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# NUMERICAL IMPLEMENTATION OF RICHARDSON EXTRAPOLATION FOR DYNAMIC PROBLEMS OF MULTILAYER CYLINDRICAL SHELLS

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***This study considers forced vibrations of a heterogeneous elastic structure in the form of a multilayer cylindrical shell consisting of rigidly connected layers and reinforced with discrete ring elements.***

***A mathematical model of vibrations of an elastic heterogeneous structure under the action of a non-stationary load has been constructed. The stressed-strained state of a multilayer cylindrical shell with discrete ring ribs was investigated using the geometrically nonlinear theory of Timoshenko-type shells and rods. The presence of a complex right-hand side and discontinuous coefficients in the spatial coordinates in the hyperbolic equations of vibrations of a heterogeneous elastic cylindrical shell (at the locations of the reinforcing ribs) necessitated the use of numerical methods for solving them. A numerical algorithm using Richardson extrapolations has been proposed for studying the constructed model.***

***For example, a three-layer reinforced cylindrical shell is considered, taking into account the discreteness of the ribs' placement under dynamic loading with rigidly clamped ribs. The proposed numerical algorithm has made it possible to investigate the stressed-strained state of a three-layer reinforced elastic structure of a cylindrical type at any given moment in time. A comparative analysis of the numerical results of the calculations revealed that, according to the standard approach, the discrepancy in the deflection values for n = 40 and n = 160 reached 31%, for n = 80 and n = 160 it was about 5%, according to Richardson's approach for n = 40 ÷ 80 and the standard approach for n = 160, this difference was about 1%.***

***A distinctive feature of this study is the use of Richardson extrapolation to identify the stressed-strained state of a three-layer reinforced cylindrical shell, which made it possible to increase the accuracy of the solution to the dynamic problem without reducing the calculation step.***

***The study's results reported in this work could be used for investigating unsteady vibrations of shell structures at research and engineering organizations***

***Keywords: multilayer shells, forced vibrations, unsteady loading, numerical methods, Richardson extrapolation***

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## 1. Introduction

Multilayer reinforced cylindrical shells occupy an important place in modern mechanical engineering and related industries due to the combination of high strength, heat resistance, as well as resistance to aggressive environments. The most typical areas of their application are rocket and space technology, shipbuilding, underwater equipment, multilayer pipelines with special functional layers, as well as medical technology and bioengineering. The need for in-depth scientific research of multilayer reinforced cylindrical shells is due to their ability to provide an optimal set of operational properties through the rational combination of various materials. This approach increases resistance to loads, allows for local reinforcement in the most loaded areas, and provides a significant reduction in the mass of structures while maintaining or even improving strength characteristics.

That is why the construction of new mathematical models of shells and shell structures, the development of methods for calculating structures are relevant for modern scientists around the world.

## 2. Literature review and problem statement

When setting the research problems of multilayer shells, various shell theories are used: classical theory (Kirchhoff-Love theory), first-order theory taking into account shear, higher-order theory taking into account nonlinear distribution of stresses and strains along the thickness, and others. Most researchers use classical shell theory – a simplified mathematical model that describes the mechanical behavior of thin shells, based on assumptions about a small shell thickness compared to other dimensions and without taking into account deformation in the thickness direction.

In study [1], the propagation of non-axisymmetric waves in a three-layer viscoelastic cylindrical shell is considered. The displacement of the outer layer is described using shell equations built on the basis of the Kirchhoff-Love hypothesis, and the behavior of the middle layer is modeled on the basis of viscoelasticity equations in a polar coordinate system. To solve the problem, approaches from the theory of elasticity and numerical methods were used, in particular the methods of Muller, Gauss, and Laplace. The results showed that with increasing thickness of the intermediate layer, the real and imaginary parts of the phase velocity of the first mode increase, while for the second mode they decrease. The issue of wave propagation in three-layer cylindrical shells remained unresolved if the outer layers are relatively thick. The reason for this was the use in the work of a simplified model built on the basis of the Kirchhoff-Love hypothesis. The application of the Kirchhoff-Love hypothesis is suitable only for thin shells.

In [2], the geometrically nonlinear response of shell structures made of magnetoelectroelastic composites was investigated. The proposed finite element model was based on the Kirchhoff-Love shell theory. A four-node shell finite element was used to model the nonlinear behavior of the structures. The discrete system of geometrically nonlinear equilibrium equations was solved using the Newton-Raphson method. A numerical analysis of the hyperboloid shell was performed; the results were compared with the available literature data to verify the effectiveness and accuracy of the proposed model, especially for thin-walled structures. High compliance of the results and adequate static response of the composite material under conditions of significant deformations and finite rotations were obtained. Such a model [2] may not be accurate enough if the shell has a larger thickness or its deformation goes beyond the assumptions of the theory of thin structures. The main reason for this was the use of a model based on the Kirchhoff-Love hypothesis in the work. The issue of studying the deformations of shells with variable thickness remained unresolved.

In study [3], the natural frequency characteristics of a functionally gradient multilayer hybrid composite cylindrical shell panel reinforced with graphene plates and carbon nanotubes were analyzed. To assess the effective material properties of the composite, a modified micromechanical model combining the Halpin-Tsai approach and the rule of mixtures was used. Based on the first-order shear deformation theory, Hamilton's principle, and the finite element method, stiffness and mass matrices of the structure were constructed. The accuracy of the proposed approach was confirmed by comparison with the results from the literature. The influence of a number of parameters, in particular the number of layers, the content of reinforcing elements, their distribution schemes, the volumetric content of carbon nanotubes, the ratio of the thickness to the length of the panel, the angle of flight, the stiffness of the Winkler elastic base and the types of boundary conditions were analyzed. The results of the study showed that three-phase cylindrical shells can effectively combine the advantages of reinforcement with graphene plates and carbon nanotubes, which significantly improves their dynamic response in the free vibration mode. Since the free vibrations of a multilayer cylindrical shell panel were investigated in [3], the issue of forced vibrations of the shell panel under the action of dynamic loading remained unresolved.

In [4], the authors analyzed the bending of a functionally gradient cylindrical nanoshell based on the nonlocal theory of elasticity and the theory of first-order shear deformation. The nanoshell is made of a combination of ceramic and metal materials reinforced with composite sheets with carbon nanotubes, which are placed along the outer radius. The structure was based on the Pasternak foundation. The equation of motion was derived using the principle of virtual work, and the properties of the reinforced composite sheets were estimated using the rule of mixtures. To verify the developed model, a comparative analysis of numerical results was performed. The influence of the core parameters, nonlocal parameter, volume fraction, and number of carbon nanotube layers, functionally gradient index, and foundation characteristics on the bending behavior of the shell was investigated. The question of the real interaction of the shell components under complex dynamic loads remained unresolved since the use of the rule of mixtures in the work did not allow local effects to be reflected.

The authors of work [5] investigated the forced oscillations of discretely reinforced five-layer cylindrical, spherical and conical shells under the action of unsteady loading. The dynamic behavior of the shells was investigated using the theory of shells and rods of the Tymoshenko type. The results of the studies showed that five-layer cylindrical shells with a less rigid filler demonstrated larger deflections and higher sensitivity to dynamic loading. In addition, for the shell with a less rigid filler, the influence of reinforcing elements was clearly observed, which was not observed with a stiffer filler. Numerical modeling demonstrated that the reinforcement of the hole in the five-layer spherical shell significantly affected the distribution of stresses and strains. At the moment of maximum loading of the unreinforced shell, significant differences in kinematic and static characteristics were observed in the hole zone: the presence of a reinforcing ring reduced local extrema several times compared to the unreinforced shell. The analysis of the results obtained for a five-layer conical shell conducted in the work allowed the authors to assess the influence of the taper angle on the symmetry of the distribution of displacements and stresses along the spatial coordinate. The issue of reliable accuracy of the solution to the problem for five-layer shells with discrete reinforcement remained unresolved since the accuracy may be low in the areas of reinforcement by ribs. The reason was the use of numerical methods by the authors in solving the problem.

The analysis of forced oscillations of a truncated elliptical conical shell arising under the action of a distributed impulse load was carried out in [6]. To solve the problem, a numerical algorithm was developed based on the finite-difference approximation of the initial equations in spatial and temporal coordinates. However, the studies carried out in the work are limited to the analysis of the dynamic behavior of the selected type of shell.

In [7], the deformation of multilayer ellipsoidal shells under the action of a non-stationary distributed load was considered. In order to increase the strength of the structure, the authors proposed reinforcing the shell with longitudinal stiffening ribs. To describe the mechanical behavior of the system, the theory of shells and rods by Tymoshenko was used, which allowed them to study the influence of longitudinal ribs on the stressed-strained state of the shell taking into account the discrete placement of the ribs. Based on the Hamilton-Ostrogradsky variational principle, a mathematical model of structural oscillations under the action of a short-term non-stationary load was built. The solution to the problem was obtained using a numerical algorithm based on the integrated-interpolation approach to constructing finite-difference schemes in spatial coordinates and an explicit finite-difference scheme in time coordinate. Analysis of the obtained dependences revealed that the presence of reinforcing ribs significantly affects the deformations of the multilayer shell. It was established that the influence of ribs on the deformed state of the reinforced ellipsoidal shell increases over time. The issues of accuracy and stability at large time steps and over a long period of modeling remained unresolved. The main reason was the use by the authors of finite-difference schemes in spatial coordinates and an explicit finite-difference scheme in time coordinate.

The authors of work [8] proposed a generalized computational model for analyzing the stability and initial post-critical behavior of cylindrical shells of the "sandwich" type with an elastic core, on which only transverse tension and compression act. The developed model is based on nonlinear equations of equilibrium of mixed form, asymptotic equations obtained by the Koiter-Budyansky method. The work proposes an analytical solution to a homogeneous problem on eigenvalues and a non-homogeneous problem for determining the values of unknown functions at the critical point. Numerical modeling showed a significant influence of internal pressure on the critical load and the nature of the shell deformation after critical deformation. However, the issue of the influence of unsteady pressure on the behavior of the shell remained unresolved. The reason could be the analytical solution to the problem proposed by the authors, which becomes more complicated under unsteady loading.

The theory of higher-order shear deformations is a development of the first-order theory and allows for a more accurate description of the operation of plates and shells. In this theory, transverse shear deformations are given by higher-order functions (quadratic, cubic, etc.), which ensures their variability over the thickness and adequate distribution of stresses in multilayer structures.

In study [9], a numerical analysis of geometrically nonlinear forced vibrations of a doubly curved sandwich shell with a honeycomb core manufactured by the method of modeling by deposited deposition was carried out. The theory of higher-order shear deformations was used to describe the operating mode of the structure. The dynamic behavior of each layer of the shell is described using five variables: three components of displacement and two components of rotation of the normal to the median surface. A system of geometrically nonlinear ordinary differential equations was obtained, which simulated the forced vibrations of the shell. The method of assumed modes was used to derive this system. The analysis of nonlinear periodic oscillations was carried out based on a numerical approach combining the method of continuation of solutions and the method of survey. The use of the method of assumed modes left unresolved the issue of local high-frequency oscillations and stress concentrations, which can significantly affect the accuracy of solutions for shells under the action of complex loads.

The authors of [10] conducted an analytical study of the bending of isotropic, layered cylindrical sandwich shells based on the theory of higher-order shear deformation. The proposed model included only four variables and did not require a shear correction factor, unlike traditional theories of this class. The work used Hamilton's principle and the Navier method to formulate the problem and solve the equations of motion. The results of the analytical model are compared with studies known in the literature. The authors used the finite element method to analyze displacements and stresses. Shells with different radii of curvature and thicknesses under the influence of uniform loading were considered. The analysis revealed that the proposed model allowed for more accurate modeling of the behavior of layered plates and shells compared to traditional approaches. The paper proposes an analytical model for static bending of layered sandwich shells, so the question of the influence of dynamic loads on the deformed state of the shells remains open.

For laminated and hybrid composites, zigzag functions are used, which reflect a sharp change in the slope of deformations between layers and increase the accuracy of modeling internal stresses.

In [11], free vibrations and losses of stability of a composite layered shell were investigated using the theory of zigzag deformations. Piecewise linear zigzag functions were used to describe the deformation state of multilayer shell structures, and the d'Alembert principle was used to derive the oscillation equations and boundary conditions. The authors conducted a study of free vibrations and losses of stability for cylindrical and spherical shells with different layering schemes, which allowed them to assess the accuracy and effectiveness of the proposed model. The results obtained were compared with three-dimensional and analytical solutions given in the literature. The comparison showed higher accuracy and better computational efficiency of the model compared to classical high-order theories. The work investigated only free vibrations and critical loads. The influence of unsteady loads on the dynamic behavior of the shell was not considered.

In [12], a model of bending of composite layered shells was proposed, based on the theory of zigzag deformations. The model differed from first-order theories in that planar linear zigzag functions along the thickness were used. This allowed the authors to exclude shear correction coefficients. Based on the principle of virtual work, equilibrium equations and boundary conditions were derived. The static properties of shells were described using solutions of Navier series. To assess the effectiveness of the model, numerical examples are given, in which the influence of layering schemes and geometric parameters was analyzed. The results of the study were compared with the three-dimensional theory of elasticity, first-order models known in the literature. Comparative analysis revealed high accuracy of the proposed model. Since work [12] considers static loads, the issue of shell deformations under real conditions when shells are under the action of dynamic loads remained unresolved.

During the loading of multilayer shell structures, local disturbances that arise in the zones of change in the physical and mechanical properties of the layers cause a significant change in the distribution of stresses and strains throughout the volume of the structure. Such complexity of the stressed-strained state necessitates the use of modern numerical methods of analysis (finite element method, finite difference method, etc.). The use of modern numerical methods of analysis ensures adequate reproduction of the mechanical behavior of multilayer shells, taking into account interlayer interactions, anisotropy of materials, and geometric nonlinearity.

The authors of study [13] analyzed the mechanical behavior of multilayer axisymmetric shells under different conditions of internal pressure based on theoretical modeling and numerical analysis using the finite element method in the ANSYS environment. Critical design parameters that affected the strength of the structure were established. The influence of the mechanical characteristics of the material, the geometric properties of the shell, and the internal pressure was studied. The studies showed the gradual nature of the development of damage in multilayer shells manufactured by the surfacing method. The authors of the work investigated the optimization of the parameters of the cellular structure, in particular the thickness of the shell, the height and thickness of the honeycomb walls and the dimensions of the cells. It was determined that the specified parameters significantly affected the reduction of stress concentration and the increase in the structural integrity of the structure. The model proposed in the work made it possible to assess the general patterns of deformations of multilayer shells, but the issues of accuracy and convergence of numerical models for complex shell structures remained open. The reason was the use of the finite element method in the ANSYS environment by the authors of the work.

In [14], the problem of optimizing the mass of layered orthotropic open shells of constant thickness under the action of an impulse load was considered. The study used an improved shell theory, in which a shell with a complex geometry was modeled by an auxiliary layered cylindrical open rectangular shell with an identical layer structure. The analytical solution was obtained in the form of a trigonometric series with the corresponding contour and boundary conditions. The adaptive optimization method using hybrid finite elements solved the problem of optimal shell design. The influence of geometric parameters on the optimal characteristics of a two-layer composite shell was analyzed. The work determined the extreme values corresponding to the optimal shapes of shell and plate structures. The use of hybrid finite elements in [14] left the issue of convergence and accuracy of numerical calculation for very thin layers or complex geometries unresolved.

In [15], an approach to modeling the process of forming the power shell of a composite fuel tank manufactured by the filament winding method is reported. To model the fiber stacking in the area of the end parts of a tank with complex geometry, the authors proposed a graphical procedure for forming the shell of a nonlinear composite tank with a small pole hole. The developed model was based on a mathematical description of the mandrel profile for forming the first layer, with subsequent iterative growth of subsequent layers. Three-dimensional CAD systems were selected for modeling, which provided parametric specification of the shell geometry and accurate representation of the structure configuration. Particular attention was paid to the pole hole area, in particular for modeling tanks with complex bottom shapes. The effectiveness of the model proposed in the work was confirmed by test results that reflected the nature of structural damage. The correctness of the model was confirmed by verification based on the manufacture of an experimental sample. Although CAD systems and parametric modeling were used, the issue of accuracy and convergence of calculations for complex geometries, thin layers, and large pole holes remained unresolved. The reason is the authors' use of a mandrel to form the first layer, followed by iterative build-up of subsequent layers.

In [16], an effective approach to the analysis of the acoustic characteristics of bi-curved multilayer composite shells reinforced with carbon nanotubes was proposed, based on three-dimensional theory and the state space method in combination with the fourth-order Runge-Kutta algorithm. The material properties of nanocomposites are described using the rule of mixtures. Variants of a homogeneous and functional-gradient distribution of nanotubes in the direction of the shell thickness are considered. The modeling of the equations of state was carried out for each $s$-th layer by combining the constitutive relations, deformation and motion equations within the state space. The assumption of a plane-wave solution allowed the authors to reduce the partial differential equations to systems of ordinary differential equations. For numerical integration, the authors used the Runge-Kutta method to construct the propagation matrix within each layer. The comparison of these matrices made it possible to form a general transfer matrix of the entire structure, on the basis of which the sound transmission losses were calculated. The work shows that the volume fraction of nanotubes, their spatial distribution, the geometric parameters of the shell, the angle of incidence of the acoustic wave and the shape of the surface significantly affect the acoustic efficiency of the structure. However, the issues of research on local high-frequency vibrations and stress concentrations remained unresolved. In addition, additional research is required on the issue of stability and accuracy of the proposed algorithm for complex configurations. The reason was the use of the rule of mixtures by the authors of the work.

In study [17], a numerical study of free vibrations of hybrid layered thin-walled cylindrical shells made of graphite composites and functionally gradient materials reinforced with carbon nanotubes was carried out. The analysis is based on the Sanders thin shell theory and the concept of artificial springs using the Rayleigh-Ritz method. Rotational effects, in particular Coriolis forces and centrifugal forces, are taken into account. It is found that optimizing the layer stacking sequence and geometric characteristics can significantly improve the dynamic performance of the shells. The work offers a numerical analysis of free vibrations of hybrid nanocomposite shells, but the issue of nonlinear behavior at large deformations for complex shells remains open. The reason was the authors' use of the Rayleigh-Ritz method, which has limitations.

It is known that extrapolation methods are widely used to solve differential equations that require high accuracy of the solution. The advantage of extrapolation methods is primarily that when using them, there is no need to recalculate the right-hand sides of the differential equations many times. In cases where the right-hand sides of the equations are quite complex, this advantage is very important. Scientists and engineers use Richardson extrapolation as a computational tool to improve the accuracy of numerical algorithms for solving systems of partial differential equations. This method improves the computational efficiency of the solution process by automatically changing the step sizes in time.

In study [18], higher-order convergence was numerically proven for the class of singularly perturbed Fredholm integrated-differential equations. To approximate the derivatives, a non-standard difference scheme was used, in which the integral term is approximated using the trapezoidal rule. The proposed numerical approach provided a uniform convergence rate that did not depend on the value of the perturbation parameter. The use of Richardson extrapolation allowed the authors to increase the accuracy of the solutions: fourth-order convergence was achieved for reaction-diffusion problems and second-order for convection-diffusion problems. The experiments confirmed the effectiveness and reliability of the theoretical results. The issue of ensuring the long-term stability of the numerical solution remained unresolved. The reason was the use by the authors of a non-standard difference scheme for approximating derivatives.

Our review of the literature [1–18] showed that despite a fairly large number of works reporting studies on vibrations of multilayer reinforced shells, certain issues remained unresolved. In the analyzed works [1, 2], simplified models of shells were used, without taking into account shear deformations and transverse stresses. The use of the rule of mixtures in [3, 4] to specify the properties of layers was a significant simplification of modeling vibrations of multilayer shells and did not reflect the real heterogeneity of the material. In papers [5–7], the authors used the Tymoshenko theory taking into account shear but limited themselves to the analysis of the dynamic behavior of the selected type of shells and the issue of high-order convergence of the selected numerical methods remained unresolved. The authors of [8] proposed an analytical solution to the problem of eigenvalues for the analysis of the stability of cylindrical shells, which complicates the use of the proposed method in modeling shells under dynamic loads. The authors of [9] proposed only free vibrations of shells, without taking into account real dynamic loads. In [10–13] the authors limited the study to static or only free vibrations, so the issue of taking into account real dynamic loads when studying vibrations of reinforced shells remained unresolved. The complexity of studying vibrations of reinforced shells led the authors of [14–18] to use various numerical methods, but the issue of increasing the accuracy of solving problems for inhomogeneous reinforced shells remained unresolved.

Therefore, it is advisable to conduct a study on determining the forced vibrations of reinforced multilayer cylindrical shells and developing a numerical algorithm that will increase the accuracy of solving the problem.

## 3. The aim and objectives of the study

The aim of our research is to increase the accuracy of solving dynamic problems by developing a numerical algorithm based on finding approximate solutions to partial differential equations using Richardson extrapolation. This will make it possible to expand existing approaches to numerical analysis of dynamic systems and use the results to improve the design of structures in various industries.

To achieve the goal, the following tasks are set:

– to state the problem of deformation of multilayer cylindrical shells taking into account the discrete placement of ribs;

– to derive the equations of oscillations of multilayer cylindrical shells supported by transverse ribs and natural boundary conditions;

– to apply the numerical method of finite-difference schemes in spatial coordinates and explicit finite-difference schemes in time coordinates to solve the problem;

– to use the Richardson extrapolation method to find approximate solutions to the problem;

– to carry out a comparative analysis of the deflection and stress obtained using the proposed methods.

## 4. The study materials and methods

The object of our study is the forced oscillations of a non-uniform elastic structure in the form of a multilayer cylindrical shell, which consists of rigidly connected layers and is reinforced by discrete ring elements.

In this study, it was hypothesized that the use of Richardson extrapolation to find approximate solutions to dynamic problems could increase the accuracy of the solution compared to the method of finite-difference schemes. It was assumed that the layers of the shell and discrete reinforcing ribs are rigidly connected to each other.

The following research methods were used in this work. At the stage of stating the problem of deformation of multilayer cylindrical shells discretely reinforced with transverse ridges, the method of imaginary construction of the object under study was used. The method of imaginary experiment was chosen because it provided the detection of the influence of reinforcing ribs on the deflection and deformation of multilayer cylindrical shells under unsteady loading without conducting expensive experimental studies.

The equation of oscillations of multilayer cylindrical shells reinforced with transverse ribs and natural boundary conditions under unsteady loading was obtained through the use of such research methods as imaginary experiment and modeling. The selected methods proved effective in detecting the influence of various factors on the deflection and deformation of multilayer cylindrical shells reinforced with ribs.

During dynamic loading of multilayer reinforced cylindrical shells, local disturbances in the region of changes in the physical and mechanical parameters of ribs led to a significant redistribution of the parameters of the stressed-strained state in the entire studied area. The complexity of the processes that arose in this case necessitated the use of a modern integrated-interpolation numerical method for solving problems of the behavior of reinforced ellipsoidal shell structures taking into account the discrete placement of ribs. This method was based on the construction of finite-difference schemes in spatial coordinates and an explicit finite-difference scheme of the “cross” type in the time coordinate. The choice of this approach allowed us to solve a system of partial differential equations in the presence of spatial discontinuities, which ensured the consideration of the discreteness of the reinforcing ribs. To implement the developed numerical algorithm, a computer search construction method was used, which was based on the application of modern computer and information technologies. In addition, when solving the problem, Richardson extrapolation was employed to find approximate solutions to the problem. Computerization of the studied object was carried out using the FortranPowerStation programming language (USA).

In order to solve the problem of comparative analysis of the accuracy of the solution of deflection and stress according to the proposed methods, the following research methods were used: graphical method, analysis, comparison. The graphical method was implemented using the MATLAB software package for numerical analysis and programming (USA). The graphical method allowed us to visualize the results of the influence of reinforcing ribs on the deflection and stress of multilayer cylindrical shells reinforced with transverse ribs when subjected to an unsteady load. The use of the graphical method made it possible to see the process of deformation of multilayer cylindrical shells reinforced with ribs in dynamics. Based on the results of this method, a comparative analysis of deflections and stresses obtained using the method of finite-difference schemes and Richardson extrapolation was carried out to confirm our research hypothesis.

## 5. Results of the development of a numerical algorithm using Richardson extrapolation

### 5. 1. Results of stating the problem of deformation of multilayer cylindrical shells taking into account the discrete placement of ribs

A non-uniform elastic shell structure was considered, which was a multilayer reinforced cylindrical shell. The research was based on the geometrically nonlinear theory of shells of the Tymoshenko type in the quadratic approximation using hypotheses for the entire package as a whole. The reinforcing elements were considered as a set of curved rods that are rigidly connected to the shell. The theory of curved rods by Tymoshenko was adopted for the calculation of the ribs.

The change in displacements along the thickness of the $m$-th layer was given by an approximation in the following form:

$$U_{1m}^{z}\left(x,y,z\right)=U_{1m}\left(x,y,z\right)+z\varphi_{1m}\left(x,y\right),$$

$$U_{2m}^{z}\left(x,y,z\right)=U_{2m}\left(x,y,z\right)+z\varphi_{2m}\left(x,y\right),$$

$$U_{3m}^{z}\left(x,y,z\right)=U_{3m}\left(x,y,z\right)+z\varphi_{3m}\left(x,y\right),$$

$$z\in\left(-\frac{h_m}{2},\frac{h_m}{2}\right), \tag{1}$$

where $U_{1m}$, $U_{2m}$, $U_{3m}$, $\varphi_{1m}$, $\varphi_{2m}$ are the components of the generalized displacement vector of the middle surface of the $m$-th layer.

The transverse shear stresses $\sigma_{13m}^{z}$ and $\sigma_{23m}^{z}$ varied along the thickness of the corresponding layer according to the formula

$$\sigma_{13m}^{z}\left(x,y,z\right)=f_{1m}\left(z\right)\sigma_{13m}^{0}\left(x,y\right),$$

$$\sigma_{23m}^{z}\left(x,y,z\right)=f_{2m}\left(z\right)\sigma_{23m}^{0}\left(x,y\right), \tag{2}$$

where functions $f_{1m}(z)$, $f_{2m}(z)$ were chosen from the condition of continuity of transverse stresses along the thickness.

The deformed state of the rib directed along the $\alpha_2$ axis was determined by the vector of displacement of the center of gravity line of the cross section:

$$U_{1j}^{yz}\left(x,y,z\right)=U_{1j}\left(x\right)+y\varphi_{1j}\left(x\right)+z\varphi_{2j}\left(x\right),$$

$$U_{2j}^{yz}\left(x,y,z\right)=U_{2j}\left(x\right)+z\varphi_{3j}\left(x\right),$$

$$U_{3j}^{yz}\left(x,y,z\right)=U_{3j}\left(x\right)-y\varphi_{3j}\left(x\right), \tag{3}$$

where $U_{1j}$, $U_{2j}$, $U_{3j}$, $\varphi_{1j}$, $\varphi_{2j}$ , $\varphi_{3j}$ are the components of the generalized vector of displacements of the center of gravity of the cross section of the $j$-th rib.

### 5. 2. Results of deriving the equations of oscillations of multilayer cylindrical shells reinforced with transverse ribs

To study the axisymmetric vibrations of a multilayer inhomogeneous elastic structure, the equations of oscillations of the layered shell in the smooth region and separately for the reinforcing ribs were compiled.

The equations of oscillations of the layered shell in the smooth region between the corresponding discrete ribs:

$$\frac{\partial T_{11}}{\partial x}+P_1=I_1\frac{\partial^2 U_1}{\partial t^2}+I_2\frac{\partial^2 \varphi_1}{\partial t^2},$$

$$\frac{\partial \overline{T}_{13}}{\partial x}+\frac{T_{22}}{R}+P_3=I_1\frac{\partial^2 U_3}{\partial t^2},$$

$$\frac{\partial M_{11}^{*}}{\partial x}-T_{13}+m_1=I_2\frac{\partial^2 U_1}{\partial t^2}+I_3\frac{\partial^2 \varphi_1}{\partial t^2},$$

$$\overline{T}_{13}=T_{13}+T_{11}\theta_1,\ \ M_{11}^{*}=M_{11}\pm h_{cm}T_{11}. \tag{4}$$

The equation of oscillations of the $j$-th annular rib at the points of discontinuities $x=x_j$ (points of projection of the centers of gravity of the cross section onto the reduced median surface of the smooth multilayer shell):

$$\left[T_{11}\right]_j=\rho_j F_j\left(\frac{\partial^2 U_1}{\partial t^2}\pm\frac{\partial^2 \varphi_1}{\partial t^2}\right),$$

$$\left[\overline{T}_{13}\right]_j-\frac{T_{22j}}{R_j}=\rho_j F_j\frac{\partial^2 U_3}{\partial t^2},$$

$$\left[M_{11}\right]_j=\rho_j F_j\left[\pm h_j\left(\frac{\partial^2 U_1}{\partial t^2}\pm h_{cj}\frac{\partial^2 \varphi_1}{\partial t^2}\right)+\frac{I_{\hat{e}\eth j}}{F_j}\frac{\partial^2 \varphi_1}{\partial t^2}\right], \tag{5}$$

where:

$$\left(T_{11},T_{22},T_{13}\right)=\sum_k\int_z\left(\sigma_{11}^{kz},\sigma_{22}^{kz},\sigma_{13}^{kz}\right)\mathrm{d}z,$$

$$M_{11}=\sum_k\int_z\left(z\sigma_{11}^{kz}\right)\mathrm{d}z,\ \ I_1=\sum_k\rho_k h_k,$$

$$I_2=\sum_k\pm\rho_k h_k h_{ck},\ \ I_3=\sum_k\rho_k\frac{h_k}{12}.$$

In equations (4), (5) the following notations are introduced: $x$, $t$ – spatial and temporal coordinates, respectively; $R$ – radius of the reduced median surface of the multilayer shell. The densities of the materials of the $k$-th shell layer and the $j$-th rib, respectively – $\rho_k$, $\rho_j$; the thicknesses of the corresponding layers of the shell – $h_k$; the distance from the median surface of the initial layer to the median surface of the $k$-th layer – $h_{ck}$. The distance from the initial median surface to the line of the center of gravity of the cross-section of the $j$-th rib is denoted by $h_{cj}$, and $x_j$ is the coordinate of the line of contact of the $j$-th rib with the multilayer shell. The geometric parameters of the $j$-th rib are $R_j$, $F_j$, $I_{kpj}$.

In the notations for the magnitudes of forces and moments, it was assumed that $\sigma_{11}^{kz}$, $\sigma_{22}^{kz}$, $\sigma_{13}^{kz}$ are the stresses along the thickness of the $k$-th layer, respectively, at $-\frac{h_k}{2}\le z\le\frac{h_k}{2}$, $k=\overline{1,3}$.

The relationship between the magnitudes of stresses and the components of deformations was obtained from the following formula:

$$\sigma_{11}^{kz}=\frac{E_1^k}{1-\nu_1^k\nu_2^k}\left(\varepsilon_{11}^{kz}+\nu_2^k\varepsilon_{22}^{kz}\right),$$

$$\sigma_{22}^{kz}=\frac{E_2^k}{1-\nu_1^k\nu_2^k}\left(\varepsilon_{22}^{kz}+\nu_1^k\varepsilon_{11}^{kz}\right),$$

$$\sigma_{13}^{kz}=G_{13}^{kz}\varepsilon_{13}^{kz}, \qquad (6)$$

where the components of the strain tensor in the coordinate system $x$, $z$ took the form:

$$\varepsilon_{11}^{kz}=\varepsilon_{11}^{k}+z\nu_{11}^{k},$$

$$\varepsilon_{22}^{kz}=k_2u_3^k,$$

$$\varepsilon_{13}^{kz}=\varphi_1^k+\theta_1^k\varepsilon_{11}^k=\frac{\partial u_1^k}{\partial x}+\frac{1}{2}\left(\theta_1^k\right)^2+k_1u_3^k,$$

$$\nu_{11}^{kz}=\frac{\partial\varphi_1^k}{\partial x},\quad \nu_{11}^{kz}=\frac{\partial\varphi_1^k}{\partial x},\quad \theta_1^k=\frac{\partial u_3^k}{\partial x}-k_1u_1^k. \qquad (7)$$

Equations (4) to (7) are supplemented with natural boundary and initial conditions, respectively.

**5. 3. Algorithm for applying the numerical method of finite-difference schemes to study vibrations of reinforced cylindrical shells**

The presence of discontinuity coefficients in the original equations of oscillations was one of the reasons for the complexity of solving boundary value problems of the theory of inhomogeneous shells, namely reinforced shells taking into account the discrete placement of ribs. According to, to solve such problems, first the solution to the problem was found in the smooth part, and then "gluing" took place on the discontinuity lines. In the problem proposed in our work, the discontinuity lines were the points of projection of the centers of gravity of the cross-section of the corresponding $j$-th edge onto the middle surface of the shell.

To construct the difference scheme when solving equations (4) to (7), the integrated-interpolation method of constructing finite-difference schemes for hyperbolic equations was used. According to this approach, equation (4) is represented in the following form in the domain $\{x_{l-1/2}\leq x\leq x_{l+1/2},\ t_{n-1/2}\leq t\leq t_{n+1/2}\}$:

$$\int_{t_{n-1/2}}^{t_{n+1/2}}\int_{x_{l-1/2}}^{x_{l+1/2}}\frac{\partial T_{11}}{\partial x}\mathrm{d}x\mathrm{d}t=\int_{t_{n-1/2}}^{t_{n+1/2}}\int_{x_{l-1/2}}^{x_{l+1/2}}\left(I_1\frac{\partial^2U_1}{\partial t^2}+I_2\frac{\partial^2\varphi_1}{\partial t^2}\right)\mathrm{d}x\mathrm{d}t,$$

$$\int_{t_{n-1/2}}^{t_{n+1/2}}\int_{x_{l-1/2}}^{x_{l+1/2}}\left(\frac{\partial\overline{T}_{13}}{\partial x}+\frac{T_{22}}{R}+P_3\left(x,t\right)\right)\mathrm{d}x\mathrm{d}t=$$
$$=\int_{t_{n-1/2}}^{t_{n+1/2}}\int_{x_{l-1/2}}^{x_{l+1/2}}I_1\frac{\partial^2U_3}{\partial t^2}\mathrm{d}x\mathrm{d}t,$$

$$\int_{t_{n-1/2}}^{t_{n+1/2}}\int_{x_{l-1/2}}^{x_{l+1/2}}\left(\frac{\partial M_{11}^*}{\partial x}-T_{13}\right)\mathrm{d}x\mathrm{d}t=$$
$$=\int_{t_{n-1/2}}^{t_{n+1/2}}\int_{x_{l-1/2}}^{x_{l+1/2}}\left(I_2\frac{\partial^2U_1}{\partial t^2}+I_3\frac{\partial^2\varphi_1}{\partial t^2}\right)\mathrm{d}x\mathrm{d}t. \qquad (8)$$

After standard transformations in ratios (8), the following difference approximations of equations (4) are obtained:

$$L_1\left(\overline{U_l^n}\right)=I_1\left(u_{1l}^n\right)_{\bar{t}t}+I_2\left(\varphi_{1l}^n\right)_{\bar{t}t},$$

$$L_2\left(\overline{U_l^n}\right)+P_3\left(x_l,t_n\right)=I_1\left(u_{3l}^n\right)_{\bar{t}t},$$

$$L_3\left(\overline{U_l^n}\right)=I_2\left(u_{1l}^n\right)_{\bar{t}t}+I_3\left(\varphi_{1l}^n\right)_{\bar{t}t}, \qquad (9)$$

where:

$$L_1\left(\overline{U_l^n}\right)=\frac{T_{11l+1/2}^n-T_{11l-1/2}^n}{\Delta x},$$

$$L_2\left(\overline{U_l^n}\right)=\frac{\overline{T}_{13l+1/2}^n-\overline{T}_{13l-1/2}^n}{\Delta x}-\frac{T_{22l+1/2}^n+T_{22l-1/2}^n}{2R},$$

$$L_3\left(\overline{U_l^n}\right)=\frac{M_{11l+1/2}^{*n}-M_{11l-1/2}^{*n}}{\Delta x}-\frac{T_{13l+1/2}^n+T_{13l-1/2}^n}{2}. \qquad (10)$$

In relations (9) $\overline{U_l^n}=\left(u_{1l}^n,u_{3l}^n,\varphi_{1l}^n\right)$, and the notation of discrete derivatives was introduced according to [5]. Based on (9), the magnitudes of forces and moments were related to the difference points in the spatial coordinate in half-integer points, and in the time coordinate – in integer points of the difference scheme

$$\left(T_{11},T_{22},\overline{T}_{13},M_{11},\right)\rightarrow\left(T_{11l\pm1/2}^n,T_{22l\pm1/2}^n,\overline{T}_{13l\pm1/2}^n,M_{11l+1/2}^n\right).$$

Based on this, equations (6), (7) were integrated, respectively, in the areas

$$\left\{x_{l-1}\leq x\leq x_l,\ t_{n-1/2}\leq t\leq t_{n+1/2}\right\}$$

and

$$\left\{x_l\leq x\leq x_{l+1},\ t_{n-1/2}\leq t\leq t_{n+1/2}\right\}.$$

Similarly, numerical integration of oscillation equations (5) for the $j$-th reinforcing element was carried out.

**5. 4. Application of Richardson extrapolation to find approximate solutions to the problem of vibrations of reinforced cylindrical shells**

As already noted, in a number of cases, when numerically solving equations (4) to (7) based on approximations (9), the convergence of numerical results deteriorated. In order to build a more effective numerical algorithm for solving this problem, the approach of finding approximate solutions using Richardson extrapolation was used [19]. The sequence of approximate approximations in the spatial coordinate was applied with a fixed difference step in the time coordinate. It was assumed that

$$\tilde{U}_{l(\Delta x)}^n=\frac{4}{3}\overline{U}_{l(\Delta x/2)}^n-\frac{1}{3}\overline{U}_{l(\Delta x)}^n, \qquad (11)$$

where $\overline{U}_{l(\Delta x)}^n$ and $\overline{U}_{l(\Delta x/2)}^n$ are numerical solutions of the oscillation equations (9), (10), and $\Delta x$, $\Delta x/2$ are the corresponding discrete steps along the spatial coordinate.

The expressions for the forces and moments are decomposed into linear and nonlinear parts:

$$T_{11}=T_{11\mathrm{L}}+T_{11\mathrm{NL}},\quad T_{22}=T_{22\mathrm{L}}+T_{22\mathrm{NL}},$$

$$\overline{T}_{13}=\overline{T}_{13\mathrm{L}}+\overline{T}_{13\mathrm{NL}},\quad M_{11}=M_{11\mathrm{L}}+M_{11\mathrm{NL}}. \qquad (12)$$

Then:

$$L_1\left(\bar{U}_l^n\right)=L_{1L}\left(\bar{U}_l^n\right)+L_{1NL}\left(\bar{U}_l^n\right),$$

$$L_2\left(\bar{U}_l^n\right)=L_{2L}\left(\bar{U}_l^n\right)+L_{2NL}\left(\bar{U}_l^n\right),$$

$$L_3\left(\bar{U}_l^n\right)=L_{3L}\left(\bar{U}_l^n\right)+L_{3NL}\left(\bar{U}_l^n\right), \quad (13)$$

$$L_{1NL}\left(\bar{U}_l^n\right)=B_{11}\frac{u_{3l+1}^n-u_{3l-1}^n}{2\Delta x}\cdot\frac{u_{3l+1}^n-2u_{3l}^n+u_{3l-1}^n}{\Delta x^2},$$

$$L_{1L}\left(\bar{U}_l^n\right)=B_{11}\frac{u_{1l+1}^n-2u_{1l}^n+u_{1l-1}^n}{\Delta x^2}+\frac{B_{12}}{R}\frac{u_{3l+1}^n-u_{3l-1}^n}{2\Delta x}+B_{13}\frac{\phi_{1l+1}^n-2\phi_{1l}^n+\phi_{1l-1}^n}{\Delta x^2},$$

$$L_{2NL}\left(\bar{U}_l^n\right)=B_{11}\frac{u_{11+1}^n-u_{1l}^n+u_{1l-1}^n}{\Delta x^2}\cdot\frac{u_{31+1}^n-u_{3l-1}^n}{2\Delta x}+B_{11}\frac{u_{11+1}^n-u_{1l-1}^n}{2\Delta x}\cdot\frac{u_{31+1}^n-u_{3l}^n+u_{3l-1}^n}{\Delta x^2}+\frac{B_{12}}{R}\left[\frac{u_{31+1}^n-u_{3l-1}^n}{2\Delta x}\right]^2+B_{13}\frac{u_{31+1}^n-u_{3l-1}^n}{2R}\times$$
$$\times\frac{u_{31+1}^n-u_{3l}^n+u_{3l-1}^n}{\Delta x^2}+B_{13}\frac{\phi_{11+1}^n-\phi_{1l}^n+\phi_{1l-1}^n}{\Delta x^2}\times\frac{u_{31+1}^n-u_{3l-1}^n}{2\Delta x}+B_{13}\frac{u_{31+1}^n-u_{3l}^n+u_{3l-1}^n}{\Delta x^2}\cdot\frac{\phi_{11+1}^n-\phi_{1l-1}^n}{2\Delta x}+B_{21}\frac{u_{31+1}^n-u_{3l}^n+u_{3l-1}^n}{\Delta x^2}\cdot\frac{u_{31+1}^n-u_{3l-1}^n}{2\Delta x},$$

$$L_{2L}\left(\bar{U}_l^n\right)=B_{21}\frac{u_{1l+1}^n-2u_{1l}^n+u_{1l-1}^n}{\Delta x^2}+\frac{B_{22}}{R}\frac{u_{3l+1}^n-u_{3l-1}^n}{2\Delta x}+B_{23}\frac{\phi_{1l+1}^n-2\phi_{1l}^n+\phi_{1l-1}^n}{\Delta x^2}+B_{31}\frac{\phi_{1l+1}^n-\phi_{1l-1}^n}{2\Delta x}+B_{31}\frac{u_{3l+1}^n-2u_{3l}^n+u_{3l-1}^n}{\Delta x^2},$$

$$L_{3NL}\left(\bar{U}_l^n\right)=D_{13}\frac{u_{31+1}^n-u_{3l-1}^n}{2\Delta x}\frac{u_{3l+1}^n-2u_{3l}^n+u_{3l-1}^n}{\Delta x^2},$$

$$L_{3L}\left(\bar{U}_l^n\right)=D_{11}\frac{\phi_{1l+1}^n-\phi_{1l}^n+\phi_{1l-1}^n}{\Delta x^2}+D_{13}\frac{u_{1l+1}^n-u_{1l}^n+u_{1l-1}^n}{\Delta x^2}. \quad (14)$$

The components of generalized vector $\bar{U}_l^n=\left(u_{ll}^n,u_{3l}^n,\phi_{ll}^n\right)$ are expanded in a Taylor series at interior points ($x_1 \pm \Delta x$), ($x_1 \pm \Delta x/2$) of the difference grids, respectively, with discrete steps along the spatial coordinate $\Delta x$ and $\Delta x/2$:

$$u_1\left(x_l\pm\Delta x\right)=u_1\left(x_1\right)\pm\Delta x u_1'\left(x_1\right)+\frac{\Delta x^2}{2!}u_1''\left(x_1\right)\pm\frac{\Delta x^3}{3!}u_1'''\left(x_1\right)+\frac{\Delta x^4}{4!}u_1^{(IV)}\left(x_1\right)\pm\frac{\Delta x^5}{5!}u_1^{(V)}\left(x_1\right)+O\left(\Delta x^6\right), \quad (15)$$

$$u_1\left(x_l\pm\frac{\Delta x}{2}\right)=u_1\left(x_1\right)\pm\frac{\Delta x}{2}u_1'\left(x_1\right)+\frac{\Delta x^2}{4\cdot 2!}u_1''\left(x_1\right)\pm\frac{\Delta x^3}{8\cdot 3!}u_1'''\left(x_1\right)+\frac{\Delta x^4}{16\cdot 4!}u_1^{(IV)}\left(x_1\right)\pm\frac{\Delta x^5}{32\cdot 5!}u_1^{(V)}\left(x_1\right)+O\left(\left(\frac{\Delta x}{2}\right)^6\right),$$

$$u_3\left(x_l\pm\Delta x\right)=u_3\left(x_1\right)\pm\Delta x u_3'\left(x_1\right)+\frac{\Delta x^2}{2!}u_3''\left(x_1\right)\pm\frac{\Delta x^3}{3!}u_3'''\left(x_1\right)+\frac{\Delta x^4}{4!}u_3^{(IV)}\left(x_1\right)\pm\frac{\Delta x^5}{5!}u_3^{(V)}\left(x_1\right)+O\left(\Delta x^6\right),$$

$$u_3\left(x_l\pm\frac{\Delta x}{2}\right)=u_3\left(x_1\right)\pm\frac{\Delta x}{2}u_3'\left(x_1\right)+\frac{\Delta x^2}{4\cdot 2!}u_3''\left(x_1\right)\pm\frac{\Delta x^3}{8\cdot 3!}u_3'''\left(x_1\right)+\frac{\Delta x^4}{16\cdot 4!}u_3^{(IV)}\left(x_1\right)\pm\frac{\Delta x^5}{32\cdot 5!}u_3^{(V)}\left(x_1\right)+O\left(\left(\frac{\Delta x}{2}\right)^6\right),$$

$$\phi_1\left(x_l\pm\Delta x\right)=\phi_1\left(x_1\right)\pm\Delta x\phi_1'\left(x_1\right)+\frac{\Delta x^2}{2!}\phi_1''\left(x_1\right)\pm\frac{\Delta x^3}{3!}\phi_1'''\left(x_1\right)+\frac{\Delta x^4}{4!}\phi_1^{(IV)}\left(x_1\right)\pm\frac{\Delta x^5}{5!}\phi_1^{(V)}\left(x_1\right)+O\left(\Delta x^6\right),$$

$$u_1\left(x_l\pm\frac{\Delta x}{2}\right)=u_1\left(x_1\right)\pm\frac{\Delta x}{2}\phi_1'\left(x_1\right)+\frac{\Delta x^2}{4\cdot 2!}\phi_1''\left(x_1\right)\pm\frac{\Delta x^3}{8\cdot 3!}\phi_1'''\left(x_1\right)+\frac{\Delta x^4}{16\cdot 4!}\phi_1^{(IV)}\left(x_1\right)\pm\frac{\Delta x^5}{32\cdot 5!}\phi_1^{(V)}\left(x_1\right)+O\left(\left(\frac{\Delta x}{2}\right)^6\right).$$

Then $L_{1NL}\left(\bar{U}_{l(\Delta x)}^n\right)$ according to (9) and (15)

$$L_{1NL}\left(\bar{U}_{l(\Delta x)}^n\right)=B_{11}\left(\begin{array}{l}u_3'\left(x_l\right)+\frac{\Delta x^2}{3!}u_3'''\left(x_l\right)+\\+\frac{\Delta x^4}{5!}u_3^{(V)}\left(x_l\right)+O\left(\Delta x^5\right)\end{array}\right)\times$$
$$\times\left(u_3''\left(x_l\right)+\frac{\Delta x^2}{12}u_3^{(IV)}+O\left(\Delta x^4\right)\right).$$

Similarly

$$L_{1NL}\left(\bar{U}_{l(\Delta x/2)}^n\right)=B_{11}\left(\begin{array}{l}u_3'\left(x_l\right)+\frac{\Delta x^2}{4\cdot 3!}u_3'''\left(x_l\right)+\\+\frac{\Delta x^4}{16\cdot 5!}u_3^{(V)}\left(x_l\right)+O\left(\Delta x^5\right)\end{array}\right)\times$$
$$\times\left(u_3''\left(x_l\right)+\frac{\Delta x^2}{48}u_3^{(IV)}+O\left(\left(\frac{\Delta x}{2}\right)^4\right)\right).$$

Then

$$\frac{4}{3}L_{1NL}\left(\bar{U}_{l(\Delta x/2)}^n\right)-\frac{1}{3}L_{1NL}\left(\bar{U}_{l(\Delta x)}^n\right)=$$
$$=B_{11}\left(\begin{array}{l}\frac{4}{3}\left(\begin{array}{l}(u_3'\left(x_l\right)+\frac{\Delta x^2}{4\cdot 3!}u_3'''\left(x_l\right)+\\+\frac{\Delta x^4}{16\cdot 5!}u_3^{(V)}\left(x_l\right)+O\left(\left(\frac{\Delta x}{2}\right)^5\right)\end{array}\right)\times\\ \times\left(u_3''\left(x_l\right)+\frac{\Delta x^2}{48}u^{(IV)}+O\left(\frac{\Delta x^4}{2}\right)\right)-\\ -\frac{1}{3}\left(\begin{array}{l}u_3'\left(x_l\right)+\frac{\Delta x^2}{3!}u_3'''\left(x_l\right)+\\+\frac{\Delta x^4}{5!}u_3^{(V)}\left(x_l\right)+O\left(\Delta x^5\right)\end{array}\right)\times\\ \times\left(u_3''\left(x_l\right)+\frac{\Delta x^2}{12}u^{(IV)}+O\left(\Delta x^4\right)\right)\end{array}\right).$$

After standard transformations, we obtain

$$\frac{4}{3}L_{1NL}\left(\bar{U}^n_{l(\Delta x/2)}\right)-\frac{1}{3}L_{1NL}\left(\bar{U}^n_{l(\Delta x)}\right)=$$
$$=B_{11}\begin{pmatrix} u_3'(x_l)u_3''-\frac{\Delta x^4}{288}u_3'''(x_l)u_3^{(IV)}(x_l)- \\ -\frac{\Delta x^4}{480}u_3''u_3^{(V)}-\frac{\Delta x^6}{4608}u_3^{(IV)}u_3^{(V)}+O(\Delta x^4)\end{pmatrix}.$$

Therefore, the difference operator $L_{1NL}\left(\bar{U}^n_l\right)$ at point $x_1$ had the fourth order of approximation.

$L_{1L}\left(\bar{U}^n_{l(\Delta x)}\right)$ based on formulae (9) and (15)

$$L_{1L}\left(\bar{U}^n_{l(\Delta x)}\right)=B_{11}\left(u_1''(x_l)+\frac{\Delta x^2}{12}u_1^{(IV)}+O(\Delta x^4)\right)+$$
$$+\frac{B_{12}}{R}\left(u_3'(x_l)+\frac{\Delta x^2}{3!}u_3'''(x_l)+\frac{\Delta x^4}{5!}u_3^{(V)}+O(\Delta x^5)\right)+$$
$$+B_{11}\left(\phi_1''(x_l)+\frac{\Delta x^2}{12}\phi_1^{(IV)}+O(\Delta x^4)\right).$$

Similarly, it is shown that

$$L_{1L}\left(\bar{U}^n_{l(\Delta x/2)}\right)=B_{11}\left(u_1''(x_l)+\frac{\Delta x^2}{48}u_1^{(IV)}+O\left(\left(\frac{\Delta x}{2}\right)^4\right)\right)+$$
$$+\frac{B_{12}}{R}\begin{pmatrix} u_3'(x_l)+\frac{\Delta x^2}{4\cdot 3!}u_3'''(x_l)+ \\ +\frac{\Delta x^4}{16\cdot 5!}u_3^{(V)}+O\left(\left(\frac{\Delta x}{2}\right)^5\right)\end{pmatrix}+$$
$$+B_{11}\left(\phi_1''(x_l)+\frac{\Delta x^2}{48}\phi_1^{(IV)}+O\left(\left(\frac{\Delta x}{2}\right)^4\right)\right).$$

Then

$$\frac{4}{3}L_{1L}\left(\bar{U}^n_{l(\Delta x/2)}\right)-\frac{1}{3}L_{1L}\left(\bar{U}^n_{l(\Delta x)}\right)=$$
$$=\frac{4}{3}\begin{pmatrix} B_{11}\left(u_1''(x_l)+\frac{\Delta x^2}{48}u_1^{(IV)}+O\left(\left(\frac{\Delta x}{2}\right)^4\right)\right)+ \\ +\frac{B_{12}}{R}\begin{pmatrix} u_3'(x_l)+\frac{\Delta x^2}{4\cdot 3!}u_3'''(x_l)+ \\ +\frac{\Delta x^4}{16\cdot 5!}u_3^{(V)}+O\left(\left(\frac{\Delta x}{2}\right)^5\right)\end{pmatrix}+ \\ +B_{11}\left(\phi_1''(x_l)+\frac{\Delta x^2}{48}\phi_1^{(IV)}+O\left(\left(\frac{\Delta x}{2}\right)^4\right)\right)\end{pmatrix}-$$
$$-\frac{1}{3}\begin{pmatrix} B_{11}\left(u_1''(x_l)+\frac{\Delta x^2}{12}u_1^{(IV)}+O(\Delta x^4)\right)+ \\ +\frac{B_{12}}{R}\begin{pmatrix} u_3'(x_l)+\frac{\Delta x^2}{3!}u_3'''(x_l)+ \\ +\frac{\Delta x^4}{5!}u_3^{(V)}+O(\Delta x^5)\end{pmatrix}+ \\ +B_{11}\left(\phi_1''(x_l)+\frac{\Delta x^2}{12}\phi_1^{(IV)}+O(\Delta x^4)\right)\end{pmatrix}=$$
$$=B_{11}u_1''(x_l)+\frac{B_{12}}{R}u_3'+B_{11}\phi_1''(x_l)-\frac{\Delta x^4}{480}u_3^{(V)}+O(\Delta x^4).$$

Therefore, the difference operator $L_{1L}\left(\bar{U}^n_l\right)$ at point $x_1$ had the fourth order of approximation.

It follows that expression $\frac{4}{3}L_1\left(\bar{U}^n_{l(\Delta x/2)}\right)-\frac{1}{3}L_1\left(\bar{U}^n_{l(\Delta x)}\right)$ approximated the first equation of system (4) with the fourth order of accuracy.

Expressions

$$\frac{4}{3}L_2\left(\bar{U}^n_{l(\Delta x/2)}\right)-\frac{1}{3}L_2\left(\bar{U}^n_{l(\Delta x)}\right)$$

and

$$\frac{4}{3}L_3\left(\bar{U}^n_{l(\Delta x/2)}\right)-\frac{1}{3}L_3\left(\bar{U}^n_{l(\Delta x)}\right)$$

also approximated the second and third equations of system (4) with the fourth order of accuracy, respectively.

Therefore, the order of accuracy in the coordinate $x$ with which (11) approximates the original equations of oscillations (4) in the smooth domain is the fourth.

**5. 5. Comparative analysis of deflection and stress values according to the method of finite-difference schemes and Richardson extrapolation**

We consider the problem of unsteady oscillations of a three-layer reinforced cylindrical shell taking into account the discreteness of the placement of ribs under dynamic loading, and the edges of the shell were rigidly clamped.

The boundary conditions took the following form at $x = 0$, $x = L$, where $L$ is the length of the shell

$$u_1=u_3=\varphi_1=0. \tag{16}$$

Zero initial conditions at $t = 0$ were assumed in the form

$$u_1=u_3=\varphi_1=0,\ \frac{\partial u_1}{\partial t}=\frac{\partial u_3}{\partial t}=\frac{\partial \varphi_1}{\partial t}=0. \tag{17}$$

The geometric and physical-mechanical parameters were assumed to be as follows:

$$h=h_1+h_2+h_3,\ \ h_1=h_3=10^{-3},\ \ \frac{h_3}{h_1}=3,$$
$$\frac{R}{h}=20,.\ \ \frac{L}{h}=80,\ \ \frac{L}{R}=4,\ \ \frac{h_j}{h}=2,\ \ F_j=h_jh,$$
$$E_1^1=E_1^3=E_j=7\cdot 10^{10}\ \text{Pa},\ \ \frac{E_1^1}{E^{fil}}=\frac{1}{100},$$
$$\nu_1^1=\nu_1^3=0.3,\ \ \nu_1^{fil}=0.4,\ \ \frac{\rho_1}{\rho_{fil}}=7,$$
$$\rho_1=\rho_3=\rho_j=2.7\cdot 10^3\ \frac{\text{kg}}{\text{m}^3}. \tag{18}$$

The normal impulse load was given in the form

$$P_3=A\cdot\sin\frac{\pi T}{t}\left[\eta(t)-\eta(t-T)\right], \tag{19}$$

where $A$ is the load amplitude; $T$ is the load duration, and these parameters in the problem were assumed to be as follows: $A = 10^6$ Pa, $T = 0.625L/c$. The reinforcing elements were located at points $x_j = 0.25L_j$, $j=1,3$.

The obtained numerical results made it possible to assess the nature of the stressed-strained state of a three-layer reinforced elastic structure of a cylindrical type at an arbitrary time point in the studied time interval. In this case, calculations were performed on the time interval $0 \le t \le 40T$. Depending on the value of the discrete step along the spatial coordinate, a comparative analysis of the calculation results was performed.

Fig. 1, 2 show the dependences of $u_3$ values on the spatial coordinate $x$ at time points $t = 8T$. In Fig. 1 curve 1 corresponded to the case $n = 40$; curve 2 – $n = 80$; curve 3 – $n = 160$; calculations were carried out according to the standard approach.

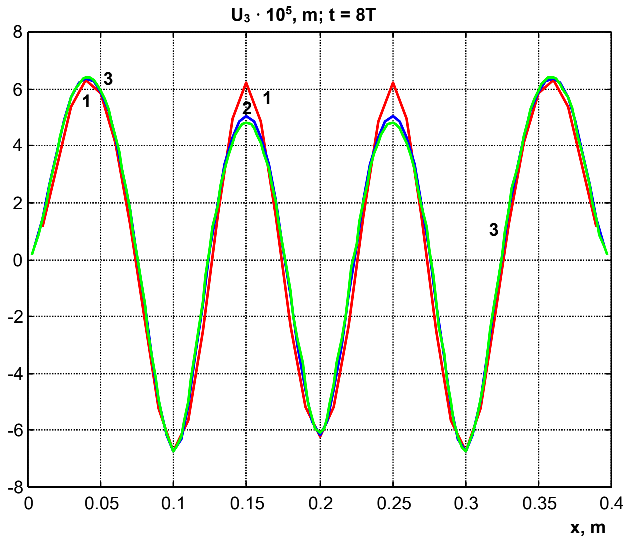


Fig. 1. Dependence of the $u_3$ value on the spatial coordinate $x$ at time $t = 8T$

Analysis of the presented graphic material in Fig. 1 revealed that the value of deflection $u_3$ in the spatial coordinate $x = 0.15m$ for the case $n = 40$ $\left(u_3\big|_{n=40} = 6.25 \cdot 10^{-5}\ \text{m}\right)$ is 1.25 times greater than the value of deflection $u_3$ for the case $n = 80$ $\left(u_3\big|_{n=80} = 5.01 \cdot 10^{-5}\ \text{m}\right)$ in the same spatial coordinate

$$\frac{u_3\big|_{n=40}}{u_3\big|_{n=80}} = \frac{6.25 \cdot 10^{-5}}{5.01 \cdot 10^{-5}} \approx 1.25.$$

In Fig. 1, the $u_3$ value of deflection in the spatial coordinate $x = 0.15m$ for the case $n = 160$ $\left(u_3\big|_{n=160} = 4.78 \cdot 10^{-5}\ \text{m}\right)$ is 1.05 times smaller than the $u_3$ value of deflection for the case $n = 80$ in the same spatial coordinate

$$\frac{u_3\big|_{n=80}}{u_3\big|_{n=160}} = \frac{5.01 \cdot 10^{-5}}{4.78 \cdot 10^{-5}} \approx 1.05.$$

The $u_3$ value of deflection in the spatial coordinate $x = 0.15m$ for the case $n = 40$ $\left(u_3\big|_{n=40} = 6.25 \cdot 10^{-5}\ \text{m}\right)$ is 1.31 times greater than the $u_3$ value of deflection for the case $n = 160$ $\left(u_3\big|_{n=80} = 4.78 \cdot 10^{-5}\ \text{m}\right)$ in the same spatial coordinate

$$\frac{u_3\big|_{n=40}}{u_3\big|_{n=160}} = \frac{6.25 \cdot 10^{-5}}{4.78 \cdot 10^{-5}} \approx 1.31.$$

In Fig. 2, curve 1 corresponded to the case $n = 160$; the calculation was carried out according to the standard approach; curve 2 – $n = 40 \div 80$; the calculation was carried out according to the Richardson approach. Our analysis of the graphic material in Fig. 2 revealed that the $u_3$ value of deflection in the spatial coordinate $x = 0.15m$ for the case $n = 160$ according to the standard approach is $\left(u_3\big|_{n=160} = 4.78 \cdot 10^{-5}\ \text{m}\right)$, which is 1.01 times greater than the $u_3$ value of deflection for the case $n = 40 \div 80$ according to the Richardson approach $\left(u_3\big|_{n=40\div80} = 4.73 \cdot 10^{-5}\ \text{m}\right)$ in the same spatial coordinate

$$\frac{u_3\big|_{n=160}}{u_3\big|_{n=40\div80}} = \frac{4.78 \cdot 10^{-5}}{4.73 \cdot 10^{-5}} \approx 1.01.$$

Fig. 3, 4 show the dependences of $\sigma_{22}$ values on the spatial coordinate $x$ at time $t = 7T$. In Fig. 3, curve 1 corresponds to the case of $n = 40$, curve 2 – $n = 80$, curve 3 – $n = 160$; calculations were carried out according to the standard approach.

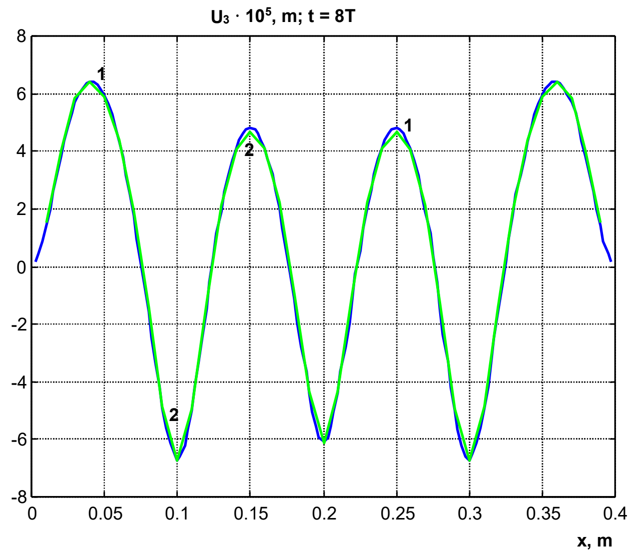


Fig. 2. Dependence of the $u_3$ value on the spatial coordinate $x$ at time $t = 8T$

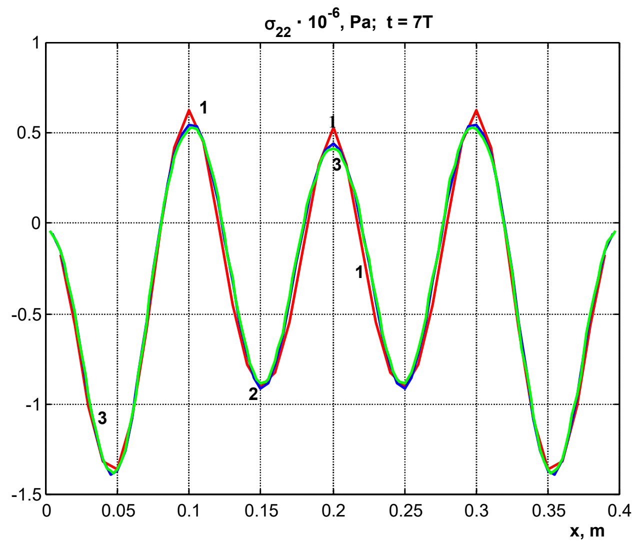


Fig. 3. Dependence of $\sigma_{22}$ value on the spatial coordinate $x$ at time $t = 7T$

Analysis of the given graphic material in Fig. 3 revealed that the $\sigma_{22}$ value of stress magnitude in the spatial coordinate $x = 0.1m$ for the case $n = 40$ is $\left(\sigma_{22}\big|_{n=40} = 0.63 \cdot 10^{6}\ \text{Pa}\right)$, and this is 1.17 times greater than the $\sigma_{22}$ stress magnitude for the case $n = 80$ $\left(\sigma_{22}\big|_{n=80} = 0.54\cdot10^6 \text{ Pa}\right)$ in the same spatial coordinate

$$\frac{\sigma_{22}\big|_{n=40}}{\sigma_{22}\big|_{n=80}} = \frac{0.63\cdot10^6}{0.54\cdot10^6} \approx 1.17.$$

In Fig. 3, the $\sigma_{22}$ value of stress magnitude in the spatial coordinate $x = 0.1$ $m$ for the case $n = 160$ is $\left(\sigma_{22}\big|_{n=160} = 0{,}53\cdot10^6 \text{ Pa}\right)$, that is, 1.02 times less than the $\sigma_{22}$ stress magnitude for the case $n = 80$ in the same spatial coordinate

$$\frac{\sigma_{22}\big|_{n=80}}{\sigma_{22}\big|_{n=160}} = \frac{0.54\cdot10^6}{0.53\cdot10^6} \approx 1.02.$$

The $\sigma_{22}$ value of stress magnitude in the spatial coordinate $x = 0.1m$ for the case $n = 40$ $\left(\sigma_{22}\big|_{n=40} = 0.63\cdot10^6 \text{ Pa}\right)$ is 1.31 times greater than the $\sigma_{22}$ stress magnitude for the case $n = 160$ $\left(\sigma_{22}\big|_{n=160} = 0.53\cdot10^6 \text{ Pa}\right)$ in the same spatial coordinate

$$\frac{\sigma_{22}\big|_{n=40}}{\sigma_{22}\big|_{n=160}} = \frac{0.63\cdot10^6}{0.53\cdot10^6} \approx 1.19.$$

In Fig. 4, curve 1 corresponded to the case $n = 160$; the calculation was carried out according to the standard approach; curve 2 – $n = 40 \div 80$; the calculation was carried out according to the Richardson approach. Our analysis of the graphic material in Fig. 4 revealed that the $\sigma_{22}$ value of stress magnitude in the spatial coordinate $x = 0.1m$ for the case $n = 160$ according to the standard approach is $\left(\sigma_{22}\big|_{n=160} = 0.530\cdot10^6 \text{ Pa}\right)$, and this is 1.01 times greater than the $\sigma_{22}$ value of stress magnitude for the case $n = 40 \div 80$ according to the Richardson approach $\left(\sigma_{22}\big|_{n=40\div80} = 0.526\cdot10^6 \text{ Pa}\right)$ in the same spatial coordinate

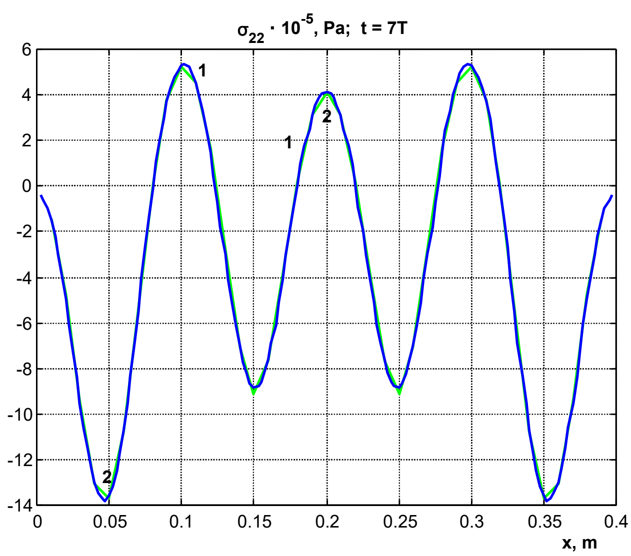


Fig. 4. Dependence of $\sigma_{22}$ value on the spatial coordinate $x$ at time $t = 7T$

So, a comparative analysis of the numerical results of our calculations revealed that at time $t = 8T$ according to the standard approach, the discrepancy in the values of deflections $u_3$ for $n = 40$ and $n = 160$ reached 31%, for $n = 80$ and $n = 160$ of the order of 5% (Fig. 1). According to the Richardson approach for $n = 40 \div 80$ and the standard approach for $n = 160$, this difference was about 1% (Fig. 2).

At time $t = 7T$ according to the standard approach, the discrepancy for the $\sigma_{22}$ values of stresses for $n = 40$ and $n = 160$ reached 19%, for $n = 80$ and $n = 160$ of the order of 2%, (Fig. 3). According to the Richardson approach for $n = 40 \div 80$ and the standard approach for $n = 160$, this difference was about 0.8% (Fig. 4).

From our graphical results, it is concluded that the use of Richardson extrapolation, in comparison with the standard approach to the numerical solution of the given equations, made it possible to achieve the required accuracy on coarser difference grids in the spatial coordinate.

The constructed numerical algorithms for solving problems of the theory of discretely reinforced multilayer cylindrical shells under the action of unsteady loading were tested on test calculations. The results of our calculations of this problem were compared with the results reported in [15]. The calculations, according to the numerical method devised in the current work, were in satisfactory agreement with the solutions in [19], which confirmed the reliability of our results.

## 6. Discussion of results of investigating the dynamic problems of layered cylindrical shells using Richardson extrapolation

Our work has considered a multilayer reinforced cylindrical shell under the action of a non-stationary load. The research was based on the geometrically nonlinear theory of shells of the Tymoshenko type in the quadratic approximation using hypotheses for the entire package as a whole. The reinforcing elements were considered as a set of curved rods that are rigidly connected to the shell. The theory of curved rods by Tymoshenko was adopted for the calculation of the ribs.

To study the axisymmetric vibrations of a multilayer inhomogeneous elastic structure, the equations of shell vibrations in the smooth region (4) and separately for the reinforcing ribs (5) were derived separately. The presence of discontinuity coefficients in the original vibration equations necessitated the need to "glue" the solutions to individual problems for the smooth region of the shell and reinforcing ribs on the discontinuity lines. The discontinuity lines were the points of projection of the centers of gravity of the cross-section of the corresponding $j$-th rib onto the middle surface of the shell.

To construct a difference scheme when solving equations (4) to (7), the integrated-interpolation method of constructing finite-difference schemes for hyperbolic equations was used. When numerically solving equations (4) to (7) based on approximations (9), the convergence of numerical results deteriorated. To construct a more efficient numerical algorithm for solving this problem, the approach of finding approximate solutions using the Richardson approximation (11) was used. The sequence of approximate approximations along the spatial coordinate was applied with a fixed difference step along the time coordinate. Our work shows that the order of accuracy along the $x$ coordinate, with which (11) approximates the original oscillation equations (4) in a smooth domain, is the fourth.

This paper considers the problem of unsteady oscillations of a three-layer reinforced cylindrical shell taking into account the discreteness of the placement of ribs under dynamic loading (19), and the ribs of the shell were rigidly clamped (16).

Geometric and physical and mechanical parameters took the form (18). The obtained numerical results made it possible to assess the nature of the stressed-strained state of a three-layer reinforced elastic structure of a cylindrical type at an arbitrary time point in the studied time interval.

As can be seen from the presented graphic material in Fig. 1–4, the accuracy of the obtained $u_3$ values of deflection and stress $\sigma_{22}$ increased with an increase in the value of discrete steps along the spatial coordinate x. The results are explained by the fact that the oscillation equations in the smooth region (7) and on the $i$-th rupture line (8) are a system of linear differential equations in partial derivatives with respect to variables $x$ and $t$. The presence of spatial discontinuities in the $x$ coordinate led to difficulties in obtaining a satisfactory solution on coarse grids. Thus, the practical convergence of the obtained results was proven. The results of the calculations were compared depending on the values of discrete steps in the $x$ spatial coordinate. Our calculations showed that satisfactory accuracy is achieved at $n = 80$.

The work has also confirmed the hypothesis that the use of Richardson extrapolation to find approximate solutions to partial differential equations could increase the accuracy of the solution to dynamic problems without increasing the calculation step.

The advantages of this study in comparison with similar known ones are as follows. Unlike the results reported in [10–12], the dynamic problem of vibrations of multilayer shells was solved on the basis of the finite difference method. It was this method that made it possible to study unsteady vibrations of discretely reinforced shells taking into account spatial discontinuities. The solution to the problem by the finite difference method showed that the proposed method, unlike that in study [4], contributed to the analysis of dynamic deformation of axisymmetric shells for various types of internal and external loads.

Our solutions were compared with the solutions by other authors. Comparative analysis of the data obtained in [15] gave satisfactory results, which confirmed the reliability of the obtained results. In addition, the conclusions given in [19] that the application of Richardson extrapolation to finding approximate solutions of differential equations would increase the accuracy of solving problems without increasing the calculation step are consistent with the conclusions of our study.

Given our research, it was possible to represent the results of studying the deflection $u_3$ and stresses $\sigma_{22}$ of multilayer cylindrical shells reinforced with transverse ribs under the influence of a non-stationary, normally distributed load. The greatest difficulties in solving the problem of deformation of reinforced cylindrical shells arose when considering the action of a non-stationary load on the shell and the presence of spatial discontinuities along the spatial coordinate. That is why the Reissner variational principle for dynamic processes was chosen to build a mathematical model for the equations of oscillations of a non-uniform structure. Computational difficulties arose due to the fact that the process of deformation of discretely supported cylindrical shells was described by a system of nonlinear partial differential equations. The complexity of solving such problems was the presence of discontinuity coefficients in the equations of oscillations along the $x$ coordinate.

We have proposed a method for overcoming these difficulties. The numerical algorithm for solving problems of discretely supported multilayer cylindrical shells was constructed as follows: solutions were sought in the smooth region of the cylindrical shell (7) and separately on the spatial discontinuity line (8). The solutions found separately for (7) and (8) were combined on the discontinuity line using kinematic conjugation conditions. This approach allowed us to obtain a solution for discretely supported multilayer cylindrical shells with different boundary conditions and under different unsteady loads.

The limitations in this study are attributed to the fact that explicit finite-difference schemes are conditionally stable. Therefore, for reinforced cylindrical shells, taking into account the discreteness of the placement of ribs, a study of the stability of difference equations was carried out. In addition, Richardson extrapolation had its limitations. Richardson extrapolation works qualitatively with a known order of error. It also has a high sensitivity to rounding errors and instability of calculations at small steps. Because of this, it was difficult to obtain high accuracy without increasing computational costs.

The disadvantage of this study was that increasing the accuracy of the solution of the dynamic problem led to computational difficulties. In addition, when the step was reduced, rounding errors accumulated, which led to a decrease in accuracy and complicated the practical implementation of the method proposed in our work.

Further advancement of this research is to develop improved numerical algorithms that would have better stability of problem solutions and would be able to independently select the optimal integration step. It is also necessary to devise methods that could reduce the impact of rounding errors at small steps, while maintaining computational stability. A promising area is the use of the finite difference method together with Richardson extrapolation to solve problems of oscillations of other types of shells of rotation with different geometric and physical-mechanical parameters of the structure under the action of other types of loading.

## 7. Conclusions

1. The problem of deforming multilayer cylindrical shells has been stated, taking into account the discrete placement of the ribs. When stating the problem, the geometrically nonlinear theory of shells of the Tymoshenko type was used in the quadratic approximation, applying hypotheses for the entire package as a whole. The reinforcing ribs were considered as a set of curved rods that are rigidly connected to the cylindrical shell. The theory of curved rods by Tymoshenko was used to describe the stressed-strained state of the ribs.

2. Taking into account all the conditions of the problem statement, a mathematical model was constructed for the equations of oscillations and natural boundary conditions of multilayer cylindrical shells, which are reinforced with transverse ribs based on the Reissner variational principle for dynamic processes. After standard transformations in the variational functional, the equations of oscillations of multilayer reinforced cylindrical shells were obtained taking into account the discreteness of the rib placement, as well as the kinematic conditions of the contact of the shell and discretely reinforced ribs. The resulting system of differential equations describing wave processes in inhomogeneous multilayer elastic structures with discrete ribs was a nonlinear equation

in partial derivatives in two spatial coordinates and a time coordinate. Owing to the model proposed in our work, it was possible to investigate the influence of geometric and physical-mechanical parameters of the structure on the stressed-strained state of the cylindrical shell under the action of a normal impulse load.

3. Taking into account the discreteness of the reinforcing ribs led to the presence of spatial discontinuities in the nonlinear equations of oscillations of a cylindrical shell. This necessitated the use of modern numerical methods and the development of appropriate algorithms. In particular, to construct a numerical algorithm for solving equations of the theory of multilayer shells taking into account the discreteness of the placement of ribs, the integrated-interpolation method of constructing finite-difference schemes in spatial coordinates and the explicit difference scheme in the time coordinate were used. This approach allowed us to take into account the presence of spatial discontinuities at the finite-difference level. The analysis confirmed the convergence of the computational process.

4. To increase the accuracy of the solution to dynamic problems, the Richardson extrapolation method in the spatial coordinate was used in our work. The analysis revealed that for the considered dynamics problems, fourth-order convergence was achieved.

5. A comparative analysis of deflection and stress was carried out on the basis of numerical examples. Analysis of the results revealed that according to the standard approach, the discrepancy in the deflection values for $n = 40$ and $n = 160$ reached 31%; for $n = 80$ and $n = 160$, it was about 5%. According to the Richardson approach for $n = 40 \div 80$ and the standard approach for $n = 160$, this difference was about 1%. Therefore, the use of the Richardson extrapolation method by the spatial coordinate made it possible to achieve the required accuracy on coarser difference grids by the spatial coordinate. Thus, the hypothesis put forward in our work has been confirmed. The results of the calculations confirmed the reliability of the proposed numerical method, which indicates the possibility of its further application for modeling the dynamic behavior of complex engineering structures.

**Conflicts of interest**

The authors declare that they have no conflicts of interest in relation to the current study, including financial, personal, authorship, or any other, that could affect the study, as well as the results reported in this paper.

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**Data availability**

All data are available, either in numerical or graphical form, in the main text of the manuscript.

**Use of artificial intelligence**

The authors declare that generative artificial intelligence tools were used exclusively for language editing, grammar checking, and technical formatting of the manuscript under full human control.

Artificial intelligence was not used to create, process, or interpret scientific data, form conclusions or other elements of the scientific results in the paper.

Tool used: ChatGPT (OpenAI GPT-5.1, version 2025).

The authors bear full responsibility for the content, reliability, and scientific correctness of the submitted material.

**Authors' contributions**

**Yuliia Meish:** Conceptualization, Supervision, Methodology; **Maryna Belova:** Writing – review & editing, Project administration; **Nataliia Arnauta:** Data curation, Writing – original draft; **Nataliia Maiborodina:** Software, Formal analysis; **Viacheslav Gerasymenko:** Resources, Visualization.

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