This study examines the heat exchange processes for thermally active and thermally sensitive individual nodes and elements in electronic devices that are subjected to thermal loads in the areas of canonical form. As a result of thermal loads, significant temperature gradients arise. To improve the accuracy of designing electronic devices and for their effective operation, linear and nonlinear mathematical models have been built to analyze their temperature regimes.

Based on the stated linear and nonlinear axisymmetric boundary value problems of heat conduction, their analytical and analytical-numerical solutions have been derived. Using these solutions has made it possible to establish the temperature distribution in spatial radial and axial coordinates for given geometric and thermophysical parameters (the chosen graphite has the ability to absorb a significant amount of heat at its thermal conductivity coefficient equal to 372 W/(m·degree)).

To effectively describe canonical heating regions, the theory of generalized functions has been used. A technique for linearizing nonlinear mathematical models has been introduced. As a result, linear second-order differential equations with partial derivatives and a singular right-hand side have been derived.

The numerical results reflect the temperature distribution in the medium along the radial and axial coordinates for the given geometric and thermophysical parameters. The number of divisions of the interval (0; r) was chosen to be 9, which made it possible to obtain numerical values of temperature with an accuracy of 10^{-6}. The resulting numerical values of temperature for the selected materials with a linear temperature dependence of the thermal conductivity coefficient differ from the results obtained for its constant value by 5%.*

The constructed mathematical models of heat transfer make it possible to analyze spatial isotropic media with respect to their thermal stability

Keywords: temperature field, thermal conductivity of material, thermal resistance of structures, thermally sensitive material, canonical region

UDC 536.24

DOI: 10.15587/1729-4061.2025.346809

CONSTRUCTION OF MATHEMATICAL MODELS OF HEAT EXCHANGE IN MODERN ELECTRONIC DEVICES WITH THERMAL ACTIVE ZONES OF CANONICAL FORM

Vasyl Havrysh
Doctor of Technical Sciences, Professor
Department of Software*

Svitlana Yatsyshyn
Corresponding author
PhD
Department of Software Engineering
Ukrainian National Forestry University
Gen. Chuprynky str., 103, Lviv, Ukraine, 79057
E-mail: yatsyshyn@nltu.edu.ua

Mykhailo Semerak
Doctor of Technical Sciences, Professor
Department of Heat Engineering and Thermal and Nuclear Power Plants*

Mihaylo Klymiuk
PhD, Deputy Head of the Institute of Postgraduate Education
Lviv State University of Life Safety
Kleparivska str., 12, Lviv, Ukraine, 79000

Fedir Honchar
PhD
Department of General Physics*
*Lviv Polytechnic National University
S. Bandery str., 12, Lviv, Ukraine, 79013

Received 03.10.2025
Received in revised form 28.11.2025
Accepted date 12.12.2025
Published date 23.12.2025

***How to Cite:** Havrysh, V., Yatsyshyn, S., Semerak, M., Klymiuk, M., Honchar, F. (2025). Construction of mathematical models of heat exchange in modern electronic devices with thermal active zones of canonical form. Eastern-European Journal of Enterprise Technologies, 6 (5 (138)), 24–33. https://doi.org/10.15587/1729-4061.2025.346809*

1. Introduction

With the rapid progress in digital technology, the power and functionality of modern electronic devices are constantly increasing. Components such as processors, microcontrollers, and graphics cards are becoming more powerful, which poses serious challenges in the field of managing their thermal regimes. High levels of heat generation lead to significant temperature gradients that cause unwanted overheating, reduced performance, and shortened device life. The relative influence of temperature on the reliability of microelectronic devices is the highest (55%) compared to other factors such as humidity, vibration, and dust. Thermal effects are the most important factors that affect the reliability indicators of electronic devices, in particular, the probability of failure-free operation and the mean time to failure. In [1], the mean time to failure of resistors, capacitors, integrated circuits, and semiconductor components was determined.

One of the important causes of these problems is the formation of non-uniform temperature fields in structural elements. For example, the high density of electronic components on a limited board area creates a significant difference in heat generation capacity and heat dissipation conditions. This, in turn, requires detailed analysis and optimization to enable stable and reliable operation. Effective heat dissipation is critically important because the device can overheat and fail without it. To solve this task, it is necessary to deeply understand the processes of thermal conductivity in electronic devices. Although materials with high thermal conductivity, such as copper and aluminum, are widely used, their properties can change under the influence of geometric parameters and microstructural defects.

Since experimental studies of the thermal state in individual components and elements of electronic devices are often impossible due to high temperatures and the tightness of structures, mathematical modeling plays a decisive role in this case. It is on the basis of mathematical models that describe complex thermophysical processes that it is possible to obtain reliable information about the temperature regimes of the device by performing certain computational procedures. For the practical implementation of these models and to analyze temperature regimes, modern software tools are used. These tools make it possible to visualize temperature fields in detail by numerical modeling and simulation, assess the influence of certain factors on their behavior, and devise effective cooling strategies, in particular, the optimal arrangement of components or the use of radiators. This approach makes it possible to identify potential overheating problems at the design stage, which significantly reduces the need for expensive physical experiments.

Consequently, it is a relevant task to conduct studies aimed at the development of mathematical models and software tools based on them for analyzing temperature regimes in modern electronic devices.

2. Literature review and problem statement

Analysis of current approaches to modeling thermal processes in thermosensitive materials reveals significant progress in the development of both analytical and analytical-numerical methods. In [2], the temperature field in an isotropic thermosensitive plate under the action of thermal radiation was investigated, taking into account the temperature dependence of thermophysical properties and the spatially inhomogeneous distribution of heat sources. The use of the Kirchhoff transform, the Green function, and linear spline approximation made it possible to reduce the problem to a recurrent nonlinear algebraic equation. At the same time, the model does not provide for describing localized surface and internal sources of the canonical form, which limits its applicability to problems with local temperature disturbances.

In [3], a thermal conductivity model for two thermosensitive layers with heat exchange with the environment was considered. The solution was obtained by the method of successive approximations using linearization and the integral Laplace transform. The influence of different types of boundary conditions was analyzed but the model is not suitable for describing pulsed and point heat sources, in particular those given by the Dirac delta function.

In [4], a generalized procedure for modeling thermal processes in layered materials based on a modified finite element method was proposed. The anisotropy of the material and the conditions of continuity of temperature and heat flux at the boundary surfaces of the layers were taken into account. Despite good adaptability to three-dimensional structures, the use of the model does not provide a correct description of local heating sources of the canonical form.

In [5], an algorithm based on the boundary element method was reported, designed to determine temperature fields and thermal stresses in functional-gradient micropolar composites with nonlinear properties. Although the application of the method makes it possible to take into account anisotropy and temperature dependence of parameters, it is not focused on modeling local temperature disturbances.

The thermal conductivity model using fractional time derivatives was analyzed in [6]. For a thermoelastic parallelepiped with a finite volume, the Fourier-Laplace transform was used. It was shown that the order of the fractional derivative significantly affects the formation of temperature fields. However, the model does not allow for the consideration of localized heat sources, and numerical calculations are accompanied by a significant accumulation of errors. In [7], a numerical method for solving the heat conductivity equation with a fractional spatial derivative of the Riemann-Liouville type in combination with temperature-dependent material parameters was proposed. Despite the effectiveness of the algorithm, its applicability is limited due to simple boundary conditions.

In [8], the problem of centrally symmetric heating of a body with a spherical hole by a harmonic heat flux was considered using the integral Fourier and Laplace transforms. The main drawback of the model is the lack of consideration of the temperature dependence of thermophysical parameters. Instead, in [9], a neural network model of temperature field reconstruction based on UNet and MLP is reported, which provides high prediction accuracy but requires large training samples and does not take into account thermal sensitivity and local temperature perturbations.

In [10], heat and mass transfer in Carro nanofluids with mobile microorganisms under the action of thermal radiation and activation energy was investigated. The reduction of partial derivative equations to the system of SDRs significantly simplifies the description of processes but makes it impossible to model local heat sources and temperature dependence of medium properties. Similarly, in [11], as a result of the analysis of thermal processes in the rail grinding zone, an analytical model with a non-uniform heat source was constructed, confirmed experimentally, but without the possibility of detailed reproduction of temperature gradients important for predicting surface defects.

In [12], issues of thermal management of electronic devices in transient regimes are highlighted; however, the research is mainly experimental in nature, which complicates the construction of generalized models. In [13], the PINN-TFI temperature field inversion method based on physically informed neural networks is presented; however, it is sensitive to data noise and does not take into account the thermally sensitive properties of the material. In [14], compact dynamic models for predicting the temperature of mobile device cases are described; however, the model does not provide for taking into account localized heat sources in canonical regions.

In [15], a numerical scheme is proposed for the one-dimensional problem of thermal conductivity in a three-layer body. Despite the simplicity of implementation, the use of the method does not allow for the estimation of spatial temperature gradients. In papers [16–19], models of thermal conductivity in homogeneous, segmentally homogeneous, and layered media with foreign inclusions of various geometric shapes were considered; however, in most cases, either the temperature dependence of the material properties was not taken into account, or there was no description of localized thermally active zones.

In work [20], a nonlinear model of thermal conductivity in a layer with a semi-through cylindrical inclusion was proposed using a linearizing function. However, the model does not allow for the description of internal heat sources concentrated inside a thin inclusion. Finally, paper [21] considered a three-dimensional model of heat and mass transfer in capillary-porous materials using the finite element method and parallel CUDA calculations. Despite high performance, an increase in the mesh density leads to a significant accumulation of errors, which limits the accuracy of modeling.

Our review of the literature demonstrates a significant number of approaches to modeling thermal processes – from classical analytical methods and integral transformations to fractional models, MFE methods, and deep neural networks. However, common limitations are observed in all papers, namely:

– lack of support for local surface and internal heat sources of canonical form;

– failure to take into account the temperature dependence of thermophysical characteristics in most models;

– accumulation of numerical errors when using integral transformations, fractional derivatives, and an excessively fine grid;

– dependence of neural network models on the volume and quality of training data.

The identified gaps justify the feasibility of building new analytical and numerical models that correctly describe spatial temperature gradients in thermosensitive materials, take into account localized heat sources of canonical form, and provide high accuracy. Construction of such models is of particular practical value for analyzing thermal processes in structural heat-sensitive elements of electronic devices with complex geometric shapes of heating sources.

3. The aim and objectives of the study

The purpose of our study is to build linear and nonlinear mathematical models for determining temperature fields in isotropic spatial environments with thermally active heating zones of a canonical form. As a result, it will be possible to increase the accuracy of determining the temperature distribution and to analyze temperature regimes in more depth, which will further affect the effectiveness of design methods for modern electronic devices.

To achieve this goal, it is necessary to solve the following problems:

– to construct a linear mathematical model of heat transfer in a layer due to heating by a heat flow;

– to build a nonlinear mathematical model of heat transfer in a heat-sensitive layer (thermophysical parameters of the material depend on temperature) due to heating by a heat flow;

– to construct a linear mathematical model of heat transfer in a heat-active layer (internal heating concentrated in the volume of the cylinder);

– to build a nonlinear mathematical model of heat transfer in a heat-active and heat-sensitive layer.

4. The study materials and methods

The object of the study is the process of heat transfer in isotropic spatial environments, the heating zones of which are geometric figures of canonical form.

Research hypothesis: if the temperature fields in the spatial environment are caused by heating in the regions of the canonical form, then they can be described by analytical and analytical-numerical solutions of linear and nonlinear axisymmetric boundary value problems of heat conduction. The heat conduction equations of these problems contain right-hand sides with the Dirac delta function, which makes it possible to describe the concentration of heating in such regions.

It is assumed that in the process of the study the spatial environment is such that the thermophysical parameters are invariant in spatial directions. The solutions to the boundary value problems of heat conduction, which correspond to linear and nonlinear heat transfer models, are determined, describing the temperature distribution in spatial radial and axial coordinates.

Asymmetric unit functions and the Dirac delta function are used to display the thermally active heating zones of the canonical form. This methodological approach makes it possible to adequately describe thermal processes caused by the heat flux acting on the boundary surface of the medium within a circular contour. It is also possible to reflect heating by internal heat sources uniformly distributed in the volume of the cylindrical region. As a result, axisymmetric boundary value problems with partial differential equations of the second order and the Dirac delta function in the right-hand side were obtained. To solve nonlinear axisymmetric heat transfer problems caused by the thermal sensitivity of the medium material, a special linearization procedure was proposed. Its essence is the preliminary application of the Kirchhoff transformation, which made it possible to linearize nonlinear differential equations and partially boundary conditions, obtaining their linear analogs and a quasi-linear boundary condition.

An isotropic layer is considered, referred to a cylindrical coordinate system ($Or\varphi z$), on the boundary surface $L_+ = \{(r, \varphi, h)$: $0 \le r < \infty$, $0 \le \varphi \le 2\pi\}$ of which in region $\Omega_0 = \{(R, \varphi, h)$: $0 \le \varphi \le 2\pi\}$ heating occurs by a heat flux with a specific density $q_0 = const$. On the other boundary surface of layer $L_- = \{(r, \varphi, -h)$: $0 \le r < \infty$, $0 \le \varphi \le 2\pi\}$, the conditions of convective heat exchange with the environment with a constant temperature t_c = const according to Newton's law are given (Fig. 1).

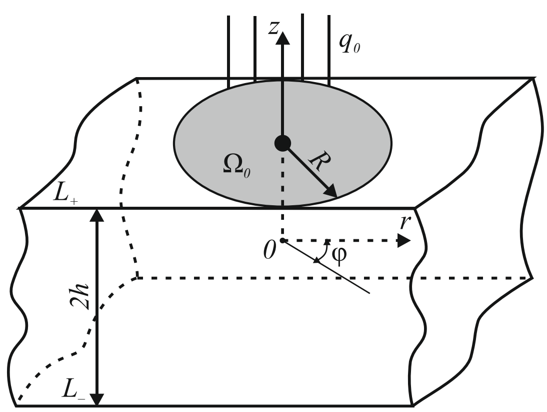


Fig. 1. Isotropic layer under the influence of heat flux

In the given medium, the temperature distribution $t(r, z)$ in spatial coordinates r and z is determined by solving the heat conduction equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right)+\frac{\partial^2\theta}{\partial z^2}=0, \quad (1)$$

under boundary conditions

$$\theta(r,z)\big|_{r\to\infty}=0, \quad \frac{\partial\theta(r,z)}{\partial r}\bigg|_{r\to\infty}=0,$$

$$\frac{\partial\theta(r,z)}{\partial z}\bigg|_{z=h}=\frac{q_0}{\lambda}S_-(R-r),$$

$$\frac{\partial\theta(r,z)}{\partial z}\bigg|_{z=-h}=-\frac{\alpha_-}{\lambda}\theta(r,z)\big|_{z=-h}, \quad (2)$$

where λ is the thermal conductivity coefficient of the layer; $\theta(r, z) = t(r, z) - t_c$; α_- is the heat transfer coefficient from the boundary surface of layer $L\_$; $S\_(\zeta)$ is the asymmetric unit function

$$S\_(\zeta)=\begin{cases}1, & \zeta\geq 0,\\ 0, & \zeta<0.\end{cases}$$

A thermosensitive layer (thermophysical parameters depend on temperature) is considered (Fig. 1).

In the given medium, the temperature field $t(r, z)$ in the spatial coordinates r and z is determined by solving the nonlinear heat conduction equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\lambda(t)\frac{\partial t}{\partial r}\right]+\frac{\partial}{\partial z}\left[\lambda(t)\frac{\partial t}{\partial z}\right]=0, \quad (3)$$

under boundary conditions

$$t(r,z)\big|_{r\to\infty}=0, \quad \frac{\partial t(r,z)}{\partial r}\bigg|_{r\to\infty}=0,$$

$$\lambda(t)\frac{\partial t(r,z)}{\partial z}\bigg|_{z=h}=q_0S_-(R-r), \quad \frac{\partial t(r,z)}{\partial z}\bigg|_{z=-h}=0, \quad (4)$$

where $\lambda(t)$ is the thermal conductivity coefficient of the thermosensitive layer.

An isotropic layer is considered, referred to a cylindrical coordinate system ($Or\varphi z$), on the boundary surface $L_+ = \{(r,\varphi,h): 0\leq r<\infty, 0\leq\varphi\leq 2\pi\}$ of which convective heat exchange with the environment with a constant temperature tc occurs according to Newton's law. The other surface of layer $L_- = \{(r,\varphi,-h): 0\leq r<\infty, 0\leq\varphi\leq 2\pi\}$ is thermally insulated (Fig. 2).

In the given medium, the temperature distribution $t(r, z)$ in spatial coordinates r and z is determined by solving the heat conduction equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right)+\frac{\partial^2\theta}{\partial z^2}=-\frac{q_0}{\lambda}S_-(R-r)\delta(z), \quad (5)$$

under boundary conditions

$$\theta(r,z)\big|_{r\to\infty}|=0, \quad \frac{\partial\theta(r,z)}{\partial r}\bigg|_{r\to\infty}=0,$$

$$\frac{\partial\theta(r,z)}{\partial z}\bigg|_{z=h}=\frac{\alpha_+}{\lambda}\theta(r,z)\big|_{z=h}, \quad \frac{\partial\theta(r,z)}{\partial z}\bigg|_{z=-h}=0. \quad (6)$$

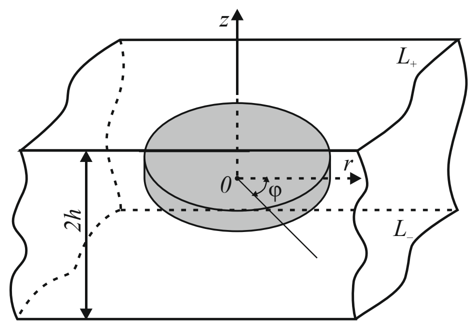


Fig. 2. Isotropic layer under the influence of internal heating

Here $\delta(\zeta)$ is the Dirac delta function; α_+ is the heat transfer coefficient from surface L_+;

$S(\zeta)$ is the symmetric unit function

$$\delta(æ)=\frac{dS(\zeta)}{d\zeta}, \quad S(\zeta)=\begin{cases}1, \zeta>0,\\ 0,5, \zeta=0,\\ 0, \zeta<0.\end{cases}$$

A thermosensitive layer is considered that is isotropic with respect to thermophysical parameters (Fig. 2).

In the given thermosensitive medium, a nonlinear heat conduction equation is considered to determine the temperature field $t(r, z)$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\lambda(t)\frac{\partial t}{\partial r}\right]+\frac{\partial}{\partial z}\left[\lambda(t)\frac{\partial t}{\partial z}\right]=-q_0S_-(R-r)\delta(z), \quad (7)$$

under boundary conditions

$$t(r,z)\big|_{r\to\infty}|=0, \quad \frac{\partial t(r,z)}{\partial r}\bigg|_{r\to\infty}=0,$$

$$\lambda(t)\frac{\partial t(r,z)}{\partial z}\bigg|_{z=h}=\alpha_+\left(t(r,z)\big|_{z=h}-t_c\right), \quad \frac{\partial t(r,z)}{\partial z}\bigg|_{z=-h}=0. \quad (8)$$

Equation (7) and boundary conditions (8) completely determine the temperature distribution in the medium in spatial coordinates r and z.

5. Results of research on mathematical models of heat transfer in media with heat-active elements of canonical form

5. 1. Linear mathematical model of heat transfer in a layer due to heating by a heat flow

The Henkel integral transformation in coordinate r is applied to equation (1) and boundary conditions (2). As a result, an ordinary homogeneous second-order differential equation with constant coefficients is obtained

$$\frac{d^2\bar{\theta}}{dz^2}-\hat{\imath}^2\bar{\theta}=0, \quad (9)$$

under boundary conditions

$$\frac{d\bar{\theta}(z)}{dz}\bigg|_{z=h}=\frac{q_0R}{\lambda\xi}J_1(R\xi), \quad \frac{d\bar{\theta}(z)}{dz}\bigg|_{z=-h}=-\frac{\alpha_-}{\lambda}\bar{\theta}(z)\big|_{z=-h}, \quad (10)$$

where $\bar{\theta}(z)$ is the transformant of function $\theta(r,z)$

$$\bar{\theta}(z)=\int_0^{\infty} rJ_0(r\xi)\theta(r,z)\mathrm{d}r;$$

$$J_\nu(x)=\sum_{n=0}^{\infty}(-1)^n\frac{(x/2)^{\nu+2n}}{n!(\nu+n)!},$$

– first-kind Bessel function of ν-th order;

ξ is the parameter of the Henkel integral transformation.

The general solution to the ordinary homogeneous differential equation (9) will be the following expression

$$\bar{\theta}(z)=c_1e^{\xi z}+c_2e^{-\xi z}, \tag{11}$$

in which integration constants c_1 and c_2 are determined using boundary conditions (10). As a result, a partial solution to problem (9), (10) is obtained

$$\bar{\theta}(z)=\frac{q_0RJ_1(R\xi)}{\lambda\xi^2P(\xi)}\left[\lambda\xi ch\xi(z+h)-\alpha_{\_}sh\xi(z+h)\right], \tag{12}$$

where $P(\xi)=\lambda\xi sh2\xi h+\alpha_{\_}ch2\xi h$.

The inverse Henkel integral transformation was applied to relation (12), which made it possible to determine the desired solution to the boundary value problem (1), (2), which is given by the following expression

$$\theta(r,z)=\frac{Rq_0}{\lambda}\int_0^{\infty}\frac{J_0(r\xi)}{\xi P(\xi)}J_1(R\xi)\begin{bmatrix}\xi ch\xi(z+h)-\\-\alpha_{\_}sh\xi(z+h)\end{bmatrix}\mathrm{d}\xi. \tag{13}$$

As a result, the temperature field in the layer, caused by heating by a heat flux concentrated in a circle on the boundary surface, is expressed by formula (13), from which the temperature value at any point of it can be derived.

According to formula (13), temperature field $t(r, z)$ in the given medium was calculated and its behavior was depicted depending on the spatial axial z (Fig. 3, *a*) and radial r (Fig. 3, *b*) coordinates for the following initial data: $q_0 = 200$ W/m^2; $h = 0.1$ m; $R = 0.05$ m, $\alpha_{\_} = 0$. The composite material ($\lambda = 0.840$ W/(degree·m)) was chosen as the layer material.

The results show that temperature $t(r, z)$, as a function of spatial coordinates, is smooth and monotonic, which confirms the correctness of our mathematical model. Numerical calculations were performed with an accuracy of 10^{-6}.

5. 2. Nonlinear mathematical model of heat transfer in a layer due to heating by a heat flow

To linearize the boundary value problem (3), (4), the Kirchhoff transformation was used

$$\vartheta(r,z)=\frac{1}{\lambda^0}\int_0^{t(r,z)}\lambda(\zeta)\mathrm{d}\zeta. \tag{14}$$

Here λ^0 is the reference coefficient of thermal conductivity of the layer material.

As a result of differentiating expression (14) with respect to variables r and z, the following relation is obtained

$$\lambda^0\frac{\partial\vartheta(r,z)}{\partial r}=\lambda(t)\frac{\partial t(r,z)}{\partial r},$$

$$\lambda^0\frac{\partial\vartheta(r,z)}{\partial z}=\lambda(t)\frac{\partial t(r,z)}{\partial z}, \tag{15}$$

taking into account which the original equation (3) and boundary conditions (4) are transformed to the following form:

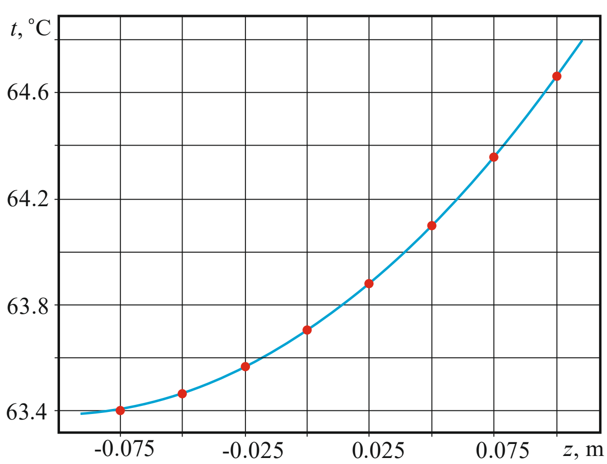
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\vartheta}{\partial r}\right)+\frac{\partial^2\vartheta}{\partial z^2}=0, \tag{16}$$

$$\vartheta(r,z)\big|_{r\to\infty}=0,$$

$$\left.\frac{\partial\vartheta(r,z)}{\partial r}\right|_{r\to\infty}=0,\quad \left.\frac{\partial\vartheta(r,z)}{\partial z}\right|_{z=-h}=0,$$

$$\left.\frac{\partial\vartheta(r,z)}{\partial z}\right|_{z=h}=\frac{q_0}{\lambda^0}S_{\_}(R-r). \tag{17}$$

As a result of the transformations, a linear homogeneous differential equation with partial derivatives of the second order with respect to function $\vartheta(r, z)$ (16) and boundary conditions (17) were obtained.



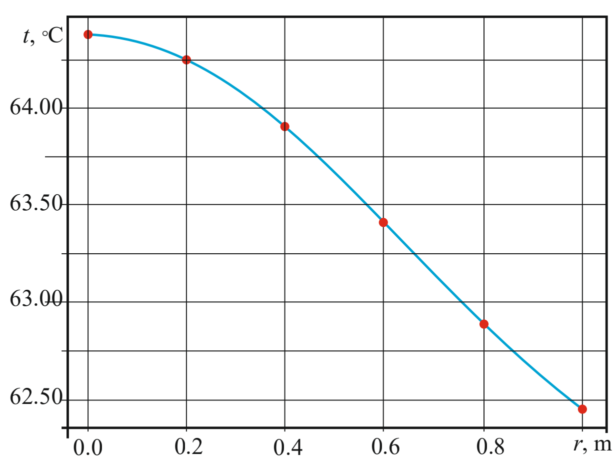


Fig. 3. Dependence of temperature $t(r, z)$ on spatial coordinates: *a* — axial coordinate z for $r = R$; *b* — radial coordinate r for $z = h$

The Henkel integral transformation was applied to equation (16) and boundary conditions (17) with respect to radial coordinate *r*. As a result, an ordinary homogeneous differential equation of the second order with constant coefficients was built

$$\frac{d^2\bar{\vartheta}}{dy^2}-\xi^2\bar{\vartheta}=0, \qquad (18)$$

under boundary conditions

$$\left.\frac{d\bar{\vartheta}(z)}{dz}\right|_{z=-h}=0,$$

$$\left.\frac{d\bar{\vartheta}(z)}{dz}\right|_{z=h}=\frac{Rq_0}{\lambda^0\xi}J_1(R\xi), \qquad (19)$$

where $\bar{\vartheta}(z)=\int_{-\infty}^{\infty} rJ_0(r\xi)\vartheta(r,z)\mathrm{d}r$ is the transformant of function ϑ (r, z).

The general solution to equation (18) is defined in the form of (11).

The use of boundary conditions (19) made it possible to obtain a partial solution to problem (18), (19) as a result of determining the constants of integration c_1 and c_2

$$\bar{\vartheta}(z)=\frac{Rq_0}{\lambda^0\xi^2}\frac{ch\xi(z+h)}{sh2\xi h}J_1(R\xi). \qquad (20)$$

The inverse Henkel integral transformation is applied to relation (20) and on this basis the expression for the Kirchhoff function $\vartheta(r, z)$ is determined in the following form

$$\vartheta(r,z)=\int_0^{\infty}\xi J_0(r\xi)\bar{\vartheta}(z)d\xi. \qquad (21)$$

The desired temperature field t (r, z) for the given medium is determined by solving a nonlinear algebraic equation obtained from the ratio of the temperature dependence of the thermal conductivity coefficient of the structural material using relations (14), (21).

The temperature distribution t $(r; h)$ (Table 1) and t $(R; z)$ (Table 2) was calculated in spatial coordinates r, z in the composite layer for a linearly varying thermal conductivity coefficient.

Table 1

Temperature change depending on spatial radial coordinate r (for $z = h$)

r, m	0.00	0.02	0.04	0.06	0.08	0.10
t, °C	68.85	68.81	68.32	67.81	67.26	66.81

Table 2

Temperature change depending on spatial radial coordinate z (for $r = R$)

z, m	–0.10	–0.075	–0.05	–0.025	0.00	0.025	0.05	0.075	0.10
t, °C	67.82	67.91	67.86	68.08	68.25	68.32	68.57	68.94	69.26

The following input data values were selected: $q_0 = 200$ W/m^2; $h = 0.1$ m; $R = 0.05$ m. Numerical calculations were performed with an accuracy of 10^{-6}.

5. 3. Linear mathematical model of heat transfer in a thermally active layer

The Henkel integral transformation in coordinate *r* was applied to equation (5) and boundary conditions (6). As a result, an ordinary inhomogeneous differential equation of the second order with constant coefficients and a singular right-hand side was obtained

$$\frac{d^2\bar{\theta}}{dz^2}-\hat{\imath}^2\bar{\theta}=-\frac{Rq_0}{\lambda\xi}J_1(R\xi)\delta(z), \qquad (22)$$

under boundary conditions

$$\left.\frac{d\bar{\theta}(z)}{dz}\right|_{z=-h}=0,\quad \left.\frac{d\bar{\theta}(z)}{dz}\right|_{z=h}=\frac{\alpha_+}{\lambda}\bar{\theta}(z)\Big|_{z=h}. \qquad (23)$$

The general solution to the ordinary homogeneous differential equation (22) will be expression (11), in which the integration constants c_1 and c_2 are determined using boundary conditions (23). As a result, a partial solution to problem (25), (26) is obtained

$$\bar{\theta}(z)=\frac{Rq_0}{\xi^2}J_1(R\xi)\times$$
$$\times\left[\frac{ch\xi(z+h)}{P(\xi)}\left(ch\xi h-\frac{\alpha_+ sh\xi h}{\lambda\xi}\right)-\frac{sh\xi z}{\lambda\xi}S(z)\right], \qquad (24)$$

where $P(\xi) = \lambda\xi sh2\xi h - \alpha_+ ch2\xi h$.

The inverse Henkel integral transformation was applied to relation (24), which made it possible to determine the desired solution to the boundary value problem (5), (6), which is given by the following expression

$$\theta(r,z)=$$
$$=Rq_0\int_0^{\infty}\frac{J_0(r\xi)}{\xi}J_1(R\xi)\left[\begin{array}{l}\dfrac{ch\xi(z+h)}{P(\xi)}\left(ch\xi h-\dfrac{\alpha_+ sh\xi h}{\lambda\xi}\right)-\\ -\dfrac{sh\xi z}{\lambda\xi}S(z)\end{array}\right]d\xi. \qquad (25)$$

As a result, the temperature field in the layer, caused by heating by an internal heat source concentrated in a thin cylinder, is expressed by formula (25), from which it is possible to obtain the temperature value at any point of it.

Due to the operation of electronic devices, high-precision temperature control is required. Overheating reduces their performance and can cause damage. With a minimal change in temperature, graphite has the ability to absorb a significant amount of heat. As a result, it is used in electronic cooling systems, which enables stable operation of the device and significantly reduces the risks associated with excessive heating.

According to formula (25), the temperature distribution θ $(R; z)$ (Fig. 4) was calculated along the spatial axial coordinate z. Graphite was chosen as the medium material. As a result of heating, its expansion is insignificant. Graphite has high thermal conductivity (the thermal conductivity coefficient is 372 W/(m·degree), which is approximately twice as high as for tungsten alloys) and is therefore resistant to thermal loads. The operation of electronic devices requires high-precision temperature control. Overheating reduces their performance and can cause damage. For a minimal change in temperature, graphite has the ability to absorb a significant amount of heat. As a result, it is used in electronic cooling systems, which enables stable operation of the device and significantly reduces the risks associated with excessive heating.

The following input values were selected: $q_0 = 200$ W/m^3; $h = 0.1$ m; $R = 0.05$ m; $\alpha_+ = 17.64$ W/(m^2·degree). Numerical calculations were performed with an accuracy of 10^{-6}.

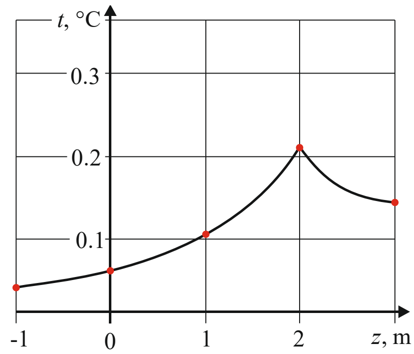


Fig. 4. Dependence of temperature $t(r, z)$ on spatial axial coordinate z for $r = R$ in an isotropic layer due to internal heating

The shape of the curve shows that the temperature distribution as a function of the spatial coordinate is smooth and monotonic, and the maximum values are observed in the area of internal heat sources.

5. 4. Nonlinear mathematical model of heat transfer in the thermally active layer

To linearize the boundary value problem (7), (8), the Kirchhoff transformation (14) was used and, taking into account expressions (15), it was transformed to the following form:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \vartheta}{\partial r}\right)+\frac{\partial^2 \vartheta}{\partial z^2}=-\frac{q_0}{\lambda^0}S_-(R-r)\delta(z), \quad (26)$$

$$\vartheta(r,z)\big|_{r\to\infty}=0, \quad \frac{\partial \vartheta(r,z)}{\partial r}\bigg|_{r\to\infty}=0, \quad \frac{\partial \vartheta(r,z)}{\partial z}\bigg|_{z=-h}=0, \quad (27)$$

$$\frac{\partial \vartheta(r,z)}{\partial z}\bigg|_{z=h}=\frac{\alpha_+}{\lambda^0}\theta(r,z)\big|_{z=h}. \quad (28)$$

As a result of the transformations, a linear inhomogeneous partial differential equation of the second order with a singular right-hand side with respect to function $\vartheta(r, z)$ (26), boundary conditions (27), and a quasilinear boundary condition (28) are obtained.

Temperature $\theta(r, h)$ is approximated as a function of the spatial radial coordinate r by a segment-constant function in the form

$$\theta(r,h)=\theta_1+\sum_{i=1}^{n-1}(\theta_{i+1}-\theta_i)S_-(r-r_i), \quad (29)$$

where $r_i \in (0; r^*)$; $r_1 \le r_2 \le \ldots \le r_{n-1}$ θ_i $(i \in (1, n))$ are unknown approximation values of temperature $\theta(x,h)$; n is the number of partitions of interval $(0; r^*)$; r^* is the value of the radial coordinate for which the temperature reaches value t_c (it is found from the corresponding linear problem).

To equation (26) and boundary conditions (27), (28) taking into account the relation (29), the Henkel integral transformation is applied in radial coordinate r. As a result, an ordinary inhomogeneous differential equation of the second order with constant coefficients and a singular right-hand side is obtained

$$\frac{d^2\bar{\vartheta}}{dz^2}-\xi^2\bar{\vartheta}=-\frac{Rq_0}{\lambda^0\xi}J_1(R\xi)\delta(z), \quad (30)$$

under boundary conditions

$$\frac{d\bar{\vartheta}(z)}{dz}\bigg|_{z=-h}=0, \quad \frac{d\bar{\vartheta}(z)}{dz}\bigg|_{z=h}=\frac{\alpha_+ A(\xi)}{\lambda^0\xi}. \quad (31)$$

Here

$$A(\xi)=(t_n-t_c)\delta_+(\xi)-\sum_{i=1}^{n-1}r_iJ_1(r_i\xi)(t_{i+1}-t_i),$$

$\delta_+(\zeta)=\frac{dS_+(\zeta)}{d\zeta}$ is the asymmetric Dirac delta function; $S_+(\zeta)$ – asymmetric unit function;

$$S_+(\zeta)=\begin{cases}1, & \zeta>0,\\ 0, & \zeta\le 0.\end{cases}$$

The general solution to the homogeneous equation (30) is defined in the form of (11).

The use of boundary conditions (31) made it possible to obtain a partial solution to problem (30), (31) as a result of determining the constants of integration c_1 and c_2:

$$\bar{\vartheta}(z)=\frac{1}{\lambda^0\xi^2}\left\{\begin{array}{l}\frac{Rq_0}{\xi}J_1(R\xi)\times\\ \times\left[\frac{ch\xi(z+h)}{sh2\xi h}ch\xi h-sh\xi zS(z)\right]+\\ +\frac{\alpha_+A(\xi)ch\xi(z+h)}{sh2\xi h}\end{array}\right\}. \quad (32)$$

The inverse Henkel integral transformation is applied to relation (32) and on this basis the expression for the Kirchhoff function $\vartheta(r, z)$ is determined in the following form

$$\vartheta(r,z)=\int_0^\infty \xi J_0(r\xi)\bar{\vartheta}(z)d\xi. \quad (33)$$

The desired temperature field $t(r, z)$ for the given medium is determined by solving a nonlinear algebraic equation obtained from the ratio of the temperature dependence of the thermal conductivity coefficient of the structural material using relations (14), (33).

The temperature distribution $\theta(R; z)$ (Table 3) was calculated along the spatial axial coordinate z in the graphite layer for a linearly varying thermal conductivity coefficient.

The following input data values were selected: $q_0 = 200$ W/m^3; $h = 0.1$ m; $R = 0.05$ m; $\alpha_+ = 17.64$ W/(m^2·degree). Numerical calculations were performed with an accuracy of 10^{-6} for the number of divisions of interval $(0; r^*)$ $n = 9$.

The results obtained for the linear temperature dependence of the thermal conductivity coefficient differ from the results obtained for the constant thermal conductivity coefficient of the composite by 5% (Tables 1, 2, Fig. 3), and for graphite by 7% (Table 3, Fig. 4).

Table 3

Temperature change depending on spatial axial coordinate z'

z, m	–1.0	0.0	1.0	2.0	3.0
t, °C	0.0473	0.0711	0.1192	0.2252	0.1609

6. Construction of mathematical models of heat transfer in media with thermally active zones of canonical form: results and summary

The boundary value problems of heat conduction have been stated in accordance with the physical nature of the processes occurring in the considered media. As a result, the heat transfer process is described by the equations of mathematical physics and boundary conditions, in the right-hand side of which the Heaviside function and its derivative appear. The nature of the temperature curves in Fig. 3, 4, constructed according to the obtained numerical values of the temperature curve based on analytical solutions (13), (25), confirms the correctness of the results. This is evidenced by the smooth behavior of the temperature field and compliance with the specified boundary conditions at the boundaries of the medium.

In our studies, the apparatus of generalized functions was used, which allowed us to correctly describe the thermally active zones of the canonical form. As a result, the obtained linear and nonlinear heat conduction equations contain the Dirac delta function in the right-hand side. A technique has been proposed that made it possible to reduce nonlinear boundary value problems (3), (4) and (7), (8) to linear ones and derive analytical solutions (21) and analytical-numerical solutions (33). The temperature distribution is determined by relations (13), (21), (25), (33) and is illustrated in Fig. 3, 4 and in Tables 1–3.

Previous work was analyzed; it was found that the process of heating the medium in regions with small geometric parameters, in which surface and internal heat sources are concentrated, was not considered. This is important since the heating of modern electronic devices is concentrated in local regions due to their miniaturization, in contrast to [2], where an isotropic plate is considered, and [3] for a two-layer medium, heating is concentrated on the entire surface. Using the Kirchhoff transformation, the nonlinear heat conduction equation and partially the boundary conditions (8) were linearized. In view of this, for full linearization, a segment-constant description of the temperature by the spatial coordinate at the layer boundary was introduced according to function (29). This approach enables the minimization of the calculation error, which could not be achieved in [4, 5, 10, 15] because of the use of purely numerical methods. Using the apparatus of generalized functions, the thermally active canonical heating zones were correctly and effectively displayed, which, in turn, made it possible to obtain analytical and analytical-numerical solutions to the heat conduction equations, the right-hand sides of which are singular.

The architecture of modern electronic devices is characterized by the concentration of individual heat-active nodes in the heating regions of the canonical form. As a result, there is a need to build mathematical models of heat transfer between nodes and their individual elements. These models can have a linear or nonlinear form for isotropic spatial environments. Although our mathematical models of heat transfer are simplified, they serve as a reliable basis for the further construction of more complex models suitable for describing heat transfer processes in spatial composite environments.

Based on the obtained analytical and analytical-numerical solutions of linear and nonlinear boundary value problems of heat transfer, the feasibility of developing computational algorithms and software for their numerical implementation has been substantiated. This will make it possible to conduct research for a number of materials used in the design of digital electronic devices regarding the influence of their thermal sensitivity on the temperature distribution.

Based on the research, it is necessary to take into account the temperature dependence of the properties of structural materials for a more accurate analysis of thermal regimes in nodes and their individual electronic devices. This significantly complicates the determination of solutions to linear and nonlinear boundary value problems of thermal conductivity. In contrast, our solutions to these problems reproduce the behavior of the temperature field as a function of spatial coordinates more adequately and closer to the real physical process.

Our research has considered a stationary process of thermal conductivity, which limits the mathematical models built that reflect the change in temperature in spatial coordinates. The use of boundary conditions of the first, second, and third kind on the boundary surfaces of the media should be considered a disadvantage.

In future studies, mathematical models of heat transfer, both linear and nonlinear, will be more complex due to spatial media, taking into account composite structural materials and their anisotropy.

7. Conclusions

1. A linear mathematical model of heat transfer between individual elements of structural units of electronic devices due to heating by a heat flux concentrated in a circle at the edge of the medium has been built. An analytical solution to the boundary value problem in the form of an improper integral (the upper limit of the integral contains infinity) has been obtained. After certain mathematical transformations, it was reduced to an integral with finite limits. As a result of using the 3/8 Newton method of numerical integration to determine the temperature distribution in spatial coordinates in the medium, the accuracy of the results has been achieved to 10^{-6}. Such accuracy is difficult to achieve using numerical methods for solving the original boundary value problem or experimental measurements. Due to the concentration of thermal heating in a circle at the boundary surface, it is effectively described using asymmetric unit functions. As a result, a sufficiently high accuracy of determining the temperature field has been achieved.

2. A nonlinear mathematical model of heat transfer between individual thermally sensitive elements of structural units of electronic devices due to heating by a heat flux concentrated in a circle at the edge of the medium has been constructed. A technique for linearizing the nonlinear boundary value problem has been introduced and on this basis an analytical solution has been obtained for the linear temperature dependence of the thermal conductivi-

ty coefficient of the medium material. A numerical experiment has been performed, as a result of which the behavior of temperature as a function of spatial coordinates was displayed. The results obtained for the selected material with a linear temperature dependence of the thermal conductivity coefficient differ from the results obtained for its constant value by 5%.

3. A linear mathematical model of heat transfer between individual elements of structural units of electronic devices with internal heating concentrated in the volume of a thin cylinder has been built. An analytical solution to the boundary value problem has been obtained and on this basis, using numerical integration of the improper integral, numerical values of temperature for selected values of thermophysical and geometric parameters with an accuracy of 10^{-6} have been given.

4. A nonlinear mathematical model of heat transfer between individual heat-sensitive elements of structural units of electronic devices with internal heating concentrated in the volume of a thin cylinder has been constructed. A technique for linearizing the nonlinear boundary value problem has been introduced and on this basis an analytical-numerical solution has been obtained for the linear temperature dependence of the thermal conductivity coefficient of the medium material. This solution made it possible to build a system of nonlinear algebraic equations under an automated mode to determine the unknown values of the temperature at the edge of the medium, the coefficients of which contain improper integrals. The coefficients were determined by numerical integration, and the solution to the system was obtained by Newton's method with an accuracy of 10^{-6}, after which the numerical values of the temperature were determined.

Conflicts of interest

The authors declare that they have no conflicts of interest in relation to the current study, including financial, personal, authorship, or any other, that could affect the study, as well as the results reported in this paper.

Funding

The study was conducted without financial support.

Data availability

All data are available, either in numerical or graphical form, in the main text of the manuscript.

Use of artificial intelligence

The authors confirm that they did not use artificial intelligence technologies when creating the current work.

Authors' contributions

Vasyl Havrysh: Conceptualization, Methodology, Formal analysis, Writing – original draft; **Svitlana Yatsyshyn**: Software, Visualization, Writing – review & editing; **Mykhailo Semerak**: Investigation, Resources, Validation; **Mykhailo Klymiuk**: Investigation, Resources, Validation; **Fedir Honchar**: Conceptualization, Methodology.

References

1. Nikitchuk, A. V. (2024). Impact of Electronic Components Thermal Resilience on the Reliability of Radio-Electronic Equipment. Visnyk NTUU KPI Seriia - Radiotekhnika Radioaparatobuduvannia, 98, 38–45. https://doi.org/10.20535/radap.2024.98.38-45
2. Protsiuk, B. V. (2023). Nonstationary Problems of Heat Conduction for a Thermosensitive Plate with Nonlinear Boundary Condition on One Surface. Journal of Mathematical Sciences, 272 (1), 135–150. https://doi.org/10.1007/s10958-023-06405-1
3. Vovk, O. M. (2025). Thermal State of Two Contacting Thermosensitive Layers Under Complex Heat Exchange. Journal of Mathematical Sciences, 287 (2), 334–345. https://doi.org/10.1007/s10958-025-07594-7
4. Kozub, H. O., Kozub, Yu. H. (2020). Modeling of thermal processes in layered bodies. Geo-Technical Mechanics, 151, 234–244. https://doi.org/10.15407/geotm2020.151.234
5. Fahmy, M. A. (2021). A new boundary element algorithm for modeling and simulation of nonlinear thermal stresses in micropolar FGA composites with temperature-dependent properties. Advanced Modeling and Simulation in Engineering Sciences, 8 (1). https://doi.org/10.1186/s40323-021-00193-6
6. Srinivas, V. B., Manthena, V. R., Bikram, J., Kedar, G. D. (2021). Fractional order heat conduction and thermoelastic response of a thermally sensitive rectangular parallelopiped. International Journal of Thermodynamics, 24 (1), 62–73. https://doi.org/10.5541/ijot.849663
7. Brociek, R., Hetmaniok, E., Słota, D. (2024). Numerical Solution for the Heat Conduction Model with a Fractional Derivative and Temperature-Dependent Parameters. Symmetry, 16 (6), 667. https://doi.org/10.3390/sym16060667
8. Povstenko, Y., Kyrylych, T., Woźna-Szcześniak, B., Yatsko, A. (2024). Fractional Heat Conduction with Heat Absorption in a Solid with a Spherical Cavity under Time-Harmonic Heat Flux. Applied Sciences, 14 (4), 1627. https://doi.org/10.3390/app14041627
9. Peng, X., Li, X., Gong, Z., Zhao, X., Yao, W. (2022). A deep learning method based on partition modeling for reconstructing temperature field. International Journal of Thermal Sciences, 182, 107802. https://doi.org/10.1016/j.ijthermalsci.2022.107802
10. Basit, M. A., Imran, M., Mohammed, W. W., Ali, M. R., Hendy, A. S. (2024). Thermal analysis of mathematical model of heat and mass transfer through bioconvective Carreau nanofluid flow over an inclined stretchable cylinder. Case Studies in Thermal Engineering, 63, 105303. https://doi.org/10.1016/j.csite.2024.105303
11. Zhou, K., Ding, H., Steenbergen, M., Wang, W., Guo, J., Liu, Q. (2021). Temperature field and material response as a function of rail grinding parameters. International Journal of Heat and Mass Transfer, 175, 121366. https://doi.org/10.1016/j.ijheatmasstransfer.2021.121366

12. Mathew, J., Krishnan, S. (2021). A Review On Transient Thermal Management of Electronic Devices. Journal of Electronic Packaging. https://doi.org/10.1115/1.4050002
13. Liu, X., Peng, W., Gong, Z., Zhou, W., Yao, W. (2022). Temperature field inversion of heat-source systems via physics-informed neural networks. Engineering Applications of Artificial Intelligence, 113, 104902. https://doi.org/10.1016/j.engappai.2022.104902
14. Liu, H., Yu, J., Wang, R. (2023). Dynamic compact thermal models for skin temperature prediction of portable electronic devices based on convolution and fitting methods. International Journal of Heat and Mass Transfer, 210, 124170. https://doi.org/10.1016/j.ijheatmasstransfer.2023.124170
15. Coronel, A., Lozada, E., Berres, S., Huancas, F., Murúa, N. (2024). Mathematical Modeling and Numerical Approximation of Heat Conduction in Three-Phase-Lag Solid. Energies, 17 (11), 2497. https://doi.org/10.3390/en17112497
16. Havrysh, V., Ovchar, I., Baranetskyj, J., Pelekh, J., Serduik, P. (2017). Development and analysis of mathematical models for the process of thermal conductivity for piecewise uniform elements of electronic systems. Eastern-European Journal of Enterprise Technologies, 1 (5 (85)), 23–33. https://doi.org/10.15587/1729-4061.2017.92551
17. Havrysh, V., Ivasyk, H., Kolyasa, L., Ovchar, I., Pelekh, Y., Bilas, O. (2017). Examining the temperature fields in flat piecewise- uniform structures. Eastern-European Journal of Enterprise Technologies, 2 (5 (86)), 23–32. https://doi.org/10.15587/1729-4061.2017.97272
18. Havrysh, V. I., Kosach, A. I. (2012). Boundary-value problem of heat conduction for a piecewise homogeneous layer with foreign inclusion. Materials Science, 47 (6), 773–782. https://doi.org/10.1007/s11003-012-9455-4
19. Gavrysh, V., Tushnytskyy, R., Pelekh, Y., Pukach, P., Baranetskyi, Y. (2017). Mathematical model of thermal conductivity for piecewise homogeneous elements of electronic systems. 2017 14th International Conference The Experience of Designing and Application of CAD Systems in Microelectronics (CADSM), 333–336. https://doi.org/10.1109/cadsm.2017.7916146
20. Havrysh, V., Kochan, V. (2023). Mathematical Models to Determine Temperature Fields in Heterogeneous Elements of Digital Devices with Thermal Sensitivity Taken into Account. 2023 IEEE 12th International Conference on Intelligent Data Acquisition and Advanced Computing Systems: Technology and Applications (IDAACS), 983–991. https://doi.org/10.1109/idaacs58523.2023.10348875
21. Sokolovskyy, Y., Nechepurenko, A., Samotii, T., Yatsyshyn, S., Mokrytska, O., Yarkun, V. (2020). Software and Algorithmic Support for Finite Element Analysis of Spatial Heat-and-Moisture Transfer in Anisotropic Capillary-Porous Materials. 2020 IEEE Third International Conference on Data Stream Mining & Processing (DSMP), 316–320. https://doi.org/10.1109/dsmp47368.2020.9204175