

This study considers the construction of tubular surfaces with a spatial axis of slope, referred to a coordinate grid of curvature lines. Such surfaces have a number of mathematical advantages compared to surfaces described by arbitrary coordinate grids. In differential geometry, this has a theoretical justification and applied value. This follows from the special role of curvature lines as geometrically privileged directions on a surface with minimal and maximal curvatures.

To parameterize a tubular surface in this way, it is necessary that the length of its axis be described by analytical dependences in a finite form. Typically, the length of spatial curves is determined by numerical integration. There is a known group of plane curves that are described by parametric equations as a function of the arc length and for which such a problem does not exist. This work proposes taking such curves as a horizontal projection of a spatial curve. The spatial curve should be constructed as a slope curve with a constant elevation angle relative to the horizontal plane. Then the spatial curve, the equations of which include the elevation angle, will be described as a function of the arc length. Its use as the axis of the tubular surface makes it possible to attribute the latter to the families of coordinate lines of curvature.

In this paper, the horizontal projection of the axis of the tubular surface is a logarithmic spiral. Parametric equations of the tubular surface in analytical form have been derived. A surface with the elevation angle of the axis  $\beta = 10^\circ$  and the radius of the generating circle  $\rho = 15$  linear units was constructed. The orthogonality of the resulting coordinate grid has been proven through the analysis of the coefficients of the first quadratic form ( $F = 0$ ), which confirms the assignment of the surface to the lines of curvature. This makes it possible to improve the accuracy of calculating the stress-strain state of shells in mechanical engineering and aerospace engineering at simultaneous minimization of computational costs

**Keywords:** curvature lines, slope curve, arc length, Frenet trihedron, orthogonal grid

# DEVISING A TECHNIQUE FOR CONSTRUCTING TUBULAR SURFACES REFERRED TO A COORDINATE GRID OF LINES OF CURVATURE

**Andrii Nesvidomin**

PhD, Associate Professor\*

ORCID: <https://orcid.org/0000-0002-9227-4652>

**Serhii Pylypaka**

Doctor of Technical Sciences, Professor, Head of Department\*

ORCID: <https://orcid.org/0000-0002-1496-4615>

**Tetiana Volina**

Corresponding author

Doctor of Technical Sciences, Associate Professor\*

E-mail: [volina@nubip.edu.ua](mailto:volina@nubip.edu.ua)

ORCID: <https://orcid.org/0000-0001-8610-2208>

**Victor Nesvidomin**

Doctor of Technical Sciences, Professor\*

ORCID: <https://orcid.org/0000-0002-1495-1718>

**Oleksandr Solarov**

PhD, Senior Researcher

Laboratory of Mechanical and Chemical Impact in Agrotechnologies\*\*

ORCID: <https://orcid.org/0000-0002-1485-0685>

**Taras Voloshko**

Senior Lecturer

Department of Transport Technologies\*\*

ORCID: <https://orcid.org/0000-0003-2605-8836>

**Taras Pylypaka**

PhD, Associate Professor

Department of Agricultural Engineering

National University of Water and Environmental Engineering

Soborna str., 11, Rivne, Ukraine, 33000

ORCID: <https://orcid.org/0009-0000-5582-1859>

**Lidiia Savchenko**

Senior Lecturer

Department of Architecture and Engineering Surveying\*\*

ORCID: <https://orcid.org/0000-0002-9444-2031>

**Oleksandr Savchenko**

PhD, Associate Professor

Department of Construction and Operation of Buildings, Roads and Road Constructions\*\*

ORCID: <https://orcid.org/0000-0003-0498-218X>

**Irina Zakharova**

PhD, Associate Professor

Department of Educational Management and Higher School Pedagogy

Sumy State Pedagogical University named after A. S. Makarenko

Romenska str., 87, Sumy, Ukraine, 40002

ORCID: <https://orcid.org/0000-0002-9693-5550>

\*Department of Descriptive Geometry, Computer Graphics and Design

National University of Life and Environmental Sciences of Ukraine

Heroyiv Oborony str., 15, Kyiv, Ukraine, 03041

\*\*Sumy National Agrarian University

Herasyma Kondratieva str., 160, Sumy, Ukraine, 40000

Received 19.12.2025

Received in revised form 27.02.2026

Accepted 10.03.2026

Published 30.04.2026

**How to Cite:** Nesvidomin, A., Pylypaka, S., Volina, T., Nesvidomin, V., Solarov, O., Voloshko, T., Pylypaka, T.,

Savchenko, L., Savchenko, O., Zakharova, I. (2026). Devising a technique for constructing tubular surfaces referred

to a coordinate grid of lines of curvature. *Eastern-European Journal of Enterprise Technologies*, 2 (1 (140)), 35–41.

<https://doi.org/10.15587/1729-4061.2026.354601>

## 1. Introduction

The primary requirement for various technical forms at the stage of their design is to increase the efficiency of the

corresponding technological processes while maintaining the operational reliability of working bodies. The effectiveness of designing such surfaces in the aviation industry, hydraulics, architecture, etc. directly depends on the availability of a math-

emathical apparatus for their description. Thus, improving the reliability of pneumatic classifiers requires the availability of an appropriate mathematical model for assessing the concentration of fine particles in a gas-dispersed flow [1]. Maintaining the operational reliability of means for preparing and distributing the feed mixture while enabling the required efficiency of these processes requires an analytical description of the movement of material by working bodies [2]. The existence of an analytical description of the design of a screw working body for surface tillage makes it possible to determining the optimal kinematic parameters of the technological process [3]. Studying the process of transporting technological material by working bodies of various designs is impossible without an analytical description of the movement of particles of technological material along them [4].

Complex technical shapes often contain tubular surfaces formed by the movement of a circle along a spatial trajectory. The assignment of a surface to a coordinate grid of curvature lines occupies a special place among existing techniques of surface parameterization as it provides consistency with the internal geometry of the surface. With such parameterization, the middle term of the first quadratic form disappears, significantly simplifying both it and the second quadratic form. In this case, two independent directions of the metric characteristics of the surface arise.

The assignment of a surface to a coordinate grid of curvature lines is especially important in analytical and numerical calculations. It makes it possible to connect the geometric characteristics of the surface with physical processes – stresses, deformations, etc. In practice, the directions of extreme stresses in the material are determined precisely by the curvature lines. The presence of such a surface parameterization makes it possible to simplify the analysis of the stress-strain state of shells and thin plates. In the machine-building, ship-building, aerospace industries, the orientation of stiffeners, reinforcement, and seams along the lines of curvature makes it possible to reduce stress concentration, increasing the reliability and durability of working bodies.

In the case of using conventional methods in the design of tubular surfaces, the axis of which is a spatial curve, difficulties arise because of the impossibility of analytically describing the length of the arc of such an axis. Thus, the issue of assigning tubular surfaces to families of coordinate lines of curvature is not only theoretically justified from the point of view of differential geometry but also practically necessary.

---

## 2. Literature review and problem statement

---

Geometric design of surfaces with predefined characteristics of linear coordinate grids is the object of many studies. For example, in [5], an approach to the design of unfolded surfaces with a given curvature line is reported. The authors analyzed the conditions under which the resulting surface is cylindrical, conical, or tangent to the edge of the return. In [6], the design of unfolded surfaces with given geodesic coordinates (the resulting surface is a cylinder) or coordinates of curvature lines (the resulting surface is tangent) is considered. A significant limitation of those papers is the possibility of applying the proposed methods to designing only a limited list of surfaces. They cannot be applied to the design of tubular surfaces because of their more complex internal geometry.

The design of geodesic tubular surfaces in three-dimensional Minkowski space is the focus of work [7]. The results

of such studies are extremely important for theoretical physics. However, their application in classical Euclidean space requires additional adaptation.

In [8], the curvatures of tubular surfaces were investigated using the Bishop frame. This is both an advantage and a disadvantage of the study. After all, objective difficulties arise in this case, associated with the transition from the description of curves in the local Bishop frame to parametric equations in a fixed coordinate system. The reason is the complexity of integrating the parameters of the Bishop frame for an arbitrary axis.

In [9], an attempt to avoid such a problem is made by using a modified orthogonal frame. The authors investigated the asymptotic and geodesic curves of focal tubular surfaces and their geometric invariants (flat and minimal surfaces, Weingarten surfaces). However, they did not consider the issue of assigning tubular surfaces to a coordinate grid of curvature lines in an analytical form.

Work [10] reports the study and interpretation of the geometric properties of harmonic evolute surfaces of tubular surfaces in Euclidean three-dimensional space. The proposed approach is based on the displacement of the point of the initial surface in the direction of the normal by a distance reciprocal to the mean curvature. The authors' attention focused precisely on the interpretation of the evolute properties of surfaces, leaving open the task of analytical parameterization of the initial surface with ensuring the orthogonality of the coordinate grid.

Finding families of surface curvature lines is simplified when one of them is known. Such cases include tubular surfaces, in which the frame of the generatrix circles is a family of curvature lines. In this case, the problem is reduced to finding a second family, that is, to the set of orthogonal trajectories perpendicular to the first family. This case also includes unfolded surfaces, in which the set of rectilinear generatrices is a family of curvature lines. So, the second family is a set of orthogonal trajectories to the first family.

Special attention is paid to the applied aspects of the assignment of curvilinear surfaces to lines of curvature. In works [11, 12], the direct influence of the exact geometric description of the surface of the working body on the system's performance in terms of calculating the deformations of elements is emphasized. The assignment of surfaces to a coordinate grid from lines of curvature plays an exceptional role in minimizing errors.

All this allows us to assert that the issue of assigning surfaces to a coordinate grid from lines of curvature is important. In this case, objective difficulties arise associated with the need to solve differential equations to derive parametric equations of the spatial axis as a function of the arc length. An option for overcoming these difficulties is the use of special classes of curves that make it possible to avoid numerical integration. The unsolved issue is the lack of an approach capable of combining the analytical advantages of slope curves with the assignment of a tubular surface to families of curvature coordinate lines.

Therefore, it is advisable to conduct a study aimed at devising a technique for constructing tubular surfaces based on slope curves.

---

## 3. The aim and objectives of the study

---

The aim of our study is to devise a technique for constructing tubular surfaces with an axial slope curve, referred to a coordinate grid of curvature lines. This will make it possible to improve the accuracy of calculating the stress-strain state of

shells in mechanical engineering and aerospace engineering while simultaneously minimizing computational costs.

To achieve this goal, the following tasks were set:

- to find a condition under which both families of a tubular surface will be families of curvature lines, and to derive parametric equations of a tubular surface in general form;
- to give a specific example of constructing a tubular surface and to show that the resulting general equations actually describe a tubular surface, referred to families of curvature coordinate lines.

---

#### 4. The study materials and methods

---

The object of our study is the process of constructing tubular surfaces with a spatial axis of slope, referred to a coordinate grid of curvature lines. The principal hypothesis assumes that the use of slope curves as axes of tubular surfaces will make it possible to obtain their parametric description in analytical form. This, in turn, will make it possible to ensure accurate orthogonality of the coordinate grid (referral to curvature lines) without using approximate numerical methods. It is assumed that such a description is possible when specifying the horizontal projection of the spatial curve by parametric equations in the function of the arc length. The simplification is that the study did not take into account cases of possible self-intersection of the surface, which can occur with large values of the axis curvature or a large radius of the generating circle.

If the axis of the tubular surface is a plane curve, then its mathematical description with the condition of reference to curvature lines does not require special calculations. An example is a torus whose surface axis is a circle. The formation of the surface occurs by the movement of the generating circle in the normal plane of another circle – the axis of the tubular surface. The axis of the tubular surface is also termed the axis of centers since all the centers of the generating circle are located on this curve (in this case, on the circle). It is important to note that with such formation of the tubular surface, both families of coordinate lines are lines of curvature. The set of generating circles and the trajectories of individual points of their movement form an orthogonal grid. This grid is a grid of lines of curvature. If the axis of the tubular surface is spatial, then its construction is carried out in a similar way; however, the resulting grid is not orthogonal. The lines of curvature are only the family of generating circles, and the second family, orthogonal to the first, must be found. This is explained by the fact that the generating circle of the surface is constructed in the normal plane of the Frenet trihedron of the surface axis. For the plane axis, the trihedron moves along it without rotation around the orthotangent, and along the spatial axis it rotates around it proportionally to the torsion. In this regard, for the spatial axis, it is necessary to make a transition from one independent variable  $v$  to another,  $u$ , according to the formula, which involves integrating the torsion  $\sigma = \sigma(s)$  depending on length  $s$  of the surface axis

$$v = -\int \sigma \cdot ds + u. \tag{1}$$

The spatial axis of the tubular surface is given by parametric equations, which include the axis curvature  $k = k(s)$  and the elevation angle  $\beta = \beta(s)$ . The equations given in [13] for a constant angle  $\beta = \text{const}$  take the following form:

$$\begin{aligned} x_0 &= \cos \beta \int \cos \left( \frac{1}{\cos \beta} \int k ds \right) ds; \\ y_0 &= \cos \beta \int \sin \left( \frac{1}{\cos \beta} \int k ds \right) ds; \\ z_0 &= s \sin \beta. \end{aligned} \tag{2}$$

At  $\beta = 0$ , the spatial slope curve (2) is transformed into a plane curve, which is described by the natural equation  $k = k(s)$ . Thus, a plane curve with a known natural equation  $k = k(s)$  can be transformed into a spatial slope curve according to equations (2) by introducing the elevation angle  $\beta = \text{const}$ .

Our study applied methods of differential geometry, in particular the theory of curves and surfaces. Mathematical modeling was based on vector analysis and the method of transition to a new independent variable through integration of the axis torsion expression. To describe the orientation of the generating circle, the orths of the accompanying Frenet trihedron were used.

We calculated coefficients for the first quadratic form of the surface and analytical transformations using symbolic calculation methods. To visualize the results and build graphic models of tubular surfaces, the MATLAB software (USA) was used.

The procedure for checking the adequacy of the proposed model was based on the analysis of coefficient  $F$  of the first quadratic form: the theoretical proof that  $F = 0$  for any point on the surface is a criterion for the orthogonality of the grid and confirmation of its belonging to the lines of curvature.

---

#### 5. Calculation and construction of a tubular surface

---

##### 5.1. Derivation of parametric equations of a tubular surface in general form

The axis of the surface, that is, the line of centers, is described by parametric equations (2) as a function of the length of its arc  $s$ . When  $s = \text{const}$ , a point is fixed on the axis, which is the center of the generating circle in the normal plane of the axis. At this point, the accompanying Frenet trihedron is located, in which the orthogonals of the main normal  $\bar{n}$  and binormal  $\bar{b}$  are located in the normal plane of the axial line (Fig. 1), and the orthogonal  $\bar{\tau}$  is tangent to it. The equations of the generating circle of radius  $\rho$  in projections onto these orthogonals of the trihedron take the form:

$$\begin{aligned} \rho_n &= \rho \cos v; \\ \rho_b &= \rho \sin v, \end{aligned} \tag{3}$$

where  $\rho = \text{const}$  – radius of the generatrix;  $v$  – independent variable (angle of rotation of a point of the circle in the normal plane around the orthogonal tangent  $\bar{\tau}$ ).

The transition from equations (3) of a circle in the normal plane of the Frenet trihedron to equations in the fixed coordinate system  $OXYZ$  is carried out using equations that include the direction cosines of the principal normal and the binormal:

$$\begin{aligned} X &= x_0 + \rho_n \cos \alpha_n + \rho_b \cos \alpha_b; \\ Y &= y_0 + \rho_n \cos \beta_n + \rho_b \cos \beta_b; \\ Z &= z_0 + \rho_n \cos \gamma_n + \rho_b \cos \gamma_b. \end{aligned} \tag{4}$$

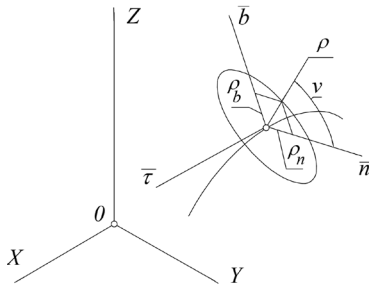


Fig. 1. Graphic illustration for constructing the generatrix of a tubular surface in the normal plane of the accompanying Frenet trihedron of the centerline

The expressions for  $A, B, C$  and the direction cosines that determine the direction of the indicated angles of the trihedron are determined through the first and second derivatives of the center line (2):

$$\begin{aligned}
 A &= y_0' z_0'' - y_0'' z_0'; \\
 B &= z_0' x_0'' - z_0'' x_0'; \\
 C &= x_0' y_0'' - x_0'' y_0'; \\
 \cos \alpha_n &= \frac{Bz_0' - Cy_0'}{\sqrt{(x_0'^2 + y_0'^2 + z_0'^2)(A^2 + B^2 + C^2)}} = \\
 &= -\sin\left(\frac{1}{\cos \beta} \int k ds\right); \\
 \cos \beta_n &= \frac{Cx_0' - Az_0'}{\sqrt{(x_0'^2 + y_0'^2 + z_0'^2)(A^2 + B^2 + C^2)}} = \\
 &= \cos\left(\frac{1}{\cos \beta} \int k ds\right); \\
 \cos \gamma_n &= \frac{Ay_0' - Bx_0'}{\sqrt{(x_0'^2 + y_0'^2 + z_0'^2)(A^2 + B^2 + C^2)}} = 0; \\
 \cos \alpha_b &= \frac{A}{\sqrt{A^2 + B^2 + C^2}} = -\sin \beta \cos\left(\frac{1}{\cos \beta} \int k ds\right); \\
 \cos \beta_b &= \frac{B}{\sqrt{A^2 + B^2 + C^2}} = -\sin \beta \sin\left(\frac{1}{\cos \beta} \int k ds\right); \\
 \cos \gamma_b &= \frac{C}{\sqrt{A^2 + B^2 + C^2}} = \cos \beta. \tag{5}
 \end{aligned}$$

If the axial line (2) is given as a function of the length of its arc  $s$ , then  $x_0'^2 + y_0'^2 + z_0'^2 = 1$  and expressions (5) are significantly simplified. The resulting equations (4) are equations of a tubular surface, since they depend on two independent variables  $s$  and  $v$ . In them, only one family of circles is a line of curvature, and the second is not. In order for the second family to be a family of lines of curvature, it is necessary to switch to a new variable  $u$  according to (1).

There is a linear relationship between the torsion  $\sigma$  and the curvature  $k$  of the slope curve:  $\sigma = k \cdot \text{tg} \beta$ . Therefore, (1) for the axial slope curve takes the following form

$$v = u - \text{tg} \beta \int k(s) ds. \tag{6}$$

When a trihedron moves along its axial line, its orthogonals of the main normal and binormal rotate around the orthogonal tangent by an angle determined by the integration of the torsion. In order for the second family to be perpendicular to the family of circles, this rotation must be eliminated. This is done using the transition (6), that is, a rotation with a "minus" sign (in the opposite direction) is performed.

Substitution of expressions (5) and (3) into equation (4) taking into account (6) after transformations and simplifications gives the following result:

$$\begin{aligned}
 X(u, s) &= \cos \beta \int \cos \psi ds - \\
 &\quad -\rho \left[ \begin{aligned} &\sin \psi \cos(u - \psi \sin \beta) + \\ &+ \sin \beta \cos \psi \sin(u - \psi \sin \beta) \end{aligned} \right]; \\
 Y(u, s) &= \cos \beta \int \sin \psi ds + \\
 &\quad +\rho \left[ \begin{aligned} &\cos \psi \cos(u - \psi \sin \beta) - \\ &- \sin \beta \sin \psi \sin(u - \psi \sin \beta) \end{aligned} \right]; \\
 Z(u, s) &= s \sin \beta + \rho \cos \beta \sin(u - \psi \sin \beta). \tag{7}
 \end{aligned}$$

In parametric equations (7), for the purpose of more compact notation, the following substitution is used

$$\psi(s) = \frac{1}{\cos \beta} \int k ds. \tag{8}$$

Thus, to construct a tubular surface, it is sufficient to have the dependence  $k = k(s)$ , but it must be such that it is possible to perform the integration of equations (2).

**5. 2. Construction and verification of a tubular surface according to the obtained equations in general form**

A tubular surface can be constructed, the horizontal projection of the axis of which is a logarithmic spiral with the dependence  $k(s) = a/s$ , where  $a$  is a constant. According to (2), the parametric equations of the axis take the following form:

$$\begin{aligned}
 x_0(s) &= \frac{s \cos^2 \beta}{a^2 + \cos^2 \beta} \times \left[ \cos \beta \cos\left(\frac{a \ln s}{\cos \beta}\right) + a \sin\left(\frac{a \ln s}{\cos \beta}\right) \right]; \\
 y_0(s) &= \frac{s \cos^2 \beta}{a^2 + \cos^2 \beta} \times \left[ \cos \beta \sin\left(\frac{a \ln s}{\cos \beta}\right) - a \cos\left(\frac{a \ln s}{\cos \beta}\right) \right]; \\
 z_0(s) &= s \sin \beta. \tag{9}
 \end{aligned}$$

Substitution in (7) of the required expressions gives parametric equations of the tubular surface related to the coordinate lines of curvature:

$$\begin{aligned}
 X(u, s) &= x_0(s) - \rho \left[ \begin{aligned} &\sin\left(\frac{a \ln s}{\cos \beta}\right) \cos(u - a \text{tg} \beta \ln s) + \\ &+ \sin \beta \cos\left(\frac{a \ln s}{\cos \beta}\right) \sin(u - a \text{tg} \beta \ln s) \end{aligned} \right]; \\
 Y(u, s) &= y_0(s) + \rho \left[ \begin{aligned} &\cos\left(\frac{a \ln s}{\cos \beta}\right) \cos(u - a \text{tg} \beta \ln s) - \\ &- \sin \beta \sin\left(\frac{a \ln s}{\cos \beta}\right) \sin(u - a \text{tg} \beta \ln s) \end{aligned} \right]; \\
 Z(u, s) &= z_0(s) + \rho \cos \beta \sin(u - a \text{tg} \beta \ln s). \tag{10}
 \end{aligned}$$

In equations (10), expressions  $x_0(s)$ ,  $y_0(s)$ ,  $z_0(s)$  are parametric equations of the surface axis and are given in (9). In Fig. 2, a tubular surface is constructed using equations (10), in which the axis is a slope curve with a rise angle of  $\beta = 10^\circ$ . The horizontal projection is a logarithmic spiral. The axis curvature is given by the dependence  $k = a/s$ , where  $a = 10$ . The radius of the generating circle  $\rho = 15$  linear units.

It is necessary to confirm that equations (7), which are parametric equations of a tubular surface, describe it in such a way that the coordinate lines are lines of curvature. For tubular surfaces, it is known that a family of circles is lines of curvature. Families of curvature lines form an orthogonal grid. If the grid of orthogonal lines in parametric equations (7) is orthogonal, then it is referred to the coordinate lines of curvature. Evidence of the orthogonality of the coordinate grids is the equality of zero of the middle term of the first quadratic form. The coefficients of the first quadratic form are determined through the partial derivatives of the parametric equations of the surface (7). They take the following form:

$$\begin{aligned} \frac{\partial X}{\partial u} &= \rho \begin{bmatrix} \sin \psi \sin(u - \psi \sin \beta) - \\ -\sin \beta \cos \psi \cos(u - \psi \sin v) \end{bmatrix}; \\ \frac{\partial Y}{\partial u} &= -\rho \begin{bmatrix} \cos \psi \sin(u - \psi \sin \beta) + \\ +\sin \beta \sin \psi \cos(u - \psi \sin v) \end{bmatrix}; \\ \frac{\partial Z}{\partial u} &= \rho \cos \beta \cos(u - \psi \sin v); \\ \frac{\partial X}{\partial s} &= \cos \beta \cos \psi [1 - \rho k \cos(u - \psi \sin v)]; \\ \frac{\partial Y}{\partial s} &= \cos \beta \sin \psi [1 - \rho k \cos(u - \psi \sin v)]; \\ \frac{\partial Z}{\partial s} &= \sin \beta [1 - \rho k \cos(u - \psi \sin v)]. \end{aligned} \quad (11)$$

The coefficients of the first quadratic shape of a tubular surface (10) are as follows:

$$\begin{aligned} E &= \left(\frac{\partial X}{\partial u}\right)^2 + \left(\frac{\partial Y}{\partial u}\right)^2 + \left(\frac{\partial Z}{\partial u}\right)^2 = \rho^2; \\ F &= \frac{\partial X}{\partial t} \cdot \frac{\partial X}{\partial u} + \frac{\partial Y}{\partial t} \cdot \frac{\partial Y}{\partial u} + \frac{\partial Z}{\partial t} \cdot \frac{\partial Z}{\partial u} = 0; \\ G &= \left(\frac{\partial X}{\partial s}\right)^2 + \left(\frac{\partial Y}{\partial s}\right)^2 + \left(\frac{\partial Z}{\partial s}\right)^2 = \\ &= \left[1 - k\rho \cos\left(u - \frac{1}{\cos \beta} \int k ds\right)\right]^2. \end{aligned} \quad (12)$$

The equality  $F = 0$  is a condition for orthogonality of the grid of coordinate lines, which for the tubular surface (7) is a sufficient condition for relating the coordinate lines to

the lines of curvature of the surface. According to the expressions of the coefficients (12), we can conclude that for all tubular surfaces with an axial slope line, the first two coefficients will be the same, and only the third depends on the dependence  $k = k(s)$ .

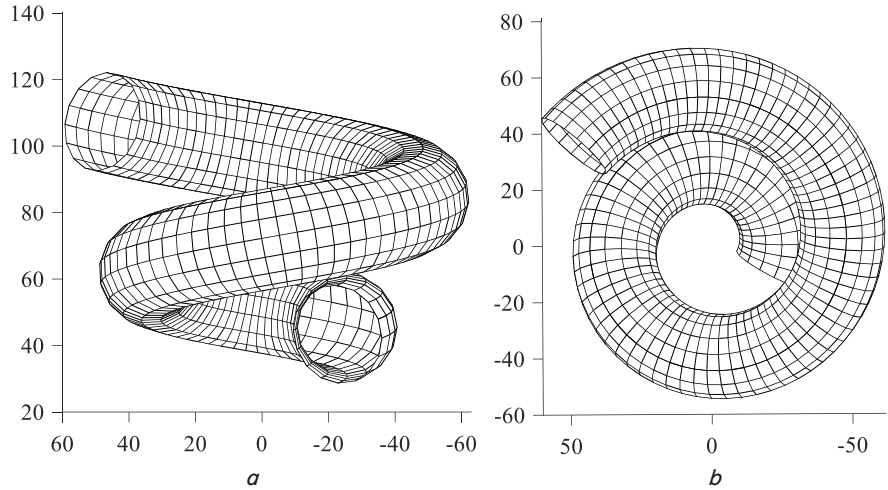


Fig. 2. Tubular surface assigned to families of curvature coordinate lines:  $a$  – frontal projection;  $b$  – horizontal projection

## 6. Discussion of results based on constructing tubular surfaces with an axial slope line

A tubular surface is formed by the movement of a circle of constant radius, which during movement is always in the normal plane of the curved line, which is the axis of the surface. To implement such a scheme of surface formation, the accompanying Frenet trihedron was used. When it moves along the axis, two orths – the main normal  $\bar{n}$  and the binormal  $\bar{b}$  – form a normal plane in which a circle with the center on the axis is located (Fig. 1). It is described by parametric equations (3) in projections onto these orths. But when the trihedron moves along the axis, the orientation of these orths relative to the fixed coordinate system  $OXYZ$  changes. The transition from the equations of the circle in the trihedron system to the equations of the surface in the fixed coordinate system  $OXYZ$  is carried out using expressions (4). These include the direction cosines of the main normal and binormal, which are determined from formulas (5). Whereas equations (3) describe a circle and are dependences of the variable  $v$ , then equation (4) are dependences of two variables  $v$  and  $s$ , where the position of the generating circle depends on the length of the arc  $s$  of the axial line. The surface constructed in this way is not referred to the coordinate lines of curvature. In order to refer it to the lines of curvature, it is necessary to pass from the variable  $v$  to the variable  $u$  according to formula (1). It is this dependence, which includes the integration of the torsion along the variable  $s$ , that requires parametric equations of the spatial axis of the surface as a function of the length of its axis. Few such spatial curves are known. The general approach to their construction according to the given dependences of the curvature  $k$  and the torsion  $\sigma$  depending on the length of the arc  $s$  leads to a system of differential equations, which in the general case cannot be solved in a finite form.

Therefore, in this work it is proposed to take for the axis a spatial slope curve that has a constant elevation angle  $\beta$ . In this case, the transition to a new variable occurs according to the new dependence (6), in which it is necessary to perform

integration not of the torsion but of the curvature, which is much simpler. In addition, the spatial axis is built on the basis of a plane curve, the parametric equations of which can be found by dependences  $k = k(s)$ . Such curves are known. For a constant angle  $\beta$  with a properly chosen dependence  $k = k(s)$ , which makes it possible to integrate dependences (2), it is possible to obtain parametric equations of the slope curve for the axis of the tubular surface. The result ( $F = 0$ ) is explained by the fact that the introduction of a new variable  $u$  through the integration of the torsion makes it possible to compensate for the rotation of the Frenet trihedron around the tangent, ensuring strict orthogonality of the grid at each step along the axis. At the same time, from a practical point of view, such a parameterization of the surface makes it possible to increase the accuracy of its metrological control, in contrast to [14, 15], where this is achieved in a much more complicated and expensive way.

Thus, the devised technique makes it possible to obtain analytical parameterization of tubular surfaces by curvature lines. The evidence base was based on the successful testing of the method for a logarithmic spiral, which was confirmed graphically and analytically.

In any case, the transition from an arbitrary family of coordinate lines to the desired one is associated with significant analytical difficulties, which, as a rule, lead to the solution of differential equations. This is the limitation of the transition from an arbitrary surface parameterization to a given one. For example, if at the first stage of the mathematical description of the surface axis by equations (2) it is not possible to find such a curvature dependence  $k = k(s)$  that these equations can be integrated, then the construction of the surface becomes impossible. The disadvantage is that the transition to a new variable according to expression (6), which also requires integration, is possible for a wide class of dependences  $k = k(s)$ , but further construction of the surface cannot be performed due to the impossibility of integrating equations (2).

The development of this research may consist in the construction of channel surfaces (for example, hydraulic systems with variable pressure), assigned to families of coordinate lines of curvature, in which the radius of the generating circle is variable.

---

## 7. Conclusions

---

1. One family of curvature lines of a tubular surface is a set of circles of constant radius. Each circle of this set is located in the normal plane of the axis of the tubular surface with the center on it. The position of a separate circle is given in the normal plane of the accompanying Frenet trihedron of the surface axis and is converted into projections on the axis of a fixed coordinate system using direction cosines. When the accompanying trihedron moves along the axis, a set of circles is formed, which are a family of curvature lines. However, the second family of lines with such a movement of the trihedron is not a family of curvature lines. Since the two families of curvature lines are mutually perpendicular, the second family can be searched for as a set of orthogonal trajectories to the family of circles. This requires a transition to another variable surface that describes a separate circle. This is due to the integration of the expression for the torsion of the axis along the length of its arc. Our paper considers a simplified version when the axis of the surface is the slope curve and from torsion it is possible to pass to curvature, which greatly facilitates the solution of the problem. Parametric equations of a tubular surface in general form have been derived.

2. According to the resulting equations, a tubular surface has been constructed, the axis of which is the slope curve obtained by transforming a plane curve into a logarithmic spiral. It is proved that families of coordinate lines are lines of curvature. For this purpose, three coefficients of the first quadratic form of the tubular surface are found on the basis of partial derivatives of general parametric equations. The average coefficient  $F$  is equal to zero, which indicates the orthogonality of the families of coordinate lines. This is sufficient to confirm that the tubular surface is assigned to two families of curvature lines. According to the obtained generalized parametric equations, a tubular surface can be constructed, assigned to curvature lines. To do this, one needs to specify the angle of elevation of the slope curve – the surface axis – and the dependence of the curvature on arc length  $k = k(s)$ , for which there is a flat curve.

---

## Conflicts of interest

---

The authors declare that they have no conflicts of interest in relation to the current study, including financial, personal, authorship, or any other, that could affect the study, as well as the results reported in this paper.

---

## Funding

---

The study was conducted within the framework of the state budget research topic "Increasing soil fertility by reducing the impact of external mechanical and chemical factors during the cultivation of agricultural crops", approved by order of the Ministry of Education and Science No. 22 and No. 23 of January 9, 2026.

---

## Data availability

---

All data are available, either in numerical or graphical form, in the main text of the manuscript.

---

## Use of artificial intelligence

---

The authors confirm that they did not use artificial intelligence technologies when creating the current work.

---

## Authors' contributions

---

**Andrii Nesvidomin:** Methodology, Software; **Serhii Pylypaka:** Conceptualization, Supervision; **Tetiana Volina:** Writing – original draft, Project administration; **Victor Nesvidomin:** Formal analysis, Data Curation; **Oleksandr Solarov:** Methodology, Writing – review & editing; **Taras Voloshko:** Resources, Writing – review & editing; **Taras Pylypaka:** Validation, Writing – review & editing; **Lidiia Savchenko:** Investigation, Data Curation; **Oleksandr Savchenko:** Validation, Visualization; **Irina Zakharova:** Writing – review & editing, Visualization.

---

## Acknowledgments

---

The authors express their gratitude to the defenders of Ukraine for the opportunity to live and engage in scientific activities.

## References

1. Lytvynenko, A., Yukhymenko, M., Pavlenko, I., Pitel, J., Mizakova, J., Lytvynenko, O. et al. (2019). Ensuring the Reliability of Pneumatic Classification Process for Granular Material in a Rhomb-Shaped Apparatus. *Applied Sciences*, 9 (8), 1604. <https://doi.org/10.3390/app9081604>
2. Novitskiy, A., Banniy, O., Novitskiy, Y. (2023). Logical-probabilistic model of the reliability of means for preparing and distributing fodder. *Naukovij Žurnal "Tehnika Ta Energetika"*, 14 (1). <https://doi.org/10.31548/machinery/1.2023.57>
3. Pylypaka, S. F., Klendii, M. B., Trokhaniak, V. I., Kresan, T. A., Hryshchenko, I. Y., Pastushenko, A. S. (2021). External rolling of a polygon on closed curvilinear profile. *Acta Polytechnica*, 61 (1), 270–278. <https://doi.org/10.14311/ap.2021.61.0270>
4. Pylypaka, S. F., Klendii, M. B., Nesvidomin, V. M., Trokhaniak, V. I. (2019). Particle motion over the edge of an inclined plane that performs axial movement in a vertical limiting cylinder. *Acta Polytechnica*, 59 (1), 67–76. <https://doi.org/10.14311/ap.2019.59.0067>
5. Li, C.-Y., Wang, R.-H., Zhu, C.-G. (2013). An approach for designing a developable surface through a given line of curvature. *Computer-Aided Design*, 45 (3), 621–627. <https://doi.org/10.1016/j.cad.2012.11.001>
6. Althibany, N. (2021). Construction of Developable Surface with Geodesic or Line of Curvature Coordinates. *Journal of New Theory*, 36, 75–87. <https://doi.org/10.53570/jnt.987265>
7. Karacan, M. K., Yayli, Y. (2008). On the geodesics of tubular surfaces in Minkowski 3-space. *Bulletin of the Malaysian Mathematical Sciences Society. Second Series*, 31 (1). Available at: <https://eudml.org/doc/54573>
8. Dogan, F., Yayli, Y. (2011). On the curvatures of tubular surface with bishop frame. *Communications Faculty Of Science University of Ankara Series A1 Mathematics and Statistics*, 60 (1), 59–69. [https://doi.org/10.1501/commua1\\_0000000669](https://doi.org/10.1501/commua1_0000000669)
9. Khalifa Saad, M., Yüksel, N., Oğraş, N., Alghamdi, F., Abdel-Salam, A. A. (2024). Geometry of tubular surfaces and their focal surfaces in Euclidean 3-space. *AIMS Mathematics*, 9 (5), 12479–12493. <https://doi.org/10.3934/math.2024610>
10. Eren, K. (2021). On the harmonic evolute surfaces of tubular surfaces in euclidean 3-space. *Journal of Science and Arts*, 21 (2), 449–460. <https://doi.org/10.46939/j.sci.arts-21.2-a12>
11. Hevko, I. B., Lyashuk, O. L., Leshchuk, R. Y., Rogatinska, L. R., Melnychuk, A. L. (2016). Investigation of the radius of bending for flexible screw sectional conveyers. *INMATEH - Agricultural Engineering*, 48 (1), 35–42. Available at: [https://inma-ita.ro/inmateh/INMATEH\\_1\\_2016/48\\_5\\_Hevko%20Iv.B.pdf](https://inma-ita.ro/inmateh/INMATEH_1_2016/48_5_Hevko%20Iv.B.pdf)
12. Lyashuk, O., Rohatynskyi, R., Hevko, I., Dmytriv, O., Tson, O., Tkachenko, I. et al. (2023). Investigation of Bulk Material Transportation by Screw Conveyer with Hinge-Pan Operating Device. *Key Engineering Materials*, 948, 169–182. <https://doi.org/10.4028/p-6qin7l>
13. Pylypaka, S., Kresan, T., Trokhaniak, O., Taras, I., Demchuk, I. (2021). Parametric equations of a spatlal curve as a function of length of the arc with given dependences of curvature and angle of ascent. *Journal for Geometry and Graphics*, 25 (2), 163–170. Available at: <https://www.heldermann-verlag.de/jgg/jgg25/j25h2pyly.pdf>
14. Ju, Y., Konoplianchenko, I., Pu, J., Zhang, Z., Dong, Q., Dumanchuk, M. (2024). Optimization of structure and properties of WC-reinforced FeCoNiCr high-entropy alloy composite coating by laser melting. *Results in Engineering*, 21, 101985. <https://doi.org/10.1016/j.rineng.2024.101985>
15. Myslyvchenko, O. M., Gaponova, O. P., Tarelnyk, V. B., Krapivka, M. O. (2020). The Structure Formation and Hardness of High-Entropy Alloy Coatings Obtained by Electrospark Deposition. *Powder Metallurgy and Metal Ceramics*, 59 (3-4), 201–208. <https://doi.org/10.1007/s11106-020-00152-7>