UDC 536.24
DOI: 10.15587/1729-4061.2026.357609

CONSTRUCTION OF MATHEMATICAL MODELS OF HEAT EXCHANGE IN ELECTRONIC DEVICES WITH SEMI-THROUGH FOREIGN ELEMENTS

Vasyl Havrysh
Doctor of Technical Sciences, Professor
Department of Software*
ORCID: https://orcid.org/0000-0003-3092-2279

Svitlana Yatsyshyn
Corresponding author
PhD, Associate Professor
Department of Software Engineering
Ukrainian National Forestry University
Gen. Chuprynky str., 103, Lviv, Ukraine, 79057
E-mail: yatsyshyn@nltu.edu.ua
ORCID: https://orcid.org/0000-0001-5200-4837

Lubov Kolyasa
PhD, Associate Professor
Department of Mathematics*
ORCID: https://orcid.org/0000-0002-9690-8042

Mykhailo Stepaniak
PhD
Department of Computerized Automatic Systems*
ORCID: https://orcid.org/0000-0003-1859-4495

Andrii Kapustianskyi
PhD
Department of Heat Engineering
and Thermal and Nuclear Power Plants*
ORCID: https://orcid.org/0000-0002-2771-2505

*Lviv Polytechnic National University
S. Bandery str., 12, Lviv, Ukraine, 79013

This study investigates heat exchange processes in isotropic spatial environments with foreign semi-through elements subjected to external and internal thermal loads.

Significant temperature gradients arise as a result of the thermal load. To establish and analyze temperature regimes for effective operation of electronic devices, mathematical models for determining temperature fields have been constructed.

Based on the formulated boundary value problems of thermal conductivity, their analytical and numerical solutions have been defined. Using these solutions, numerical calculations of the temperature distribution in spatial coordinates for given geometric and thermophysical parameters have been performed.

For an effective description of the thermal conductivity coefficient for inhomogeneous spatial media, asymmetric unit functions were used. A technique for segment-constant approximation of temperature as a function of spatial coordinates on the surfaces of foreign elements has been introduced. As a result, second-order differential equations with partial derivatives and discontinuous and singular coefficients have been derived.

The numerical results reflect temperature distribution in the media in spatial coordinates for the given geometric and thermophysical parameters. The number of partitions of the intervals (0; h), (–H; H), (0; R) was chosen to be equal to 9. That has made it possible to obtain numerical values of temperature with an accuracy of 10^{-6}. The constructed mathematical models of heat transfer make it possible to analyze spatial isotropic media with foreign through-going elements in terms of their thermal stability

Keywords: temperature field, thermal conductivity of the material, thermal resistance of structures, heat transfer, semi-through foreign elements

Received 29.01.2026
Received in revised form 06.04.2026
Accepted 14.04.2026
Published 30.04.2026

How to Cite: *Havrysh, V., Yatsyshyn, S., Kolyasa, L., Stepaniak, M., Kapustianskyi, A. (2026). Construction of mathematical models of heat exchange in electronic devices with semi-through foreign elements. Eastern-European Journal of Enterprise Technologies, 2 (5 (140)), 36–43. https://doi.org/10.15587/1729-4061.2026.357609*

1. Introduction

Modern electronic devices operate under an accelerated mode, which leads to the emergence of intense temperature fields and their gradients. In certain local areas of processors and microcontrollers, high-intensity heat flows are generated. This leads to overheating of their components and, as a result, partial or complete failure of the device. Studies show that almost half of all microcircuits are destroyed due to overheating. This occurs on boards due to heating from neighboring elements that emit heat, the thermal power of the elements and devices themselves, as well as uneven heat removal from thermally active areas. The number of failures due to the operation of electronic devices is closely related to temperature conditions. Thus, for germanium elements at 140°C, this number is 7.5 times greater than at 20°C. It is even greater for silicon elements. It was found that the relative influence of temperature is the highest (55%) compared to humidity (19%), vibration (20%), and dust (6%).

In the process of designing electronic devices, developers choose metals with high thermal conductivity, in particular, copper or aluminum to increase the thermal resistance of structures.

Although these materials conduct heat well, their influence on this process may depend on the shape and presence of defects inside individual device nodes. Experimentally analyzing the heating in individual elements of the device due to operation is impossible due to the complex geometric structure of the device or excessively high temperatures. In this case, mathematical models of the heat transfer process are built, which take into account the geometric structure of the electronic device and the physical process. Using them, it could be possible to predict the behavior of temperature gradients due to the operation of devices. And this would make it possible to develop algorithms and software tools on their basis for research on the selection of effective structural materials, the geometric design of individual elements and nodes, the installation of effective heat dissipation elements and their placement. As a result, it will be possible to increase the reliability and durability of electronic devices even at the design stage, without conducting expensive experiments.

Therefore, it is a relevant task to construct mathematical models of heat transfer for isotropic spatial environments with heterogeneous elements of various geometric shapes.

2. Literature review and problem statement

In [1], the effectiveness of the boundary element method for determining numerical solutions of direct and inverse potential theory problems in a bounded segmentally homogeneous object of arbitrary shape, the components of which are in perfect contact, was noted. To solve the inverse problem, an iterative algorithm for identifying the main physical and geometric parameters of foreign elements based on redundant potential or flux data at the object boundary was introduced. To verify the effectiveness of the algorithm, a computational experiment was performed for the electrical exploration problem using an artificial constant electric field and the electrical profiling method. Local areas are modeled using relatively simple flat classical geometric figures. The use of such models in electrical exploration problems using the electrical profiling method for detecting minerals is not effective enough. To increase the accuracy and reliability of the results, it is necessary to build mathematical models based on the solution to non-stationary boundary value problems.

Work [2] considers the numerical modeling of heat conduction processes in complex inhomogeneous structures with sharply varying thermophysical parameters in space. The authors introduce a non-classical meshless method for solving the heat conduction equation, which involves dividing the region into its separate sections and using the interpolation method using polyharmonic functions. This method allows for a more accurate description of temperature fields in structures with internal inclusions and with thermal conductivity coefficients that change sharply. The effectiveness of the given method for analyzing thermal regimes is confirmed by numerical experiments. It can be used to analyze temperature regimes in modern electronic devices.

Study [3] reports modeling of heat transfer processes in complex technical structures using modern numerical methods. The finite element method, finite volume method, and computational fluid dynamics methods were used to analyze temperature states. The study emphasizes that for thermal management tasks in electronic devices, it is imperative to take into account the complexity of the design and the presence of local heat sources. This technique provides detailed reproduction of thermal regimes and contributes to the rational design of electronic devices.

In [4], the process of heat transfer in solids is described. To determine the approximate solution of the heat conduction equation, a combination of the Boltzmann lattice method with the Runge-Kutta method was used. The method makes it possible to relate the temperature distribution to the geometric and thermophysical parameters of the structural material. It can be used for thermal modeling in electronic devices.

Analysis of papers [2–4] confirms that mathematical models built by using numerical methods make it possible to effectively describe thermal processes in complex environments. However, in modern electronic devices, in particular mobile ones, the geometric dimensions of local heating areas and foreign inclusions are small. This complicates their accurate reproduction in models constructed on the basis of mesh methods or the finite element method. Averaging the physical parameters of the environment in such cases does not make it possible to adequately reflect the influence of foreign inclusions on the behavior of the temperature field.

In [5], the thermal contact resistance between rough surfaces was investigated, taking into account their microgeometry and thermophysical properties of materials. An experimental-theoretical approach was presented to determine the thermal contact resistance for surfaces with an uneven contact structure. The influence of the thermal conductivity of materials, contact pressure, and roughness parameters on the intensity of heat transfer through the contact zone was established. The results could be used to analyze thermal regimes in complex structures and electronic systems.

In [6], a mathematical model of heat transfer through imperfect contact surfaces in multilayer structures is described. Using thermal contact resistance, the effect of surface micro-unevenness and contact pressure is effectively taken into account. This approach makes it possible to predict the temperature distribution in complex technical systems with many contact surfaces. The results could be used to optimize thermal regimes in engineering and electromechanical devices.

In the models [5, 6] built for complex structures, ideal thermal contact between components is taken into account. In complex engineering systems, due to heating, imperfect thermal contact between dissimilar structural elements is observed. Therefore, the models are simplified and require improvement.

In study [7], an experimental analysis of thermal contact resistance between flat and curved surfaces was performed, taking into account changes in temperature, contact pressure, and geometric parameters of the surfaces. The results of the above analysis made it possible to identify patterns that reproduce changes in thermal contact resistance and their effect on the intensity of heat exchange in the contact zone. The results are essential for the development of heat-resistant systems and complex engineering structures.

In work [8], the thermal contact resistance between objects with a regular surface texture was studied, taking into account the influence of a heat-conducting intermediate medium. The constructed mathematical model makes it possible to effectively describe heat transfer through contact surfaces, taking into account the geometric shape of micro-irregularities and the characteristics of the intermediate medium. As a result of numerical experiments, the influence of the parameters of the texture and thermal conductivity of the medium on the behavior of the heat flow and the phenomenon of thermal rectification was revealed, which is critically important for multilayer structures.

In [9], a mathematical model of the thermal process in systems with contact surfaces under transient thermal loads was built. The model is based on a description of each element of the system with thermal resistance and heat capacity, taking into account the interaction of heterogeneous materials in the contact zone. Numerical modeling confirms the significant influence of thermophysical parameters of structural materials in the contact zone on the formation of spatial temperature distribution, which makes it possible to analyze the thermal stability of complex electronic systems.

In [10], modern methods of artificial intelligence were applied to predict temperature fields in microelectronic systems. The model was constructed based on neural networks and, with its use, it is possible to take into account local heat sources. The application of this model makes it possible to detect uneven distribution of thermal loads and features of heat transfer in integrated circuits. Numerical experiments performed on the basis of this model confirmed the high accuracy of predicting the behavior of temperature fields. On this basis, it is possible to effectively detect zones of local overheating, which is critically important for ensuring the reliability of electronic components.

Studies [7–10] demonstrate significant progress in predicting and modeling thermal processes in complex electronic systems. However, the constructed mathematical models reported in those studies do not allow for an effective quantitative assessment of the reduction in peak temperatures and amplitudes of temperature fluctuations. And this is a determining criterion for increasing the reliability and thermal stability of modern electronic components with semi-through foreign elements.

In [11], a mathematical model was built that allows for the prediction of transient temperature processes in power electronic systems. The model takes into account variable thermal loads and the interaction between structural components with different thermophysical parameters. It is based on generalized heat conduction equations, which provides an effective determination of the spatial distribution of temperature fields without the need to use full CFD (computational fluid dynamics) or FEM (finite element method) models, which significantly reduces computational costs. Numerical studies confirm that the presented method allows for the prediction of temperature regimes for electronic systems under variable thermal load conditions with high accuracy.

In [12], a reduced-order model is described, which is intended for modeling thermal processes in electronic components taking into account surface thermal radiation. The reported method combines the finite element method with the reduction of the model order, which makes it possible to save computational time while maintaining high accuracy for predicting temperature distributions. Numerical experiments confirm that the developed model enables effective determination of the behavior of the temperature field under different thermal load conditions. It also provides the ability to analyze the thermal regimes of electronic systems in real time, which is critically important for increasing the reliability of components containing local intense zones and semi-through foreign elements.

In studies [11, 12], techniques are described that are suitable for predicting dynamic thermal processes. However, the consideration of individual foreign components in such models remains insufficiently developed, in particular when it is necessary to establish the limiting heating indicators and temperature drop limits. These factors often determine the durability and stability of modern microelectronic systems.

A number of studies describe mathematical modeling of temperature regimes in thermosensitive media with foreign inclusions. In particular, in [13], a nonlinear model was proposed, which took into account thermosensitive elements with semi-through foreign inclusions and linearization was applied to reduce the axisymmetric nonlinear boundary value problem to a quasi-linear one. However, the model does not provide for internal thermal heating in the volume of a thin cylinder, which reduces its suitability for predicting the behavior of temperature fields under real operating conditions of microelectronic devices.

In [14], a linear and nonlinear mathematical model of thermal conductivity for an isotropic layer with a semi-through cylindrical inclusion and internal heat sources was built. At the same time, in the cited work, attention focuses on the formulation and analytical study of the model, while the numerical analysis of temperature fields and the influence of system parameters was not carried out in detail.

Thus, our review of [13, 14] demonstrates that the existing approaches allow us to estimate the general trends of temperature distribution in thermally sensitive environments. However, they do not provide sufficient accuracy for objects with local internal and external thermal loads, semi-through foreign elements and non-uniform distribution of thermophysical properties. This necessitates devising more effective analytical-numerical models for predicting temperature fields in modern microelectronic systems, which is the main goal of our study.

Analysis of related literature [1, 5–10] reveals the existence of a significant problem associated with the lack of theoretically justified approaches to the formalization of heat conduction processes in heat-active media containing internal foreign semi-through elements of arbitrary geometric shape. Existing mathematical models do not adequately reflect the heat transfer between structural elements of modern electronic devices, taking into account local heating in inhomogeneous areas, which limits their practical applicability for multifunctional composite structures. The main factor determining this problem is the complexity of the mathematical description of thermal processes in media with a complex geometric structure and inhomogeneous properties. However, this is only part of the difficulty – it is even more difficult to carry out real tests of such models. Experiments are expensive because the exact reproduction of conditions requires enormous effort.

All this allows us to state that it is advisable to conduct a study aimed at building analytical and numerical models of heat transfer for isotropic spatial environments with inhomogeneous elements of various geometric shapes. The models to be constructed make it possible to predict temperature regimes in modern electronic devices made of composite materials, which creates the prerequisites for increasing their efficiency, reliability, and durability of operation.

3. The aim and objectives of the study

The purpose of our study is to determine the temperature fields in spatial environments with foreign semi-through elements due to external and internal heating. The mathematical models to be constructed will make it possible to increase the accuracy in predicting temperature regimes and provide effective means for optimizing the design of modern electronic devices.

To achieve the set goal, it is necessary to solve the following tasks:

– to build a mathematical model of heat transfer in an isotropic plate with a semi-through inclusion of a parallelepiped shape, which is heated by a heat flux at the boundary surface;

– to construct a mathematical model of heat transfer in an isotropic layer with a semi-through inclusion of a cylindrical shape, in the area of which internal heat sources are uniformly concentrated.

4. The study materials and methods

The object of our study is the heat transfer processes in isotropic spatial media with foreign semi-through elements that are subjected to external and internal thermal loads.

The hypothesis of the work assumes that the temperature fields in such media, caused by the combined effect of internal and external heating, can be represented by analytical-numerical solutions of the corresponding boundary value problems of heat conduction. In these problems, inhomogeneous differential equations with partial derivatives of the second order contain discontinuous and singular coefficients that arise due to the presence of foreign semi-through elements.

The study assumes the isotropic nature of the spatial media, that is, the thermophysical parameters are constant along all coordinate axes. The mathematical models of heat transfer built for this system are simplified since the analysis of temperature fields and modes is carried out exclusively in spatial coordinates.

The method of asymmetric unit functions is used to model thermal conductivity in isotropic media with foreign semi-through elements. This approach makes it possible to describe the effective thermal conductivity of a material of heterogeneous structures as a single whole in the form of appropriate relations and form a single inhomogeneous second-order differential equation with partial derivatives, which contains discontinuous and singular coefficients.

In addition, a segment-constant approximation of the temperature field on the boundary surfaces of foreign elements is introduced, which provides the possibility of using integral Fourier and Henkel transforms for analytical and numerical solution of boundary problems of heat conduction.

An isotropic plate with a thickness of 2δ with thermally insulated front surfaces $|z| = \delta$, which contains a foreign semi-through inclusion with a length h and a width $2H$, is considered, which is referred to the Cartesian rectangular coordinate system ($Oxyz$). In the region $\Omega_0 = \{(x, h, z) : |x| \le H, |z| \le \delta\}$ of the boundary surface $L_+ = \{(x, h, z) : |x| \le \infty, |z| \le \delta\}$ of the plate, a heat flux is concentrated, the surface density of which is $q_0 = \text{const}$, and on the other surface of the plate $L_- = \{(x, -l, z) : |x| \le \infty, |z| \le \delta\}$ the conditions of convective heat exchange with the environment according to Newton's law with constant temperature $t_c = \text{const}$ are given. On the boundary surfaces of the inclusion $K_{\pm H} = \{(\pm H, y, z) : 0 \le y \le h, |z| \le \delta\}$, $K_0 = \{(x, 0, z) : |x| \le H, |z| \le \delta\}$, there is an ideal thermal contact $t_0(\pm H, y) = t_1(\pm H, y)$, $\lambda_0 \cdot \partial t_0(x, y)/\partial x = \lambda_1 \cdot \partial t_1(x, y)/\partial x$ for $|x| = H$, $0 \le y \le h$, $|z| \le \delta$ and $t_0(x, 0) = t_1(x,0)$, $\lambda_0 \cdot \partial t_0(x, y)/\partial y = \lambda_1 \cdot \partial t_1(x, y)/\partial y$ for $y = 0$, $|x| = H$, $|z| \le \delta$ (0 – for inclusion, 1 – for plate) (Fig. 1).

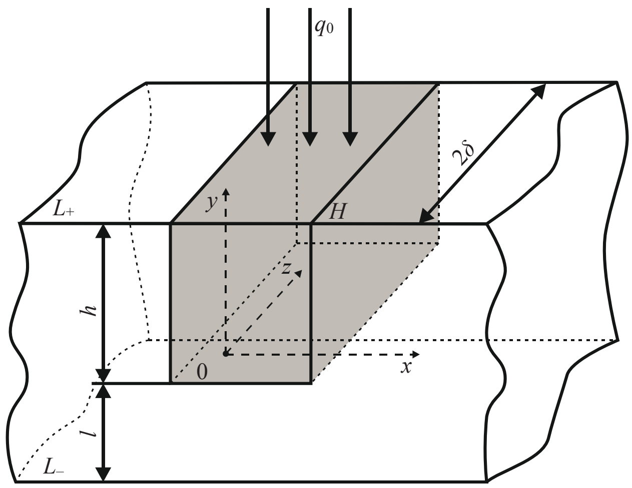

Fig. 1. Isotropic plate with a half-through inclusion under the action of heat flux

In the above structure, the temperature distribution $t(x, y)$ in spatial coordinates x and y is determined by solving the heat conduction equation

$$div\left[\lambda(x,y)\, grad\, t(x,y)\right] = 0, \tag{1}$$

with boundary conditions:

$$t(x,y)\Big|_{|x|\to\infty} = t_c, \quad \frac{\partial t(x, y)}{\partial x}\Bigg|_{|x|\to\infty} = 0,$$

$$\lambda_1 \frac{\partial t(x, y)}{\partial y}\Bigg|_{y=-l} = \alpha_-\left(t(x,y)\Big|_{y=-l} - t_c\right),$$

$$\lambda_0 \frac{\partial t(x, y)}{\partial y}\Bigg|_{y=h} = q_0 S_-\left(H - |x|\right), \tag{2}$$

where $\lambda(x, y)$ – thermal conductivity coefficient of a non-uniform plate:

$$\lambda(x,y) = \lambda_1 + (\lambda_0 - \lambda_1) S_-\left(H - |x|\right) S_-(y); \tag{3}$$

$$S_\pm(\zeta) = \begin{cases} 1, \zeta > 0, \\ 0.5 \mp 0.5, \zeta = 0, \\ 0, \zeta < 0, \end{cases}$$

where λ_1 and λ_0 are the thermal conductivity coefficients of the plate and inclusion materials, respectively; α_- is the heat transfer coefficient from the surface L_-; $S_\pm(\zeta)$ are asymmetric unit functions.

5. Results of investigating mathematical models of heat transfer in media with foreign elements

5. 1. Mathematical model of heat transfer in a plate due to heating by a heat flux at the boundary surface

The function $T(x, y) = \lambda(x, y)\theta(x, y)$ was introduced and differentiated for variables x and y taking into account the expression for the thermal conductivity coefficient $\lambda(x, y)$ (3). As a result, we obtain

$$\lambda(x,y)\frac{\partial\theta(x,y)}{\partial x}= =\frac{\partial T(x,y)}{\partial x}-(\lambda_0-\lambda_1)\begin{bmatrix}\theta(x,y)\big|_{x=-H}\delta_-(x++h)-\\ -\theta(x,y)\big|_{x=H}\delta_+(x-h)\end{bmatrix}S_-(y),$$

$$\lambda(x,y)\frac{\partial\theta(x,y)}{\partial y}=\frac{\partial T(x,y)}{\partial y}- -(\lambda_0-\lambda_1)\theta(x,y)\big|_{y=0}S_-(H-|y|)\delta_-(y), \qquad (4)$$

where $\theta(x, y) = t(x, y) - t_c$; $\delta_\pm(\zeta) = (dS\pm(\zeta))/d\zeta$ – asymmetric Dirac delta functions.

As a result of substituting expressions (4) into relation (1), a second-order partial differential equation with discontinuous and singular coefficients is obtained

$$\Delta T-(\lambda_0-\lambda_1)\left\{\begin{matrix}\begin{bmatrix}\theta(-H,y)\delta_-'(x+H)-\\ -\theta(H,y)\delta_+'(x-H)\end{bmatrix}S_-(y)+\\ +\theta(x,0)\delta_-'(y)S_-(H-|x|)\end{matrix}\right\}=0, \qquad (5)$$

where Δ is the Laplace operator in the Cartesian rectangular coordinate system

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

As a result of such transformations, the desired temperature field in the given medium is completely determined from equation (5) under boundary conditions (2).

The unknown functions $\theta(x, 0)$, $\theta(\pm H, y)$ are approximated by the spatial coordinates x and y by segment-constant functions in the form:

$$\theta(x,0)=\theta_1+\sum_{k=1}^{m-1}(\theta_{j+1}-\theta_j)S_-(x-x_k),$$

$$\theta(\pm H,y)=\theta_1+\sum_{j=1}^{n-1}(\theta_{j+1}-\theta_j)S_-(y-y_j). \qquad (6)$$

Here, $y_j \in (0; h)$; $y_1 \le y_2 \le \ldots \le y_{n-1}$; $x_k \in (-H; H)$; $x_1 \le x_2 \le \ldots \le x_{m-1}$; $\theta_j(j = 1\ldots n)$, $\theta_k(j = 1\ldots m)$ are unknown approximation values of temperature $\theta(\pm H, y)$, $\theta(x,0)$; n and m are the number of partitions of intervals $(0; h)$, $(-H; H)$.

To solve equation (5) under boundary conditions (2) using expressions (6), the integral Fourier transform with respect to the x coordinate is applied. As a result of such transformation, the original boundary value problem is transformed into an ordinary second-order differential equation with constant coefficients. In this case, the right-hand side of the equation contains discontinuous and singular coefficients that reflect the influence of semi-through foreign elements:

$$\frac{d^2\bar{T}}{dy^2}-\xi^2\bar{T}= =\frac{1}{\sqrt{2\pi}\xi}\left\{(\lambda_0-\lambda_1)\begin{bmatrix}(2\theta_mC(\xi)+iB(\xi))\dfrac{\delta_-'(y)}{\xi}-\\ -2\xi^2C(\xi)A(\xi,y)\end{bmatrix}\right\}, \qquad (7)$$

with boundary conditions

$$\left.\frac{d\bar{T}(y)}{dy}\right|_{y=-l}=\frac{\alpha_-}{\lambda_1}\bar{T}(y)\big|y=-l,$$

$$\left.\frac{d\bar{T}(y)}{dy}\right|_{y=h}=\sqrt{\frac{2}{\pi}}\frac{q_0}{\xi}\sin\xi h, \qquad (8)$$

where $\bar{T}(y)$ – transform of a function $T(x, y)$

$$\bar{T}(y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\xi x}T(x,y)\mathrm{d}x;$$

ξ – integral Fourier transform parameter, $i^2 = -1$; $C(\xi) = \sin H\xi$:

$$A(\xi,y)=\theta_1S_-(y)+\sum_{j=1}^{n-1}(\theta_{j+1}-\theta_j)S_-(y-y_j);$$

$$B(\xi)=\sum_{k=1}^{m-1}(\theta_{k+1}-\theta_k)\left(e^{i\xi x_k}-e^{i\xi x_{k-1}}\right).$$

The general solution to equation (7) is derived by the method of variation of constants

$$\bar{T}(y)=c_1e^{\xi y}+c_2e^{-\xi y}+ +\frac{\lambda_0-\lambda_1}{\xi\sqrt{2\pi}}\left\{\begin{matrix}2C(\xi)\left[\theta_1(1-\mathrm{ch}\xi y)S_-(y)+A_1(\xi,y)\right]+\\ +B_1(\xi)(1-\mathrm{ch}\xi y)S_-(y)\end{matrix}\right\}.$$

Here c_1 and c_2 are the integration constants:

$$A_1(\xi,y)=\sum_{j=1}^{n-1}(\theta_{j+1}-\theta_j)(1-\mathrm{ch}\xi(y-y_j))S_-(y-y_j);$$

$$B_1(\xi)=2\theta_mC(\xi)+i\sum_{k=1}^{m-1}(\theta_{k+1}-\theta_k)\left(e^{i\xi x_k}-e^{i\xi x_{k-1}}\right).$$

Boundary conditions (8) were used to find the constants of integration and, on this basis, the solution to problem (7), (8) was obtained

$$\bar{T}(y)=\frac{1}{\xi\sqrt{2\pi}}\left\{\begin{matrix}(\lambda_0-\lambda_1)\begin{bmatrix}2C(\xi)\begin{pmatrix}\theta_1A_2(\xi,y)+\\ +A_3(\xi,y)\end{pmatrix}+\\ +B_1(\xi)B(\xi,y)\end{bmatrix}-\\ -2q_0P_1(\xi,y)C(\xi)\end{matrix}\right\}, \qquad (9)$$

where:

$$P_1(\xi,y)=\frac{P(\xi,y)}{\xi P(\xi)},$$

$$A_2(\xi,y)=(1-\mathrm{ch}\xi y)S_-(y)-\frac{P(\xi,y)}{P(\xi)}\mathrm{sh}\xi h;$$

$$B(\xi,y)=\mathrm{ch}\xi yS_-(y)+\frac{P(\xi,y)}{P(\xi)}\mathrm{sh}\xi h,$$

$$A_3(\xi,y)= =\sum_{j=1}^{n-1}(\theta_{j+1}-\theta_j)\begin{bmatrix}(1-\mathrm{ch}\xi(y-y_j))S_-(y-y_j)-\\ -\dfrac{P(\xi,y)}{P(\xi)}\mathrm{sh}\xi(h-y_j)\end{bmatrix};$$

$$P(\xi)=(\lambda_1\xi-\alpha_-)e^{-\xi(l+h)}-(\lambda_1\xi+\alpha_-)e^{\xi(l+h)},$$

$$P(\xi,y)=(\lambda_1\xi+\alpha_-)e^{\xi(l+y)}+(\lambda_1\xi-\alpha_-)e^{-\xi(l+y)}.$$

The inverse integral Fourier transform was applied to relation (9) and as a result, the solution to the boundary value problem (1), (2) was derived in the following form:

$$T(x,y)= =\frac{1}{\pi}\int_0^\infty\frac{1}{\xi}\left\{\begin{matrix}(\lambda_0-\lambda_1)\begin{bmatrix}F(\xi,x)\begin{pmatrix}\theta_1A_2(\xi,y)+\\ +A_3(\xi,y)\end{pmatrix}+\\ +B(\xi,x)B(\xi,y)\end{bmatrix}-\\ -P_1(\xi,x,y)\end{matrix}\right\}\mathrm{d}\xi, \qquad (10)$$

where:

$$P_1(\xi,x,y)=q_0F(\xi,x)P_1(\xi,y),$$

$$F(\xi,x)=2C(\xi)\cos\xi x,$$

$$B(\xi,x)=2\theta_m C(\xi)\cos\xi x+$$
$$+\sin\xi x\sum_{k=1}^{m-1}(\theta_{k+1}-\theta_k)(\cos\xi x_k-\cos\xi x_{k-1})-$$
$$-\cos\xi x\sum_{k=1}^{m-1}(\theta_{k+1}-\theta_k)(\sin\xi x_k-\sin\xi x_{k-1}).$$

The unknown approximation values θ_j ($j = 1...n$), θ_k ($k = 1...m$), temperatures $\theta(\pm H, y)$, $\theta(x, 0)$, are determined by solving the system of $n + m$ linear algebraic equations obtained from expression (10).

As a result of the transformation performed, the determined temperature field in a plate with a semi-through foreign inclusion, caused by a locally applied heat flux at the boundary surface, is described by formula (10). From this expression, it is possible to derive numerical values of the temperature at an arbitrary point of the "plate-inclusion" structure, which makes it possible to analyze the temperature regimes in this structure and to evaluate the influence of the inclusion on the temperature distribution.

Analysis of numerical results. According to formula (10), the calculation and numerical analysis of the temperature distribution $\theta(x, y)$ in the plate were performed. The following initial data were selected: the material of the plate and inclusions is silicon ($\lambda_1 = 154.7$ W/(m · deg)) and silver ($\lambda_0 = 419$ W/(m · deg)) at a temperature of $t = 27$°C; $q_0 = 200$ W/m^2; $l = h = 0.005$ m; $H = 0.01$ m; $\alpha_- = 17.64$ W/(m^2 · deg). The change in temperature $\theta(x, y)$ depending on the spatial coordinates x and y is illustrated. From the behavior of the curves (Fig. 2, *a*) it is seen that the temperature as a function of the spatial coordinate y is smooth and monotonic and reaches maximum values at the boundary surface of the plate L_+, on which the heat flux is concentrated. Fig. 2, *b* shows the behavior of temperature depending on the coordinate x and determines the point (0; 0.005) at which the maximum value is reached. If the absolute value of the spatial coordinate x increases, the temperature decreases.

As can be seen from Fig. 2, *a, b*, there is no temperature jump on the inclusion surfaces K_0, K_R (the conditions of ideal thermal contact are met), which indicates the adequacy of the mathematical model to the real physical process. The number of partitions n and m of intervals (0; h) and (–H; H) is chosen to be equal to nine. As a result, the numerical experiment was performed with an accuracy of 10^{-6}.

5. 2. Mathematical model of heat transfer in a layer due to internal heating by a cylindrical source

The model of heat transfer in an isotropic layer with a semi-through cylindrical inclusion, in the region of which internal heat sources are uniformly concentrated, is described in paper [14]. As a result of analysis of (1) to (15) in [14], the temperature field in a layer with a semi-through thermally active cylindrical inclusion is described by formula (15) in [14], which makes it possible to determine the temperature at any point of the "layer-inclusion" structure. At the same time, the cited work did not perform a numerical analysis of the temperature distribution in the layer.

Analysis of numerical results. According to formula (15) in [14], the calculation and numerical analysis of the temperature distribution $\theta(r, z)$ in the layer were performed. Initial data: layer and inclusion material – silicon ($\lambda_1 = 154.7$ W/(m · deg)) and silver ($\lambda_0 = 419$ W/(m·deg)) at temperature $t = 27$°C; $q_0 = 200$ W/m^3; $l = h = 0.005$ m; $R = 0.01$ m; $\alpha_+ = 17.64$ W/(m^2 · deg). The determined numerical results for temperature are in Tables 1, 2.

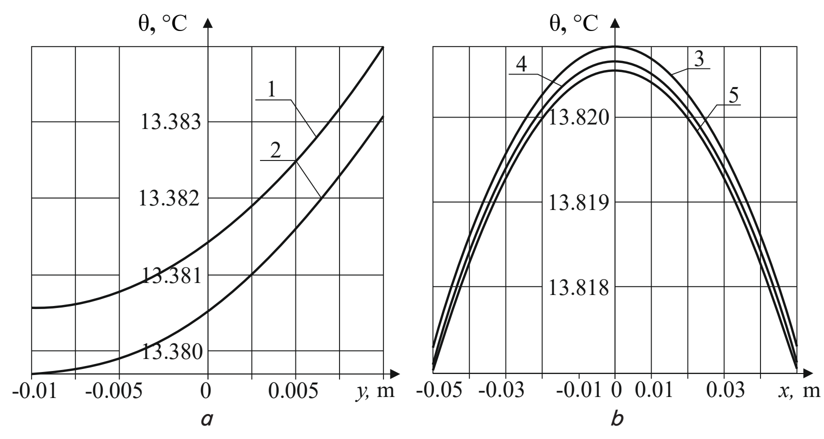

Fig. 2. Temperature $\theta(x, y)$ dependence on a – y coordinate for given values of the x coordinate; b – x coordinate for given values of the y coordinate; 1 – $x = 0$; 2 – $x = 0.01$; 3 – $y = 0.005$; 4 – $y = 0$; 5 – $y = -0.005$

Table 1

Temperature change depending on the spatial radial coordinate r (for $z = 0$)

r, m	0.00	0.02	0.04	0.06	0.08	0.10
t, °C	15.853	15.851	15.849	15.847	15.845	15.843

Table 2

Temperature change depending on spatial axial coordinate z (for $r = R$)

z, m	–0.005	–0.004	–0.003	–0.002	–0.001	0.000	0.001	0.002	0.003
t, °C	15.353	15.355	15.357	15.359	15.361	15.363	15.357	15.353	15.349

The number of partitions n and m of intervals $(0; h)$ and $(0; R)$ is chosen to be nine. As a result, the numerical experiment is performed with an accuracy of 10^{-6}.

6. Discussion of results based on construction of mathematical models of heat transfer in spatial environments with heating in canonical regions

The boundary value problems of heat conductivity are formulated taking into account the physical essence of the processes occurring in isotropic spatial environments with the presence of foreign semi-through elements. The integration of such elements into the environment leads to the fact that the generalized differential equations of heat conductivity contain discontinuous and singular coefficients, which reflects local changes in physical parameters due to foreign inclusions. The appearance of the curves in Fig. 2, which are constructed on the basis of the determined numerical values of temperature as a function of spatial coordinates, obtained using the analytical solution of the boundary value problem, indicates the correspondence of the results to the physical process. This is confirmed by the smoothness of the temperature as a function of spatial coordinates (the conditions of ideal thermal contact are met) on the surfaces of conjugation of inhomogeneous environments (plates and inclusions) $K_{\pm H}=\{(\pm H, y, z): 0 \le y \le h, |z| \le \delta\}$, $K_0=\{(x, 0, z): |x| \le H, |z| \le \delta\}$ and the fulfillment of the given boundary conditions.

The reported technique is based on the use of the theory of generalized functions, which made it possible to describe the thermophysical parameters of inhomogeneous media with canonical foreign elements. As a result, a single generalized differential equation of heat conductivity of the second order with discontinuous and singular coefficients and the corresponding boundary conditions on the boundary surfaces of the media were obtained. To determine the analytical-numerical solutions to boundary problems (1), (2) of our work and (1), (2) in [14], the method of segment-constant approximation of temperature as a function of spatial coordinates on the boundary surfaces of foreign inclusions (6), (10) [14] was applied. This approach made it possible to effectively use the integral Fourier and Hankel transforms, which provided the determination of analytical-numerical solutions in the form of (10) and (15) in work [14]. The temperature distribution is geometrically depicted in Fig. 2 and given in Tables 1, 2.

It is worth noting that existing approaches to solving boundary value problems of thermal conductivity for isotropic media, in particular presented in [1], do not make it possible to accurately reflect local temperature changes, which is critical for electrical exploration problems. Numerical modeling methods, such as FEM, are accompanied by the accumulation of errors [2–6]. The proposed analytical-numerical technique makes it possible to minimize these errors as they are limited only to the stages of numerical integration and segment-constant approximation of the temperature on the surfaces of foreign inclusions. And this provides a high level of accuracy of the results, unattainable when using traditional experimental or numerical methods [7–12].

The use of generalized functions also makes it possible to adequately describe the geometric shapes of inhomogeneous media and locally concentrated heating zones, resulting in differential equations of heat conduction with partial derivatives and singular coefficients.

In modern electronic devices, the presence of localized thermally active nodes of canonical form prompts the need to construct mathematical models of heat transfer between their individual structural elements. Such models can take a linear or nonlinear form for isotropic spatially inhomogeneous media. Although the proposed models are simplified, they serve as a foundation for the further development of complex nonlinear models of heat transfer in thermally sensitive composite media.

Based on our analytical-numerical solutions to boundary value problems of heat transfer, it is possible to develop computational algorithms and software for their numerical implementation. This will allow for further research into materials used in modern digital electronic devices to improve thermal stability and increase durability.

It is recommended to take into account the presence of foreign elements in structures for the analysis of thermal regimes, which significantly increases the complexity of solving boundary value problems of thermal conductivity, but provides a more accurate reflection of physical processes and the behavior of the temperature field as a function of spatial coordinates.

Our study was performed for a stationary thermal conductivity process, so the models are limited to determining the temperature only by spatial coordinates. The boundary value problems were solved for a medium with one foreign element, which does not reduce the generality of the study.

The disadvantage of the study is the simplification of models, which do not take into account nonlinear effects of heat transfer in heterogeneous structures, as well as the lack of experimental verification. The disadvantage associated with the simplification of models will be eliminated in further studies by devising more complex models. As for the experimental data, there are certain difficulties associated with reproducing localized thermal processes in real designs of electronic devices.

Further research will focus on the construction of mathematical models of heat transfer for media with several foreign elements, for a non-stationary heat conduction process, as well as taking into account the influence of thermal radiation.

7. Conclusions

1. A mathematical model of heat transfer in the structural units of electronic devices containing a foreign element of a parallelepiped shape, under the action of a heat flux locally concentrated in a rectangular region on the boundary surface of the medium, has been built. The solution to the corresponding boundary value problem is an analytical-numerical expression in the form of an improper integral with an infinite upper limit. In order to determine the approximate values of the temperature on the surfaces of the foreign element, a system of linear algebraic equations has been formed, the coefficients of which contain integrals with infinite limits. As a result of performing certain mathematical transformations, these integrals have been reduced to integrals with finite limits. Further, the use of numerical integration according to the 3/8 Newton method provided the determination of the numerical values for coefficients in the system of linear equations, which was solved by the iterative method. The determination of the temperature distribution in the space of the medium obtained in this way was achieved with an accuracy of 10^{-6}, which significantly exceeds the possibilities of using traditional numerical methods or experimental measurements to solve similar boundary value problems.

2. A mathematical model of heat transfer in structural units of electronic devices with a foreign cylindrical element in which internal heat sources are evenly distributed has been constructed. An analytical-numerical solution has been obtained for the corresponding boundary value problem, on the

basis of which, using numerical integration of the improper integral, the temperature values for selected geometric and thermophysical parameters have been determined with an accuracy of 10^{-6}. A numerical experiment confirms that to ensure such accuracy, the segment-constant approximation of the temperature on the conjugation surfaces of inhomogeneous materials requires nine divisions of the corresponding intervals.

Conflicts of interest

The authors declare that they have no conflicts of interest in relation to the current study, including financial, personal, authorship, or any other, that could affect the study and the results reported in this paper.

Funding

The study was conducted without financial support.

Data availability

All data are available in the main text of the manuscript.

Use of artificial intelligence

The authors confirm that they did not use artificial intelligence technologies when creating the current work.

Authors' contributions

Vasyl Havrysh: Conceptualization, Methodology, Formal analysis, Writing – original draft; **Svitlana Yatsyshyn**: Software, Visualization, Writing – review & editing; **Mykhailo Stepaniak**: Investigation, Resources, Validation; **Andrii Kapustianskyi**: Investigation, Resources, Validation; **Lubov Kolyasa**: Conceptualization, Methodology.

References

1. Zhuravchak, L. M., Zabrodska, N. V. (2025). Solving inverse problem of the potential theory by the cascade algorithm and the near-boundary element method. Mathematical Modeling and Computing, 12 (4), 1243–1253. https://doi.org/10.23939/mmc2025.04.1243
2. Bartwal, N., Shahane, S., Roy, S., Vanka, S. P. (2023). Simulation of heat conduction in complex domains of multi-material composites using a meshless method. Applied Mathematics and Computation, 457, 128208. https://doi.org/10.1016/j.amc.2023.128208
3. Łach, Ł., Svyetlichnyy, D. (2025). Advances in Numerical Modeling for Heat Transfer and Thermal Management: A Review of Computational Approaches and Environmental Impacts. Energies, 18 (5), 1302. https://doi.org/10.3390/en18051302
4. Channouf, S., Benhamou, J., Jami, M. (2024). Investigating convective and conductive heat transfer in square and circular heated bodies: A novel approach using coupled Runge-Kutta and lattice Boltzmann method. Thermal Science and Engineering Progress, 49, 102441. https://doi.org/10.1016/j.tsep.2024.102441
5. Bi, D., Jiang, M., Chen, H., Liu, S., Liu, Y. (2020). Effects of thermal conductivity on the thermal contact resistance between non-conforming rough surfaces: An experimental and modeling study. Applied Thermal Engineering, 171, 115037. https://doi.org/10.1016/j.applthermaleng.2020.115037
6. Shen, F., Li, Y.-H., Güler, M. A., Wu, H.-D., Shen, W.-W., Ke, L.-L. (2025). A high-efficiency prediction method for thermal contact resistance of rough surfaces. International Communications in Heat and Mass Transfer, 167, 109325. https://doi.org/10.1016/j.icheatmasstransfer.2025.109325
7. Jiang, G., Chen, W., Chen, J., Yang, W. (2026). Experimental Investigation of Thermal Contact Resistance at Flat/Curved Surface Interfaces Under Various Temperature, Pressure, and Surface Roughness Levels. Technologies, 14 (1), 41. https://doi.org/10.3390/technologies14010041
8. Chumak, K. A., Martynyak, R. M. (2018). Effective Thermal Contact Resistance of Regularly Textured Bodies in the Presence of Intercontact Heat-Conducting Media and the Phenomenon of Thermal Rectification. Journal of Mathematical Sciences, 236 (2), 160–171. https://doi.org/10.1007/s10958-018-4103-7
9. Silva, D. (2022). Modeling the Transient Response of Thermal Circuits. Applied Sciences, 12 (24), 12555. https://doi.org/10.3390/app122412555
10. Chandra, S., Chowdhury, S. S., Roy, K. (2025). 2D-ThermAl: Physics-Informed Framework for Thermal Analysis of Circuits using Generative AI. IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems. https://doi.org/10.1109/tcad.2025.3642715
11. Padmanabhan, N. (2024). A Transient Thermal Model for Power Electronics Systems. SoutheastCon 2024, 1294–1299. https://doi.org/10.1109/southeastcon52093.2024.10500091
12. Zorzetto, M., Torchio, R., Lucchini, F., Massei, S., Robol, L., Dughiero, F. (2024). Reduced Order Modeling for Thermal Simulations of Electric Components With Surface-to-Surface Radiation. IEEE Access, 12, 178117–178126. https://doi.org/10.1109/access.2024.3507367
13. Havrysh, V., Kochan, V. (2023). Mathematical Models to Determine Temperature Fields in Heterogeneous Elements of Digital Devices with Thermal Sensitivity Taken into Account. 2023 IEEE 12th International Conference on Intelligent Data Acquisition and Advanced Computing Systems: Technology and Applications (IDAACS), 983–991. https://doi.org/10.1109/idaacs58523.2023.10348875
14. Havrysh, V., Kolyasa, L. (2026). Mathematical modeling and analysis of heat transfer in structures with foreign elements. Naukovyi Visnyk Natsionalnoho Hirnychoho Universytetu, 1, 34–42. https://doi.org/10.33271/nvngu/2026-1/034