\xi\,\mathrm{sh}\,\xi(y-l)S(y-l).$$

Here c_1 and c_2 are the constants of integration.

The boundary conditions (8) were used, and, on this basis, the constants of integration were found and a partial solution to problem (7), (8) was derived

$$\bar{\theta}(y)=\frac{q_0}{\xi\sqrt{2\pi}}\sin H\xi\left[\begin{array}{l}\dfrac{\mathrm{ch}\,\xi(y+l)}{P(\xi)}-\\-\dfrac{2}{\lambda\xi}\mathrm{sh}\,\xi(y-l)S(y-l)\end{array}\right], \qquad (9)$$

where $P(\xi)=\lambda\xi\mathrm{sh}2\xi l-\alpha\mathrm{ch}2\xi l$.

The inverse integral Fourier transform is applied to relation (9) and, as a result, the solution to problem (1), (2) is obtained in the following form

$$\theta(x,y)=\frac{q_0}{\pi}\times$$
$$\times\int_0^{\infty}\frac{\cos\xi x}{\xi}\sin H\xi\left[\begin{array}{l}\dfrac{\mathrm{ch}\,\xi(y+l)}{P(\xi)}-\\-\dfrac{2\mathrm{sh}\,\xi(y-l)}{\lambda\xi}S(y-l)\end{array}\right]\mathrm{d}\xi. \qquad (10)$$

As a result, the desired temperature field in the plate, caused by surface heating, is expressed by formula (10), from which the temperature value at any point is obtained.

According to formula (10), numerical calculations of the temperature distribution $\theta(x;0)$ (Fig. 3, *a*) and $\theta(0.05;y)$ (Fig. 3, *b*) in the spatial coordinates x, y in the plate for a constant value of the thermal conductivity coefficient for silicon ($\lambda=67.9$ W/(m·degree) at a temperature $t=27$°C) were performed. The following input data values were selected: $q_0=200$ W/m^3; $l=0.1$ m; $H=0.05$ m; $\alpha=17.64$ W/(m^2 · degree). Numerical calculations were performed with an accuracy of 10^{-6}.

The behavior of the curves demonstrates that the temperature as a function of spatial coordinates is smooth and monotonic and reaches maximum values in the region where near-surface heat sources are concentrated.

5. 2. Nonlinear mathematical model of heat transfer in an isotropic thermosensitive plate with near-surface heating

The Kirchhoff transformation is considered

$$\vartheta(x,y)=\frac{1}{\lambda^0}\int_0^{t(x,y)}\lambda(\zeta)\mathrm{d}\zeta. \qquad (11)$$

Here, λ^0 is the reference coefficient of thermal conductivity of the plate material.

Expression (11) is differentiated with respect to variables x and y and, as a result, the following relation is obtained:

$$\lambda^0\frac{\partial\vartheta(x,y)}{\partial x}=\lambda(t)\frac{\partial t(x,y)}{\partial x},$$
$$\lambda^0\frac{\partial\vartheta(x,y)}{\partial y}=\lambda(t)\frac{\partial t(x,y)}{\partial y}, \qquad (12)$$

taking into account which the original equation (3) and boundary conditions (4) are transformed to the following form:

$$\Delta\vartheta=-\frac{q_0}{\lambda^0}S_-(H-|x|)\delta(y-l), \qquad (13)$$

$$\vartheta(x,y)\big|_{|x|\to\infty}=0;\ \left.\frac{\partial\vartheta(x,y)}{\partial x}\right|_{|x|\to\infty}=0;\ \left.\frac{\partial\vartheta(x,y)}{\partial y}\right|_{y=-l}=0, \qquad (14)$$

$$\left.\frac{\partial\vartheta(x,y)}{\partial y}\right|_{y=l}=\frac{\alpha}{\lambda^0}\left(t(x,y)\big|_{y=l}-t_c\right). \qquad (15)$$

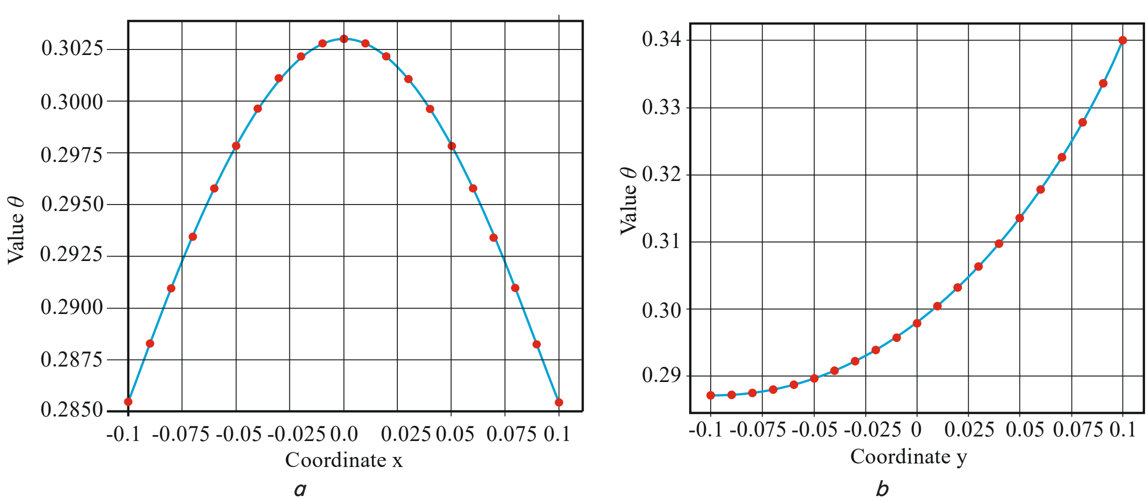

Fig. 3. Temperature dependence $\theta(x, y)$ in an isotropic plate with near-surface heating: *a* — on the spatial coordinate x for $y=0$; *b* — on the spatial coordinate y for $x=0.05$

As a result of the transformations, linear differential equations with partial derivatives of the second order with respect to function $\theta(x, y)$ with a discontinuous and singular right-hand side (13) and boundary conditions (14) and a quasilinear boundary condition (15) were obtained.

The temperature $t(x, l)$ was approximated as a function of the spatial coordinate x by a segment-constant function in the form

$$t(x,l)=t_1+\sum_{j=1}^{m-1}(t_{j+1}-t_j)S_-(x-x_j), \tag{16}$$

where $x_j \in (0; x^*)$; $x_1 \le x_2 \le \ldots \le x_{m-1}$; $t_j(j=\overline{1,m})$ – unknown approximate values of temperature $t(x, l)$; m – number of partitions of interval 0; x^*); x^* – value of abscissa for which temperature reaches value t_c (it is found from corresponding linear problem).

The integral Fourier transform in coordinate x is applied to equation (13) and boundary conditions (14), (15), taking into account relation (16). As a result, we obtain an ordinary differential equation of the second order with constant coefficients and singular right-hand side

$$\frac{d^2\overline{\vartheta}}{dy^2}-\xi^2\overline{\vartheta}=-\sqrt{\frac{2}{\pi}}\frac{q_0}{\lambda^0\xi}\sin H\xi\delta(y-l), \tag{17}$$

and linear boundary conditions

$$\left.\frac{d\overline{\vartheta}(y)}{dy}\right|_{y=-l}=0,\ \left.\frac{d\overline{\vartheta}(y)}{dy}\right|_{y=l}=\frac{\alpha D(\xi)}{\sqrt{2\pi}\lambda^0\xi}, \tag{18}$$

where $\overline{\vartheta}(y)=1/\sqrt{2\pi}\int_{-\infty}^{\infty}e^{i\xi x}\vartheta(x,y)\mathrm{d}x$ is the transformant of function $\theta(x, y)$;

$$D(\xi)=i\sum_{j=1}^{m-1}(t_{j+1}-t_j)\left(e^{i\xi x_j}-e^{i\xi x_{j-1}}\right).$$

The general solution to equation (17) is obtained in the form

$$\overline{\vartheta}(y)=c_1e^{\xi y}+c_2e^{-\xi y}-$$
$$-\sqrt{\frac{2}{\pi}}\frac{q_0}{\lambda^0\xi^2}\sin H\xi\,\mathrm{sh}\xi(y-l)S(y-l),$$

and using boundary conditions (18), the integration constants c_1, c_2 are determined and, as a result, there is a solution to problem (17), (18)

$$\overline{\vartheta}(y)=\frac{1}{\sqrt{2\pi}\lambda^0\xi^2}\left[\begin{array}{l}q_0\sin H\xi\times\\ \times\left(\begin{array}{l}D(\xi,y)-2\mathrm{sh}\xi\times\\ \times(y-l)S(y-l)\end{array}\right)+\\ +\alpha D(\xi,y)D(\xi)\end{array}\right]. \tag{19}$$

Here, $D(\xi,y)=\dfrac{\mathrm{ch}\xi(l+y)}{\mathrm{sh}2\xi l}$.

The inverse integral Fourier transform is applied to relation (19) and the expression for the linearizing function $\theta(x, y)$ is defined in the following form:

$$\vartheta(x,y)=\frac{1}{\pi\lambda^0}\int_0^{\infty}\frac{1}{\xi^2}\left\{\begin{array}{l}q_0\sin H\xi\cos\xi x\times\\ \times\left[\begin{array}{l}D(\xi,y)-2\mathrm{sh}\xi\times\\ \times(y-l)S(y-l)\end{array}\right]+\\ +\alpha D(\xi,y)D(\xi,x)\end{array}\right\}\mathrm{d}\xi, \tag{20}$$

where

$$D(\xi,x)=\sin\xi x\sum_{j=1}^{m-1}(t_{j+1}-t_j)(\cos\xi x_j-\cos\xi x_{j-1})-$$
$$-\cos\xi x\sum_{j=1}^{m-1}(t_{j+1}-t_j)(\sin\xi x_j-\sin\xi x_{j-1}).$$

As a result of substituting the expression of the temperature dependence of the thermal conductivity coefficient of the medium material into relations (11), (20), as well as certain mathematical transformations, a system of nonlinear algebraic equations is built for determining the unknown approximate values $t_j(j=\overline{1,m})$ of temperature $t(x, l)$.

The desired temperature field $t(x, y)$ for the given structure is determined using the obtained nonlinear algebraic equation taking into account the temperature dependence of the thermal conductivity coefficient of structural materials of the plate in relations (11), (20), and by performing certain mathematical transformations.

The dependence of the thermal conductivity coefficient on the temperature of the structural material of the plate is given in the form

$$\lambda=\lambda^0(1-kt), \tag{21}$$

where k is the temperature coefficient of thermal conductivity of the plate material.

Using relations (11), (21), the following expression for determining temperature $t(x, y)$ is obtained

$$t(x,y)=\frac{1}{k}\left(1-\sqrt{1-2k\vartheta(x,y)}\right). \tag{22}$$

In the temperature range [0°C; 1127°C], the temperature dependence of the thermal conductivity coefficient of silicon was obtained by interpolation in the form

$$\lambda(t)=67.9\frac{\mathrm{W}}{\mathrm{deg\cdot m}}\left(1-0.0005\frac{1}{\mathrm{deg}}t\right), \tag{23}$$

which is a special case of relation (21).

According to formula (22), numerical calculations of the temperature distribution $\theta(x; 0)$ (Table 1) and $\theta(0,05; y)$ (Table 2) were performed in spatial coordinates x, y in the plate for a linearly varying thermal conductivity coefficient (relation (23)).

Table 1

Temperature change depending on the spatial coordinate x (for $y = 0$)

x, m	–0.1	–0.07	–0.04	0	0.04	0.07	0.1
θ, °C	0.29122	0.29932	0.30564	0.30906	0.30564	0.29932	0.29122

Table 2

Temperature change depending on the spatial coordinate y (for $x = 0.05$)

y, m	–0.1	–0.05	–0.02	0	0.0125	0.038	0.1
θ, °C	0.2922	0.2954	0.2998	0.3039	0.3072	0.3150	0.4370

The following input data values were selected: $q_0 = 200$ W/m^3; $l = 0.1$ m; $H = 0.05$ m; $\alpha = 17.64$ W/(m^2·deg). Numerical calculations were performed with an accuracy of 10^{-6} for the number of partitions of the interval $(0; x^*)$ $m = 9$.

5. 3. Linear mathematical model of heat transfer in an isotropic plate with internal heating

The integral Fourier transform along the x coordinate is applied to equation (5) and boundary conditions (2). As a result, a non-homogeneous ordinary differential equation of the second order with constant coefficients and a singular right-hand side is obtained

$$\frac{d^2\bar{\theta}}{dy^2} - \xi^2\bar{\theta} = -\sqrt{\frac{2}{\pi}}\frac{q_0}{\lambda\xi}\sin H\xi\delta(y), \qquad (24)$$

under boundary conditions (8).

The general solution to equation (24) is defined as

$$\bar{\theta}(y) = c_1 e^{\xi y} + c_2 e^{-\xi y} - \sqrt{\frac{2}{\pi}}\frac{q_0}{\lambda\xi^2}\sin H\xi \,\mathrm{sh}\,\xi(y) S(y).$$

The boundary conditions (8) were used and, on this basis, the integration constants c_1, c_2 were found; a partial solution to problem (8) to (24) was obtained.

$$\bar{\theta}(y) = \sqrt{\frac{2}{\pi}}\frac{q_0}{\lambda\xi^2}\sin H\xi\left[\frac{2\,\mathrm{ch}\,\xi(y+l)}{P(\xi)}P_1(\xi) - \mathrm{sh}\,\xi y S(y)\right], \qquad (25)$$

where $P_1(\xi) = \alpha\,\mathrm{sh}\,\xi l - \lambda\xi\,\mathrm{ch}\,2\xi l$.

The inverse integral Fourier transform is applied to relation (25) and, as a result, the solution to problem (2) to (5) is obtained in the following form

$$\theta(x,y) = -\frac{2q_0}{\pi\lambda}\int_0^\infty \frac{\cos\xi x}{\xi^2}\sin H\xi\left[\begin{array}{l}\frac{\mathrm{ch}\,\xi(y+l)}{P(\xi)}P_1(\xi)+\\ +\mathrm{sh}\,\xi y S(y)\end{array}\right]\mathrm{d}\xi. \qquad (26)$$

As a result, the desired temperature field in the plate, caused by internal local heating, is expressed by formula (26), from which the temperature value at any point is obtained.

According to formula (26), numerical calculations of the temperature distribution $\theta(0.05; y)$ (Fig. 4) along the spatial coordinate y in the plate were performed for a constant value of the thermal conductivity coefficient for silicon ($\lambda = 67.9$ W/(m · degree) at a temperature $t = 27$°C). The following input data values were selected: $q_0 = 200$ W/m^3; $l = 0.1$ m; $H = 0.05$ m; $\alpha = 17.64$ W/(m^2 · degree). Numerical calculations were performed with an accuracy of 10^{-6}.

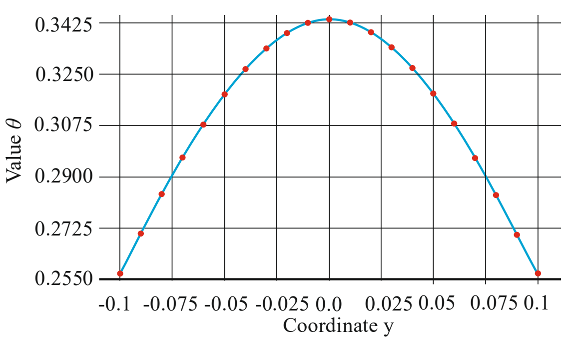

Fig. 4. Dependence of temperature $\theta(x, y)$ on spatial coordinate y for $x = 0.05$ in an isotropic plate with internal heating

The behavior of the curve demonstrates that the temperature as a function of the spatial coordinate is smooth and monotonic and reaches maximum values in the region where internal heat sources are concentrated.

5. 4. Nonlinear mathematical model of heat transfer in an isotropic plate with internal heating

A thermosensitive plate is considered, isotropic with respect to thermophysical parameters (Fig. 2).

As a result of using relations (11), (12), equation (6) is transformed to the following form

$$\Delta\vartheta = -\frac{q_0}{\lambda^0}S_-(H-|x|)\delta(y). \qquad (27)$$

The integral Fourier transform along the x coordinate is applied to this equation and a second-order ordinary differential equation with constant coefficients and a singular right-hand side is obtained

$$\frac{d^2\bar{\vartheta}}{dy^2} - \xi^2\bar{\vartheta} = -\sqrt{\frac{2}{\pi}}\frac{q_0}{\lambda^0\xi}\sin H\xi\delta(y), \qquad (28)$$

whose general solution will be as follows

$$\bar{\vartheta}(y) = c_1 e^{\xi y} + c_2 e^{-\xi y} - \sqrt{\frac{2}{\pi}}\frac{q_0}{\lambda^0\xi^2}\sin H\xi\,\mathrm{sh}\,\xi(y)S(y).$$

Using boundary conditions (18) makes it possible to determine the integration constants c_1, c_2 and, as a consequence, the solution to problem (18) to (28)

$$\bar{\vartheta}(y) = \frac{1}{\sqrt{2\pi}\lambda^0\xi^2}\left[\begin{array}{l}2q_0\sin H\xi\left(\begin{array}{l}D(\xi,y)\,\mathrm{ch}\,\xi l-\\ -\mathrm{sh}\,\xi y S(y)\end{array}\right)+\\ +\alpha D(\xi,y)D(\xi)\end{array}\right]. \qquad (29)$$

The inverse integral Fourier transform is applied to relation (29) and the expression for the linearizing function $\theta(x, y)$ is defined in the following form:

$$\vartheta(x,y) = \frac{1}{\pi\lambda^0}\int_0^\infty \frac{1}{\xi^2}\left\{\begin{array}{l}2q_0\sin H\xi\cos\xi x\times\\ \times\left[\begin{array}{l}D(\xi,y)\,\mathrm{ch}\,\xi l-\\ -\mathrm{sh}\,\xi y S(y)\end{array}\right]+\\ +\alpha D(\xi,y)D(\xi,x)\end{array}\right\}\mathrm{d}\xi. \qquad (30)$$

As a result of substituting the expression of the temperature dependence of the thermal conductivity coefficient of the medium material into the relations (11), (22) and certain mathematical transformations, a system of nonlinear algebraic equations is obtained for determining the unknown approximate values of temperature $t(x, l)$.

The desired temperature field $t(x, y)$ for the given structure is determined using the obtained nonlinear algebraic equation taking into account the temperature dependence of the thermal conductivity coefficient of the structural materials of the plate in relations (11), (22) and by performing certain mathematical transformations.

According to formula (22), taking into account relation (23), numerical calculations of the temperature distribution $\theta(0.05; y)$ (Table 3) along the spatial coordinate y in the plate for a linearly varying thermal conductivity coefficient were performed. The following input data values were selected: $q_0 = 200$ W/m^3; $l = 0.1$ m; $H = 0.05$ m; $\alpha = 17.64$ W/(m^2 · degree). Numerical calculations were performed with an accuracy of 10^{-6} for the number of partitions of the interval (0; x^*) $m = 9$.

Table 3

Temperature change depending on the spatial coordinate y (for $x = 0.05$)

y, m	–0.1	–0.07	–0.04	0	0.04	0.07	0.1
θ, °C	0.2619	0.3026	0.3331	0.3505	0.3331	0.3026	0.2619

The results obtained for the selected medium material (silicon) with a linear temperature dependence of the thermal conductivity coefficient differ from the results obtained for a constant thermal conductivity coefficient by 2% (Tables 1–3, Fig. 3, 4). Their insignificant difference is explained by the fact that the value of the temperature coefficient of thermal conductivity for silicon, as shown by relation (23), is small.

The experimental values of temperature at points with coordinates (0.05; y; 0) are given in Table 4.

Table 4

Experimental temperature values at points with coordinates (0.05; y; 0)

y, m	–0.1000	–0.0500	0.0125	0.0380	0.1000
θ, °C	0.3312	0.2468	0.2568	0.3532	0.2968

The obtained numerical calculations of temperature differ from the experimental values by 15% (Fig. 3, b; Table 4).

6. Results of the construction of mathematical models of heat transfer in flat media with local heating: discussion

The boundary value problems of heat conduction have been stated in accordance with the physical process considered in the above media. As a result, the differential equations of heat conduction and boundary conditions containing discontinuous and singular functions in the right-hand sides describe the heat transfer process. The form of curves in Fig. 3, 4, which are constructed on the basis of the determined numerical values of temperature as a function of spatial coordinates, obtained using analytical-numerical solutions of the boundary value problems (10), (26), indicates the correctness of our results. This is confirmed by the smoothness of the temperature function in spatial coordinates and the fulfillment of the specified boundary conditions at the edges of the plate.

In our studies, the theory of generalized functions was used, which made it possible to effectively describe local near-surface and internal heating, as a result of which the obtained partial differential equations contain discontinuous and singular right-hand sides. For linearization of nonlinear boundary value problems (3), (4), and (4) to (6), a linearization method is presented, which made it possible to analytically obtain analytical-numerical solutions (20), (30). The temperature distribution is determined by relations (10), (22), (26), and is displayed in Fig. 3, 4; in Tables 1–3.

It should be noted that the above-analyzed works did not consider an approach for linearizing boundary value problems of thermal conductivity for thermosensitive media in an analytical-numerical way. Unlike [3], in which a porous medium was considered, and [4], a layered medium, the boundary value problems were not linearized in an analytical way. As a result of using numerical methods for linearization, significant errors accumulated. In our studies, the use of the Kirchhoff transformation made it possible to linearize the differential equations of thermal conductivity and partially the boundary condition (4). In this regard, for its complete linearization, an approximation of the temperature by the spatial coordinate by a segment-constant function (16) at the edge of the plate was introduced. This approach leads to obtaining a minimum error in the results, which was not achieved in [9] and [10] due to the use of experimental and numerical methods, respectively. The use of generalized functions makes it possible to effectively describe thermally active heating zones. This leads to the solution of partial differential equations of heat conduction with discontinuous and singular right-hand sides.

Since the architecture of modern electronic devices locally concentrates individual thermally active nodes, in particular surface and internal uniformly distributed ones, there is a need to construct mathematical models of heat transfer between their individual elements. These models can be linear or nonlinear for isotropic flat media. The given mathematical models of heat transfer are simplified, but they make it possible to construct more complex mathematical models for flat composite media on their basis.

Based on the obtained analytical-numerical solutions to both linear and nonlinear boundary value problems of heat transfer, it is proposed to develop computational algorithms and software tools for their numerical implementation. This will make it possible to conduct research into a number of materials used in the design of digital electronic devices regarding the influence of their thermal sensitivity on the temperature distribution.

It is proposed to take into account the thermal sensitivity of structural materials for the analysis of thermal regimes in electronic devices, which significantly complicates the process of solving the corresponding linear and nonlinear boundary value problems of heat conduction. However, the sought solutions to these problems describe the behavior of temperature as a function of spatial coordinates somewhat more adequately to the real physical process.

This study was performed for a stationary process of heat conduction, as a result of which the models built are limited as they make it possible to determine the temperature change only by spatial coordinates. Heat transfer problems contain only boundary conditions of the first, second, and third kind at the boundary surfaces of the media, which is a drawback, although this does not reduce the generality of the research.

In the future, research may involve the construction of mathematical models of heat transfer for inhomogeneous flat media with foreign thermally active elements, for the unsteady process of heat conduction and for more complex boundary conditions, in particular for thermal radiation.

7. Conclusions

1. A linear mathematical model of heat transfer between individual elements of structural units of electronic devices with local near-surface heating has been constructed. An analytical-numerical solution to the boundary value problem in the form of an improper integral (the upper limit of the integral contains infinity) has been obtained. After certain mathematical transformations, it was reduced to an integral with finite limits. As a result of using the 3/8 Newton method of

numerical integration to determine the temperature distribution in spatial coordinates in the environment, an accuracy of results of 10^{-6} has been achieved. Such accuracy is difficult to achieve using numerical methods for solving the initial boundary value problem or experimental measurements. Due to the local concentration of thermal heating (a sufficiently small heating area, almost point-like, described by the Dirac delta function), it is impossible to build a mathematical model for determining the temperature field using numerical methods.

2. A nonlinear mathematical model of heat transfer between individual thermally sensitive elements of structural assemblies of electronic devices with local near-surface heating has been constructed. A method for linearizing a nonlinear boundary value problem has been introduced and, on this basis, an analytical-numerical solution has been obtained for the linear temperature dependence of the thermal conductivity coefficient of the structure material. A numerical experiment was performed, as a result of which the behavior of temperature as a function of spatial coordinates was displayed. The results obtained for the selected material with a linear temperature dependence of the thermal conductivity coefficient differ from the results obtained for its constant value by 2%.

3. A linear mathematical model of heat transfer between individual elements of structural units of electronic devices with locally concentrated internal heating has been built. An analytical-numerical solution to the boundary value problem was obtained and, on this basis, using numerical integration of the improper integral, numerical values of temperature for selected values of thermophysical and geometric parameters with an accuracy of 10^{-6} have been given.

4. A nonlinear mathematical model of heat transfer between individual heat-sensitive elements of structural units of electronic devices with locally concentrated internal heating has been constructed. A method of linearization of the nonlinear boundary value problem has been introduced, and, on this basis, an analytical-numerical solution was obtained for the linear temperature dependence of the thermal conductivity coefficient of the structure material. This solution made it possible to form a system of nonlinear algebraic equations under an automated mode to determine unknown values of the temperature at the edge of the medium, the coefficients of which contain improper integrals. The coefficients were determined by numerical integration, and the solution to the system was obtained by Newton's method with an accuracy of 10^{-6}, after which the numerical values of the temperature were determined.

Conflicts of interest

The authors declare that they have no conflicts of interest in relation to the current study, including financial, personal, authorship, or any other, that could affect the study, as well as the results reported in this paper.

Funding

The study was conducted without financial support.

Data availability

All data are available, either in numerical or graphical form, in the main text of the manuscript.

Use of artificial intelligence

The authors confirm that they did not use artificial intelligence technologies when creating the current work.

References

1. Cui, Y., Li, M., Hu, Y. (2020). Emerging interface materials for electronics thermal management: experiments, modeling, and new opportunities. Journal of Materials Chemistry C, 8 (31), 10568–10586. https://doi.org/10.1039/c9tc05415d
2. Pannucharoenwong, N., Rattanadecho, P., Echaroj, S., Hemathulin, S., Nabudda, K. (2020). The Investigation of Heat Absorber on the Efficiency of Slanted Double-Slope Solar Distillation Unit. International Journal of Heat and Technology, 38 (1), 171–179. https://doi.org/10.18280/ijht.380119
3. Zhang, Z., Sun, Y., Cao, X., Xu, J., Yao, L. (2024). A slice model for thermoelastic analysis of porous functionally graded material sandwich beams with temperature-dependent material properties. Thin-Walled Structures, 198, 111700. https://doi.org/10.1016/j.tws.2024.111700
4. Zhang, Z., Zhou, D., Fang, H., Zhang, J., Li, X. (2021). Analysis of layered rectangular plates under thermo-mechanical loads considering temperature-dependent material properties. Applied Mathematical Modelling, 92, 244–260. https://doi.org/10.1016/j.apm.2020.10.036
5. Filymonenko, N. M., Filimonenko, K. V. (2020). Analysis of capability for improvement of basic mathematical model of electrode of ferro-alloy furnace. Scientific journals of Volodymyr Dahl East Ukrainian National University, 7 (263), 53–57. https://doi.org/10.33216/1998-7927-2020-263-7-53-57
6. Sadiq Al-Baghdadi, M. A. R., Noor, Z. M. H., Zeiny, A., Burns, A., Wen, D. (2020). CFD analysis of a nanofluid-based microchannel heat sink. Thermal Science and Engineering Progress, 20, 100685. https://doi.org/10.1016/j.tsep.2020.100685
7. Liu, J., Zhang, Y., Zhang, D., Jiao, S., Zhang, Z., Zhou, Z. (2020). Model development and performance evaluation of thermoelectric generator with radiative cooling heat sink. Energy Conversion and Management, 216, 112923. https://doi.org/10.1016/j.enconman.2020.112923
8. Hu, S., Li, C., Zhou, Z., Liu, B., Zhang, Y., Yang, M. et al. (2023). Nanoparticle-enhanced coolants in machining: mechanism, application, and prospects. Frontiers of Mechanical Engineering, 18 (4). https://doi.org/10.1007/s11465-023-0769-8
9. Peng, X., Li, X., Gong, Z., Zhao, X., Yao, W. (2022). A deep learning method based on partition modeling for reconstructing temperature field. International Journal of Thermal Sciences, 182, 107802. https://doi.org/10.1016/j.ijthermalsci.2022.107802

10. Ren, Y., Huo, R., Zhou, D., Zhang, Z. (2022). Thermo-Mechanical Buckling Analysis of Restrained Columns Under Longitudinal Steady-State Heat Conduction. Iranian Journal of Science and Technology, Transactions of Civil Engineering, 47 (3), 1411–1423. https://doi.org/10.1007/s40996-022-01020-7
11. Zhou, Y., Wu, S., Long, Y., Zhu, P., Wu, F., Liu, F., Murugadoss, V. et al. (2020). Recent Advances in Thermal Interface Materials. ES Materials & Manufacturing, 7, 4–24. https://doi.org/10.30919/esmm5f717
12. Chu, Y.-M., Shah, F., Khan, M. I., Kadry, S., Abdelmalek, Z., Khan, W. A. (2020). Cattaneo-Christov double diffusions (CCDD) in entropy optimized magnetized second grade nanofluid with variable thermal conductivity and mass diffusivity. Journal of Materials Research and Technology, 9 (6), 13977–13987. https://doi.org/10.1016/j.jmrt.2020.09.101
13. Sebald, G., Komiya, A., Jay, J., Coativy, G., Lebrun, L. (2020). Regenerative cooling using elastocaloric rubber: Analytical model and experiments. Journal of Applied Physics, 127 (9). https://doi.org/10.1063/1.5132361
14. Liu, H., Yu, J., Wang, R. (2023). Dynamic compact thermal models for skin temperature prediction of portable electronic devices based on convolution and fitting methods. International Journal of Heat and Mass Transfer, 210, 124170. https://doi.org/10.1016/j.ijheatmasstransfer.2023.124170
15. Ghannad, M., Yaghoobi, M. P. (2015). A thermoelasticity solution for thick cylinders subjected to thermo-mechanical loads under various boundary conditions. International Journal of Advanced Design & Manufacturing Technology, 8 (4). Available at: https://sanad.iau.ir/journal/admt/Article/534941?jid=534941
16. Havrysh, V. I. (2015). Nonlinear Boundary-Value Problem of Heat Conduction for a Layered Plate with Inclusion. Materials Science, 51 (3), 331–339. https://doi.org/10.1007/s11003-015-9846-4
17. Havrysh, V. I. (2017). Investigation of Temperature Fields in a Heat-Sensitive Layer with Through Inclusion. Materials Science, 52 (4), 514–521. https://doi.org/10.1007/s11003-017-9984-y
18. Havrysh, V. I., Kosach, A. I. (2012). Boundary-value problem of heat conduction for a piecewise homogeneous layer with foreign inclusion. Materials Science, 47(6), 773–782. https://doi.org/10.1007/s11003-012-9455-4
19. Gavrysh, V., Tushnytskyy, R., Pelekh, Y., Pukach, P., Baranetskyi, Y. (2017). Mathematical model of thermal conductivity for piecewise homogeneous elements of electronic systems. 2017 14th International Conference The Experience of Designing and Application of CAD Systems in Microelectronics (CADSM), 333–336. https://doi.org/10.1109/cadsm.2017.7916146