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CONSTRUCTION OF MATHEMATICAL MODELS OF HEAT EXCHANGE IN SEMI-INFINITE ENVIRONMENTS WITH FOREIGN INCLUSIONS

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This study investigates heat transfer processes in isotropic semi-infinite 3D media with foreign inclusions, which are heated by internal sources and heat flow.

As a result of the thermal load during the operation of devices, significant temperature gradients arise. To analyze the temperature regimes and establish the effective operation of these devices, mathematical models for determining temperature fields have been constructed.

Based on the stated boundary value problems of thermal conductivity, their analytical solutions have been derived, which make it possible to obtain the temperature distribution and the behavior of temperature gradients in a heterogeneous medium. Using these solutions, numerical calculations of the temperature distribution in spatial coordinates for given geometric and thermophysical parameters have been performed.

For an effective description of the thermophysical parameters of heterogeneous semi-infinite 3D media, a symmetric unit function and the Dirac delta function have been used. As a result, second-order differential equations with partial derivatives and singular coefficients have been derived.

The numerical results reflect the temperature distribution in semi-infinite 3D media in spatial coordinates for the given geometric and thermophysical parameters. The numerical values of temperature for the selected half-space material (ceramics VK94-I) and inclusions (silicon, molybdenum) were obtained with an accuracy of 10-6.

The application of the constructed mathematical models of heat transfer contributes to investigating the thermal stability in semi-infinite 3D media with foreign inclusions. Using these models makes it possible to predict temperature regimes in devices, which is a prerequisite for improving their reliability and durability

Keywords: temperature field, thermal conductivity of the material, thermal stability of structures, convective heat transfer, heat flow

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1. Introduction

Heat exchange processes are of great importance for the intensification of heat-energy, energy, and chemical-technological processes in industry with high efficiency; their consideration is important for various physical processes occurring in nature and technology. The temperature regime of the environment and residential premises depends on these processes, they determine the course of the work process in various technological installations, etc. Therefore, it is not surprising that the theory of heat exchange has developed intensively, especially in recent decades due to the needs of public utilities, nuclear power, astronautics, and many other areas of human activity. Methods of thermal protection of high-speed aircraft, primarily reusable space aircraft, in the cores of reactors, in magnetohydrodynamic generators (installations for direct conversion of heat into electrical energy), in gas turbine installations, etc. are in demand. For technical devices that operate at low temperatures, heat exchange processes are being studied, in particular in installations using the superconductivity effect, for example, in magnets, where significant electromagnetic fields arise. Research is continuing on the design of cryosurgical instruments, which are used for operations with rapid freezing of individual areas of living tissue. Progress in this area of studies is largely associated with the correct organization of heat exchange processes both in the instrument itself and in the tissue of a living organism. Research is being carried out on the fabrication of installations for drying certain food products, the successful implementation of which depends on the correct organization of the sublimation and desublimation processes.

In this case, mathematical models of the heat transfer process are constructed, which take into account the geometric structure and the physical process. Using these models, it is possible to predict the behavior of temperature gradients as a result of their operation. Subsequently, algorithms and software tools can be developed on their basis for practical use, as well as certain studies can be performed. In particular, to select effective structural materials, to design geometric structures of individual elements and assemblies, perform experiments on the installation of effective heat dissipation elements and their placement. As a result, it will be possible to increase the reliability and durability of structures at the design stage, without conducting expensive experiments.

Therefore, the construction of mathematical models of heat transfer for isotropic semi-infinite 3D environments with foreign elements of various geometric shapes is a relevant task.

2. Literature review and problem statement

In [1], a three-dimensional model for solving the problem of thermal conductivity in a steady state in a semi-infinite region containing a cubic inhomogeneity was reported. Analytical relations were derived that determine the temperature distribution in spatial coordinates in the given medium. Subsequently, the mathematical model was modified for inhomogeneities of an arbitrary canonical form. The model does not take into account the presence of internal heat sources, which significantly limits its application for the analysis of thermal processes in electronic devices, where heat generation in a small volume of the inclusion is a determining factor in the formation of the temperature field.

Paper [2] considers the construction of a continuum multiphase mathematical model of non-stationary thermal conductivity, in which each component of the composite is described by its own temperature field, and the interphase interaction is taken into account by heat exchange parameters. Based on the dynamic homogenization method, a closed system of coupled thermal conductivity equations was built that adequately describe the local thermal imbalance between the phases. The results correlate well with the numerical experiment. At the same time, a significant limitation of the model is the lack of detailed consideration of internal heat sources, which are concentrated in inhomogeneities with small volumes. This narrows the application of the given model for thermal problems of microelectronics. In such problems, Joule heat release, local thermal generation in foreign elements, and other thermally active volumetric sources are decisive for the formation of thermal fields and their gradients due to the functioning of devices.

Work [3] considers analytical-numerical modeling of heat transfer in an elastic half-space with ellipsoidal inclusions under the conditions of frictional heating and spatially inhomogeneous intrinsic temperature gradients. The mathematical model is based on the Fourier equation of unsteady thermal conductivity and the right-hand side due to heating by heat sources due to friction. The disadvantages of the model are the use of the classical Fourier law, which does not make it possible to take into account the finite rate of heat propagation and possible non-local effects. The model also simplified the frictional heating as a given source without a detailed thermomechanical connection with the conditions of thermomechanical contact. In addition, the temperature dependence of the thermophysical properties of materials, phase transitions, and possible microcontact effects on the surfaces of inclusions are not taken into account, which limits the applicability of the model to micro- and nanoscale tasks, in particular in microelectronics and high-intensity tribological systems.

Paper [4] reports an analytical study of steady-state thermal conductivity in a two-dimensional medium with inhomogeneities that do not significantly affect the behavior of the temperature field on the external surface. The mathematical model is built on the basis of the steady-state Laplace equation, in which inhomogeneities are described by special boundary conditions of conjugation. The limitations of the model are that it considers only the steady-state regime of thermal conductivity, which does not make it possible to analyze transient (dynamic) heat transfer processes. In addition, the influence of internal heat sources, which are critical for many practical applications, in particular in microelectronics and energy systems, is not taken into account. The model is based on the linear Fourier law and does not take into account nonlinear effects, temperature dependence of the materials of the medium, and possible interface thermal resistances, which limits its applicability to real multiphase and microstructured systems.

Work [5] highlights the construction of a mathematical model for detecting and reconstructing inclusions in an inhomogeneous heat-conducting plate due to the action of external heat flows and internal heat sources. The model is based on the quasi-stationary Fourier heat conduction equation with spatially varying heat conduction coefficients taking into account internal heat generation and boundary conditions of heat exchange with the environment. To identify foreign inclusions, the inverse problem is solved using the topological optimization method, which makes it possible to restore the geometric parameters of inhomogeneities based on temperature fields and heat flows. The model built describes internal heat sources in a simplified form and does not take into account complex physical processes, in particular, Joule heating in electronic structures and thermochemical reactions. The model specifies classical conditions for external heat transfer, which do not allow it to be used in problems with uncertain or variable convective flows.

Study [6] considers the construction of a mathematical model of harmonic heat transfer in media with inhomogeneities using analytical methods. A mathematical model of heat transfer in the frequency domain has been built, where the non-stationary Fourier process of heat conduction is transformed into a harmonic form with complex temperature amplitudes. The thermal behavior of an infinite medium with an inclusion has been determined, which is described by analytical solutions that reflect the influence of geometric and thermophysical parameters of the inclusion on the harmonic temperature distribution. The method of potentials has been used, which made it possible to formulate generalized conditions for inhomogeneities in a heat-conducting medium.

The model is limited as it was constructed on the basis of the linear Fourier law and the harmonic regime, which assumes stationary sinusoidal processes and does not make it possible to describe complex transient or pulsed thermal phenomena.

Work [7] reports development of a combined method for modeling stationary (quasi-stationary) heat transfer in inhomogeneous media with a number of foreign inclusions of arbitrary canonical form. To build a mathematical model, the classical Fourier heat conduction equation with thermal conductivity coefficients dependent on spatial coordinates was used. The influence of inhomogeneities was taken into account in the boundary conditions of an ideal thermal contact. To solve the boundary problem, analytical solutions of fundamental equations were used using discretization of foreign inclusions of complex geometric shape. This makes it possible to effectively model multicomponent structures without a complete transition to computational procedures. The model does not take into account the temperature dependence of the thermophysical properties of the structural materials of the medium. A random distribution of inclusions is not assumed, which reduces the universality of the model for real natural and engineering materials.

Paper [8] considers the construction of a combined model of three-dimensional non-stationary thermal conductivity due to heating by a dynamic heat source of arbitrary shape. The model is built on the basis of the non-stationary Fourier heat conductivity equation taking into account heating by heat sources depicted by an integral representation and superposition of elementary solutions. The method makes it possible to effectively determine temperature fields for complex trajectories and geometric shapes of heat sources, without taking into account the full numerical discretization of the area. This is important in laser heating of the medium, additive manufacturing and heat treatment of materials. The use of the model ensures the efficiency of obtaining numerical results and their high accuracy compared to the use of traditional numerical methods. The model does not take into account non-local and wave effects of heat transfer for high-speed heating processes. In addition, the thermophysical properties of the material are constant, which limits the accuracy for high temperature regimes. The model does not take into account phase transitions, evaporation, thermomechanical deformations, and complex multi-physical processes characteristic of real additive manufacturing technologies.

Papers [1–8] report the results of studies on heat conduction processes in heterogeneous media with foreign inclusions of various geometric shapes. It is shown that the use of analytical and combined analytical-numerical approaches makes it possible to derive temperature field distributions in stationary and non-stationary regimes, to take into account the influence of geometric and thermophysical parameters of inhomogeneities, as well as study temperature perturbations due to the presence of inclusions and external heat sources. The effectiveness of the constructed models is established; their adequacy is confirmed by comparison with experimental results.

At the same time, the task to build analytical models of heat transfer in spatial semi-infinite media with small thermally active inhomogeneities remains unsolved. Most analyzed models do not take into account localized internal heat sources in small-volume inhomogeneities, as well as the temperature dependence of thermophysical parameters. In addition, nonlocal heat transfer effects, multi-scale processes, and complex conditions of thermal interaction on the surfaces of conjugation of inhomogeneous materials remain out of consideration. The reason is the significant mathematical complexity in determining analytical solutions for three-dimensional inhomogeneous media with a complex structure, as well as the need to take into account a significant number of interconnected physical factors.

Papers [9, 10] show that modern numerical methods, in particular the lattice Boltzmann method in combination with the Runge-Kutta method, as well as finite element and finite volume methods, provide effective modeling of heat transfer processes in complex technical structures. These approaches make it possible to take into account the geometric features of structures, thermophysical properties of materials, and the presence of local heat sources, which is important for the analysis of thermal regimes of electronic devices. At the same time, it is possible to form the basis for the construction of classical 3D mathematical models of heat transfer in heterogeneous media.

However, a number of limitations of these approaches have been identified in the context of constructing 3D heat transfer models with a certain accuracy. The main drawback is the significant computational complexity of mesh methods in modeling small-scale local heating areas and foreign inclusions typical of modern micro- and mobile devices. In addition, the use of averaged physical parameters of the environment leads to the loss of information about microstructural effects, which reduces the accuracy of reproducing temperature gradients. Also, existing approaches have a limited ability to correctly take into account the multi-scale nature of heat transfer processes, which complicates their use for building universal 3D mathematical models with a high level of spatial detail.

In work [11], a mathematical model of the thermomechanical behavior of multilayer rectangular plates is given, taking into account the temperature dependence of the physical and mechanical properties of the material. The reported method is based on a combination of heat conduction equations and equations of deformable solid mechanics, which makes it possible to analyze the relationship between temperature fields and the stress-strain state of structures. The constructed model makes it possible to increase the accuracy of modeling thermal processes in comparison with classical linear statements due to taking into account the nonlinear dependence of structural parameters on temperature.

At the same time, it was found that the above technique contains a number of limitations in the context of construction of modern 3D mathematical models of heat transfer in complex media. In particular, the model is focused mainly on layered plates, and its use does not provide an adequate description of three-dimensional thermal processes in semi-infinite media with complex geometry and the presence of foreign inclusions. In addition, the use of simplified geometric assumptions and averaged structural parameters limits the possibility of taking into account local heat transfer effects that arise in the vicinity of inclusions and areas of conjugation of heterogeneous media. The model also does not take into account the multi-scale nature of thermal processes and the influence of local heat sources of a complex 3D configuration.

Study [12] reports modeling the thermal conductivity using fractional derivatives and temperature-dependent material parameters. The technique is aimed at increasing the accuracy of the description of non-stationary heat transfer processes taking into account the economy of the medium memory for materials of complex architecture with a heterogeneous structure. To derive a numerical solution to the corresponding fractional differential equation, certain computational methods were used that make it possible to analyze the dynamics of temperature field with time variation and evaluate the influence of the temperature dependence of physical parameters.

At the same time, the model does not provide for the heat transfer process in semi-infinite media with foreign inclusions. In particular, the case of a 3D model with inclusions and interface effects for heterogeneous materials is not taken into account, which is critical for microstructured composites. In addition, the use of averaged medium parameters does not make it possible to take into account local concentrations of heat fluxes on the surfaces of the mating surfaces. The use of fractional models in large-scale 3D computational schemes limits their use for full spatial modeling of heat transfer in semi-infinite media with canonical inclusions.

The results of heat transfer studies using modern numerical methods and generalized mathematical approaches are reported in [9–12]. It is shown that the finite element method, the finite volume method, the lattice Boltzmann method, and models based on fractional derivatives provide high accuracy in modeling complex heat transfer processes. The use of these approaches makes it possible to take into account the presence of local heat sources and the inhomogeneous structure of materials.

However, the task associated with the effective modeling of temperature fields in spatial semi-infinite media with small-scale thermally active inclusions remains unsolved. The reason is the significant computational complexity of numerical methods, the need to apply small computational grids, significant costs of computational resources, and the complexity of taking into account multi-scale heat transfer effects. This limits the possibility of effective analysis of thermal regimes of modern microelectronic devices.

In study [13], a mathematical model for determining the temperature field in a thermally sensitive layered plate with a rectangular foreign inclusion is given. The use of the constructed model does not make it possible to analyze temperature regimes in a 3D thermally sensitive semi-infinite media with a parallelepipedal thermally active inclusion.

A number of studies report mathematical modeling of temperature regimes in thermally sensitive media with foreign inclusions. In particular, in [14] a nonlinear model was proposed, which takes into account thermally sensitive elements with semi-through foreign inclusions and linearization was applied to reduce the axisymmetric nonlinear boundary value problem to a quasi-linear one. However, this model does not provide for internal thermal heating in the semi-infinite region. This reduces its suitability for predicting the behavior of temperature fields under real operating conditions of microelectronic devices for semi-infinite environments.

In [15, 16], an analytical model for determining temperature fields and analyzing thermal regimes was built without a detailed study on the influence of thermophysical and geometric parameters of the medium and foreign inclusions.

The results of studies on temperature regimes in thermosensitive media with foreign inclusions are described in papers [13–16]. It is shown that the constructed mathematical models make it possible to determine temperature distributions and analyze the influence of geometric and thermophysical parameters of inclusions on thermal processes in media with temperature dependence of the properties of structural materials.

However, the models built do not allow for the analysis of heat transfer in thermosensitive semi-infinite media in the presence of small thermally active inclusions. And they do not provide for full consideration of internal thermal heating, which is characteristic of the functioning of modern electronic systems; there is no analysis of the influence of small geometric parameters of thermally active elements on the formation of temperature fields. The reason is the complexity of building nonlinear spatial models with localized heat sources and temperature dependence of material properties.

An option for overcoming these difficulties may be the use of an analytical technique based on the construction of three-dimensional mathematical models of heat transfer for isotropic semi-infinite media with small inhomogeneous thermally active elements. This approach makes it possible to obtain analytical dependences for determining temperature fields, reduce computational costs compared to the use of traditional numerical methods, and ensure high accuracy of modeling local thermal processes.

The above gives grounds to argue that it is advisable to conduct research into the construction of analytical mathematical models of heat transfer in isotropic spatial semi-infinite media with small thermally active inhomogeneities. The use of such models will contribute to increasing the efficiency of designing modern electronic devices, as well as ensuring their reliability and durability during operation.

3. The aim and objectives of the study

The purpose of our study is to build mathematical models of heat transfer for isotropic spatial semi-boundary environments with small inhomogeneous heat-active elements. The mathematical models to be constructed will provide increased accuracy in predicting temperature regimes and effective means for optimizing the design of modern electronic devices.

To achieve this aim, the following objectives were accomplished:

– to build a mathematical model of heat transfer in an isotropic half-space with a foreign inclusion of a parallelepiped shape, in the region of which heat sources are concentrated;

– to build a mathematical model of heat transfer in an isotropic half-space with a foreign inclusion of a parallelepiped shape, which is heated by a heat flux at the boundary surface.

4. The study materials and methods

The object of our study is the heat transfer processes in isotropic semi-infinite 3D media with foreign inclusions, which are heated by internal sources and heat flow.

The principal hypothesis assumes that the temperature distribution in spatial coordinates in such media, caused by combined heating, can be determined by analytical solutions to the corresponding boundary value problems of heat conduction. In these problems, inhomogeneous differential equations with partial derivatives of the second order contain singular coefficients and the right-hand side, which arise as a result of the presence of foreign inclusions.

The study assumes the isotropic nature of the spatial medium, i.e., the thermophysical parameters are constant along the spatial coordinate axes. The constructed mathematical models of heat transfer are simplified as the determination of temperature fields and the analysis of thermal regimes are carried out for foreign inclusions, the geometric parameters of which are sufficiently small.

To model thermal conductivity in the above media with foreign inclusions, the singular Dirac delta function is used. Although the local inhomogeneity of the layer, described by the Dirac delta function, is formally concentrated at the origin of the coordinate system, it is characterized by finite dimensions, which are related to the volume of the foreign inclusion. As a result, the finite dimensions of the inclusion are effectively taken into account.

A spatial medium described by an isotropic half-space is considered, which contains a parallelepiped inclusion with a volume $V_0 = 8hbd$ at a distance l, in the region of which uniformly concentrated internal heat sources with a power $q_0 = \text{const}$ operate. On the boundary surface of the half-space $K = \{(x, y, -d - l)$:):$|x| < \infty$, $|y| < \infty\}$ there is convective heat exchange with the surrounding environment with a constant temperature $t_c = \text{const}$. The conditions of ideal thermal contact are given on the surfaces of the inclusion. The above structure is assigned to the Cartesian rectangular coordinate system (x, y, z) with the origin O at the center of the inclusion (Fig. 1).

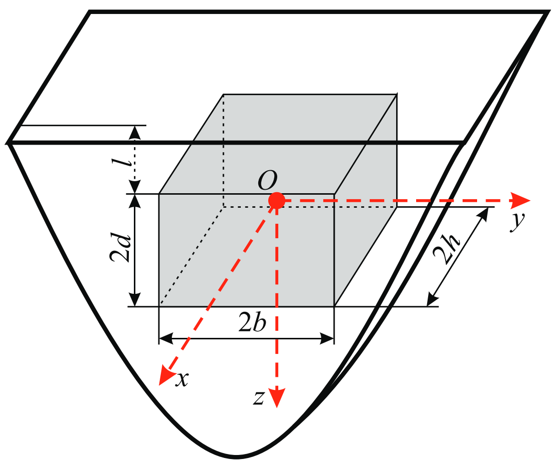


Fig. 1. Isotropic half-space with thermally active inclusion

In the above structure, the temperature distribution $t(x, y, z)$ in spatial coordinates x, y, and z is determined by solving the heat conduction equation

$$\operatorname{div}\left[\lambda(x,y,z)\operatorname{grad}\theta(x,y,z)\right] = -q_0 N(x,y,z), \tag{1}$$

under boundary conditions:

$$\theta(x,y,z)\big|_{|x|\to\infty} = \theta(x,y,z)\big|_{|y|\to\infty} = \theta(x,y,z)\big|_{z\to\infty} = 0,$$
$$\lambda_1 \frac{\partial\theta(x,y,z)}{\partial z}\bigg|_{z=-d-l} = \alpha\,\theta(x,y,z)\big|_{z=-d-l}, \tag{2}$$

where $\lambda(x, y, z) = \lambda_1 + (\lambda_0 - \lambda_1)\, N(x, y, z)$ is the thermal conductivity coefficient of the inhomogeneous half-space; λ_1, λ_0 are the thermal conductivity coefficients of the half-space material and the inclusion, respectively; α is the heat transfer coefficient from the surface K; $\theta(x, y, z) = t(x, y, z) - t_c$; $N(x, y, z) = S(h - |x|)S(b - |y|)S(d - |z|)$; $S(\zeta)$ is the symmetric unit function

$$S(\zeta) = \begin{cases} 1, & \zeta > 0,\\ 0{,}5, & \zeta = 0,\\ 0, & \zeta < 0.\end{cases}$$

Given that the geometric parameters of the inclusion b, h, d are small compared to distance l from its boundary surface $\Pi = \{(x, y, -d)$:$|x| \le h$, $|y| \le b\}$ to the boundary surface K of the half-space, to determine the temperature field $\theta(x, y, z)$ in the given medium, a second-order inhomogeneous differential equation with partial derivatives and singular coefficients and the right-hand side is obtained:

$$\Delta\theta + \frac{\Lambda_0}{\lambda_1}\left[\begin{aligned} &\frac{\partial\theta}{\partial y}\bigg|^{*}_{x=0}\delta'(x)\delta(y,z) + \\ &+\frac{\partial\theta}{\partial y}\bigg|^{*}_{y=0}\delta'(y)\delta(x,z) + \\ &+\frac{\partial\theta}{\partial z}\bigg|^{*}_{z=0}\delta'(z)\delta(x,y)\end{aligned}\right] = -\frac{Q_0}{\lambda_0}\delta(x,y,z). \tag{3}$$

Here $\delta(x,y,z) = \delta(x)\delta(y)\delta(z)$; $\delta(\zeta) = dS(\zeta) / d\zeta$ is the symmetric Dirac delta function; $\Lambda_0 = \lambda_0 V_0$ is the combined thermal conductivity coefficient of the inclusion; $Q_0 = q_0 V_0$ is the combined power of the operating internal heat sources:

$$\frac{\partial\theta}{\partial\zeta}\bigg|^{*}_{\zeta=0} = \frac{1}{2}\left[\frac{\partial\theta}{\partial\zeta}\bigg|_{\zeta=+0} + \frac{\partial\theta}{\partial\zeta}\bigg|_{\zeta=-0}\right];$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2};$$

Δ is the Laplace operator in a Cartesian rectangular coordinate system.

The case is considered when a spatial medium is described by an isotropic half-space containing at a distance l an inclusion of a parallelepiped shape with a volume $V_0 = 8hbd$. This medium is heated by a heat flux with a specific density $q_0 = \text{const}$ on the boundary surface of the half-space $K = \{(x, y, -d - l)$:):$|x| < \infty$, $|y| < \infty\}$ (Fig. 2).

The temperature distribution $\theta(x, y, z)$ in spatial coordinates in the given medium is determined by a second-order homogeneous differential equation with partial derivatives and singular coefficients:

$$\Delta\theta + \frac{\Lambda_0}{\lambda_1}\left[\begin{aligned} &\frac{\partial\theta}{\partial x}\bigg|^{*}_{x=0}\delta'(x)\delta(y,z) + \\ &+\frac{\partial\theta}{\partial y}\bigg|^{*}_{y=0}\delta'(y)\delta(x,z) + \\ &+\frac{\partial\theta}{\partial z}\bigg|^{*}_{z=0}\delta'(z)\delta(x,y)\end{aligned}\right] = 0, \tag{4}$$

under boundary conditions:

$$\theta(x,y,z)\big|_{|x|\to\infty} = \theta(x,y,z)\big|_{|y|\to\infty} = \theta(x,y,z)\big|_{z\to\infty} = 0,$$

$$\lambda_1 \frac{\partial\theta(x,y,z)}{\partial z}\bigg|_{z=-d-l} = -q_0\delta(x,y). \tag{5}$$

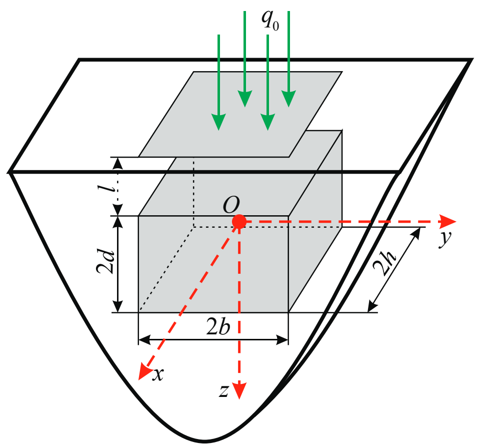


Fig. 2. Isotropic half-space with inclusion under the action of heat flux

Temperature gradients in the given semi-infinite 3D medium are completely determined by the obtained equation (4) and boundary conditions (5).

5. Results of research on mathematical models of heat transfer in semi-infinite media with foreign inclusions

5. 1. Mathematical model of heat transfer in a half-space due to heating by an internal source

The integral Fourier transform in coordinates x and y is applied to problem (2), (3). As a result, an ordinary second-order differential equation with constant coefficients and a singular right-hand side is obtained

$$\frac{d^2\bar{\theta}(z)}{dz^2}-\gamma^2\bar{\theta}(z)=-P_1\delta(z)-P_2\delta'(z), \qquad (6)$$

under boundary conditions:

$$\bar{\theta}(z)\Big|_{z\to 0}=0,\quad \lambda_1\frac{d\bar{\theta}(z)}{dz}\Bigg|_{z=-d-l}=\alpha\bar{\theta}(z)\Big|_{z=-d-l}, \qquad (7)$$

where

$$P_1=\frac{Q_0}{2\pi\lambda_1};\quad P_2=\frac{\Lambda_0}{2\pi\lambda_1}\frac{\partial\theta(0,0,z)}{\partial z}\Bigg|_{z=0}^{*}.$$

The solution to the boundary value problem (6), (7) is defined in the form

$$\bar{\theta}(z)=\frac{1}{2}\left\{\begin{array}{l}\frac{P_1}{\gamma}\left[F_1(z)-F_2(z+d_1)\right]-\\ -P_2\left[F_2(z+d_1)+F_1(z)\operatorname{sign}(z)\right]\end{array}\right\}. \qquad (8)$$

Here:

$$F_1(z)=\exp(-\gamma|z|);\ F_2(z)=\frac{\alpha-\lambda_1\gamma}{\alpha+\lambda_1\gamma};\ d_1=2(d+l);$$

$$\operatorname{sign}(\zeta)=\begin{cases}1, \zeta>0,\\ 0, \zeta=0,\\ -1, \zeta<0.\end{cases}$$

$\bar{\theta}(y)$ – transformant of a function $\theta(x, y)$

$$\bar{\theta}(y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\xi x}\,\theta(x,y)\,\mathrm{d}x;$$

ξ – parameter of the integral Fourier transform, $i^2=-1$;

The inverse integral Fourier transform is applied to relation (8) and an expression for the desired dimensionless temperature $T(X, Y, Z)$ is obtained

$$T(X,T,Z)=\frac{\lambda_1 l}{Q_0}\theta(x,y,z)=$$
$$=\frac{1}{2}\left\{\begin{array}{l}\frac{1}{2\pi}\left[\begin{array}{l}\varphi^{-\frac{1}{2}}(X,Y,Z)+\varphi^{-\frac{1}{2}}(X,Y,Z+D)-\\ -Bi\psi(X,Y,Z+D)\end{array}\right]+\\ +P_2\left[\begin{array}{l}Z\varphi^{-\frac{3}{2}}(X,Y,Z)-(Z+D)\varphi^{-\frac{3}{2}}(X,Y,Z)+\\ +Bi\psi_1(X,Y,Z+D)\end{array}\right]\end{array}\right\}, \qquad (9)$$

where:

$X=x/l;\ Y=y/l;\ Z=z/l;\ D=d_1/l;$

$\varphi(X, Y, Z)=X^2+Y^2+Z^2;$

$Bi=\alpha_z l/\lambda_1$ – Biot criterion:

$$\psi(X,Y,Z)=2\exp(BiZ)\times$$
$$\times\int_Z^{\infty}\exp(-BiZ)\varphi^{-\frac{1}{2}}(X,Y,Z)\mathrm{d}Z;$$

$$\psi_1(X,Y,Z)=2\exp(BiZ)\times$$
$$\times\int_Z^{\infty}Z\exp(-BiZ)\varphi^{-\frac{3}{2}}(X,Y,Z)\mathrm{d}Z;$$

$$\frac{\partial\theta(0,0,z)}{\partial z}\Bigg|_{z=0}^{*}=$$
$$=\frac{\frac{1}{d^2}+\frac{1}{4(d+l)^2}+\frac{Bi}{l}\left(\frac{Bi}{l}\psi(0,0,D)-\frac{1}{l+d}\right)}{l\left\{4\pi+\frac{\Lambda_0}{\lambda_1}\left[\frac{1}{4(d+l)^3}+\frac{Bi}{l}\left(\frac{Bi}{l^2}-\frac{1}{2(d+l)^2}\right)\right]\psi_1(0,0,D)\right\}}.$$

In the partial case for $Bi=0$, relation (9) will take the following form

$$T(X,T,Z)=$$
$$=\frac{1}{2}\left\{\begin{array}{l}\varphi^{-1/2}(X,Y,Z)\left[\frac{1}{2\pi}+ZP_2\varphi^{-1}(X,Y,Z)\right]+\\ +\varphi^{-1/2}(X,Y,Z+D)\times\\ \times\left[\frac{1}{2\pi}-(Z+D)P_2\varphi^{-1}(X,Y,Z+D)\right]\end{array}\right\}. \qquad (10)$$

Here

$$\frac{\partial\theta(0,0,z)}{\partial z}\Bigg|_{z=0}^{*}=\frac{\frac{1}{d^2}+\frac{1}{4(d+l)^2}}{l\left(4\pi+\frac{\Lambda_0}{4\lambda_1(d+l)^3}\psi_1(0,0,D)\right)}.$$

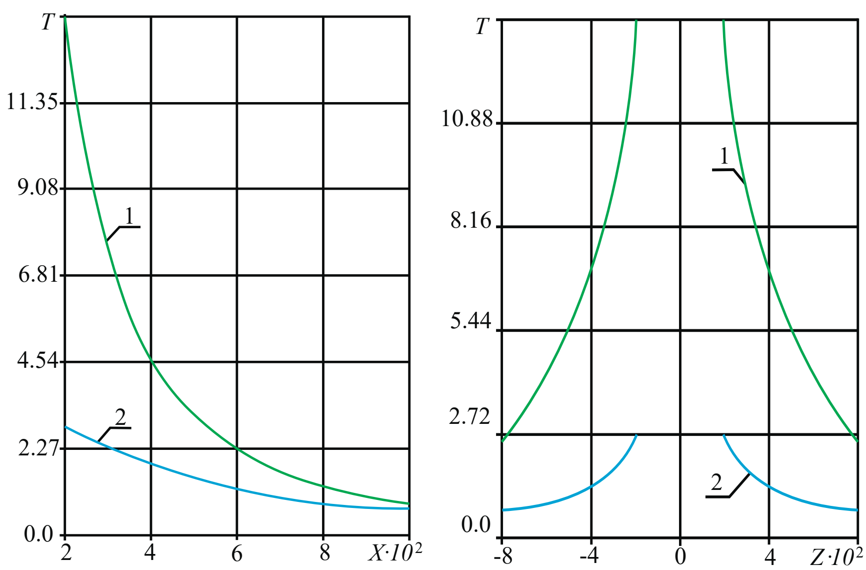
Analysis of numerical results. According to formulae (9), (10) for $Y = 0$, numerical calculations were performed and the behavior of the temperature field $T(X,Y,Z)$ was investigated. The following initial data were selected: medium material – VK94-I ceramics; inclusion material – molybdenum ($\lambda_0 = 162$ W/(deg·m)]; $h / l = d / l = d / l = 0.02$. The numerical results that make it possible to establish the change in the dimensionless temperature $T(X, Y, Z)$ along the spatial dimensionless coordinates X and Z for $Bi = 1$ are shown in Fig. 3; and values of the dimensionless temperature $T(X, Y, Z)$ for $-22 \leq Z \cdot 10^2 \leq -2$ and $Bi = 0; 1; 5$ are given in Table 1.

Table 1

Dependence of the dimensionless temperature $T(X, Y, Z)$ on the spatial dimensionless coordinate Z ($-22 \leq Z \cdot 10^2 \leq -2$) for different values of the Biot criterion

$-Z \cdot 10^2$	*Bi*		
	0	1	5
22	0.6502	0.5924	0.5769
18	0.8498	0.7928	0.7781
14	1.2075	1.1514	1.1371
10	1.9731	1.9177	1.9038
6	4.2007	4.1460	4.1324
2	13.6778	13.6231	13.6094

From Fig. 3 it is seen that the dimensionless temperature $T(X, Y, Z)$, as a function of the dimensionless spatial coordinates X, Y, Z, monotonically increases with decreasing values of coordinates X and $|Z|$. It reaches its greatest value in the region of concentrated internal heat sources (for a half-space with a foreign inclusion – curve 1; for a homogeneous half-space – curve 2). It is seen that the presence of a foreign thermally active inclusion leads to a significant increase in temperature. For values of the dimensionless spatial coordinate $|Z| \leq 0.12$, the symmetry of the temperature distribution relative to plane $Z = 0$ is observed.



a *b*

Fig. 3. Dependence of the dimensionless temperature $T(X, Y, Z)$ on the dimensionless coordinates: a — X for $Z = 0.02$; b — Z for $X = 0.02$; 1 — for a half-space with an inclusion; 2 — for a homogeneous half-space

Fig. 4 illustrates the dependence of dimensionless temperature $T(X, Y, Z)$ on the spatial dimensionless coordinate Z for different values of the Biot criterion.

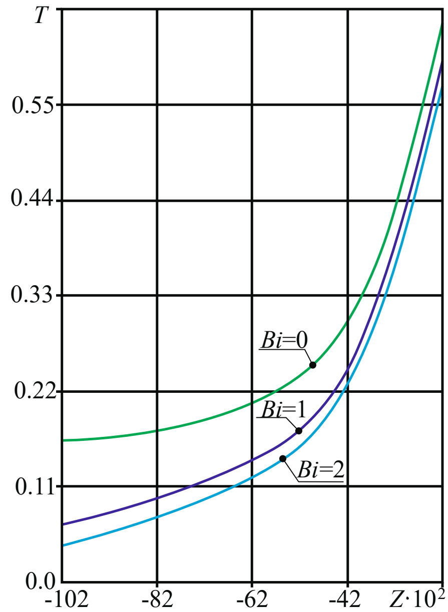


Fig. 4. Dependence of dimensionless temperature T(X,Y,Z) for $Z = 0.02$ on the spatial dimensionless coordinate Z for different values of the Biot criterion

It is seen that with the increase in heat transfer, the temperature decreases, which confirms the consistency of the mathematical model with the physical one.

5. 2. Mathematical model of heat transfer in a half-space due to heating by a heat flux at the boundary surface

The integral Fourier transform in the coordinates x and y is applied to equation (4) and boundary conditions (5). As a result, an ordinary second-order differential equation with constant coefficients and a singular right-hand side and boundary conditions are obtained:

$$\frac{d^2\bar{\theta}(z)}{dz^2}-\gamma^2\bar{\theta}(z)=P_2\delta'(z), \qquad (11)$$

$$\left.\frac{d\bar{\theta}(z)}{dz}\right|_{z=-d-l}=P_1,\left.\bar{\theta}(z)\right|_{z\to\infty}=0. \qquad (12)$$

The solution to boundary value problem (11), (12) is defined in the form

$$\bar{\theta}(z)=\frac{1}{2}\times \times\left\{\begin{array}{l}P_2\left[\begin{array}{l}F_1(z)\mathrm{sign}(z)-\\-\exp(-\gamma z_1)\end{array}\right]-\\-P_1\dfrac{\exp(-\gamma z_2)}{\gamma}\end{array}\right\}, \qquad (13)$$

where

$$z_1 = z + 2(d+l); \quad z_2 = z + d + l.$$

The inverse integral Fourier transform was applied to relation (13) and the relation for the desired dimensionless temperature $T(X, Y, Z)$ was obtained

$$T(X,Y,Z) = \frac{\lambda_1 h}{Q_0}\theta(x,y,z) = = \frac{1}{2}\left[\begin{matrix}\frac{1}{\pi}\varphi^{-1/2}(X,Y,Z) + \\ +P_2\left(Z\varphi^{-3/2}(X,Y,Z) - Z_1\varphi^{-3/2}(X,Y,Z_1)\right)\end{matrix}\right]. \quad (14)$$

Here:

$X = x / h$; $Y = y / h$; $Z = z / h$; $Z_1 = z_1 / h$; $Z_2 = z_1 / h$;

$$\left.\frac{\partial\theta(0,0,z)}{\partial z}\right|_{z=0}^{*} = \frac{-8(d+l)}{h\left[\frac{\Lambda_0}{\lambda_1} + 16\pi(d+l)^3\right]}.$$

Analysis of numerical results. According to formula (14) for $Y = 0$, numerical calculations were performed and the behavior of temperature field $T(X, Y, Z)$ was investigated. The medium material was VK94-I ceramics; the inclusion material was tungsten ($\lambda_0 = 130$ W/(deg·m)). $d / h = 1$; $b / h = 2$; $l / h = 50$. The numerical results that make it possible to establish the change in the dimensionless temperature $T(X, Y, Z)$ along the spatial dimensionless coordinate X for $Z = 1$ are shown in Fig. 5. The maximum temperature value is reached in the vicinity of zero; with an increase in the value of the dimensionless spatial coordinate X, the temperature increases.

Fig. 6 shows dependences of the $T(X, Y, Z)$ values on the spatial dimensionless coordinate Z for $X = 1$.

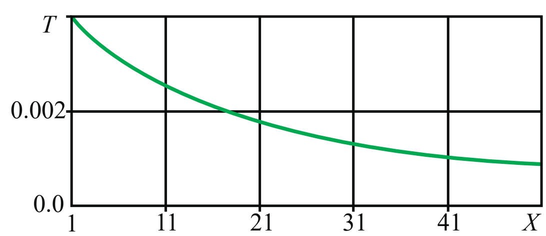


Fig. 5. Dependence of the dimensionless temperature $T(X, Y, Z)$ for $Z = 1$ along the spatial dimensionless coordinate X

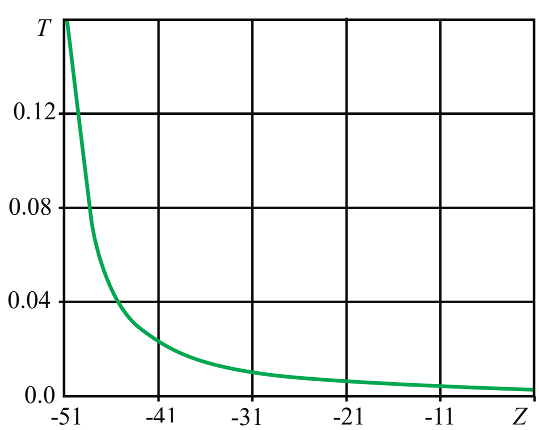


Fig. 6. Dependence of the dimensionless temperature $T(X, Y, Z)$ for $X = 1$ on the spatial dimensionless coordinate Z

It can be seen that the temperature increases with decreasing Z values and reaches its highest value at the boundary surface of the half-space.

6. Discussion of results related to the construction of mathematical models of heat transfer in spatial semi-infinite inhomogeneous media

Our results are explained by the fact that the generalized equations of heat conduction and the corresponding boundary conditions adequately reflect the physical processes of heat transfer in 3D semi-infinite inhomogeneous media with foreign inclusion. In the constructed mathematical model, the inhomogeneity of the semi-infinite medium is taken into account due to the presence of singular coefficients in the heat conduction equations that arise as a result of using the generalized Dirac delta function. This approach has made it possible to describe localized heat-active regions with small geometric parameters and obtain a single generalized second-order differential equation of heat conduction with partial derivatives. The derived analytical solutions to the boundary value problems in the form of formulae (9), (14), determined using the integral Fourier transform, made it possible to determine temperature fields depending on spatial coordinates. Analysis of the curves shown in Fig. 3, 5, 6 confirms the correspondence of the results to the physical process of heat transfer as temperature disturbances are localized mainly in the vicinity of the foreign thermally active inclusion. In addition, Fig. 4 shows the influence of heat transfer from the boundary surface of the medium for different values of the Biot criterion, which confirms the sensitivity of the model to changes in the boundary conditions of heat transfer.

A feature of our approach compared to existing methods is the use of symmetric generalized unit and delta Dirac functions to describe small-sized foreign inclusions in semi-infinite 3D media. Analysis of the literature [1–12] revealed that most of the known models are focused mainly on finite regions or simplified heat transfer conditions without detailed consideration of local internal heat sources and small-sized thermally active inclusions. In particular, in [1, 2, 4, 6, 7, 9–12], numerical modeling methods are mainly used, while in [3, 5, 8] traditional experimental approaches to heat transfer research are applied. In papers [13–15], asymmetric unit and delta Dirac functions are used to build mathematical models, while in our study the symmetric form of such functions is used, which made it possible to more effectively describe inhomogeneous semi-infinite regions with local inclusions. Unlike traditional numerical approaches, the use of the devised analytical method provides high accuracy of results as the errors are mainly associated only with the stages of numerical integration and do not exceed 10^{-6}. Such accuracy is difficult to achieve through experimental or numerical methods because of discretization, approximation, and measurement errors.

The limitations of our study are that the models considered only the stationary process of heat conduction, so the results reflect the temperature distribution exclusively in spatial coordinates without taking into account the time dependence. The model was built for isotropic 3D semi-infinite media with one foreign inclusion, and the geometric parameters of the inclusions are considered small enough that the use of the generalized singular Dirac delta function is an effective way to take them into account. In addition, the results are adequate within the linear statement of the problem and in the absence of significant nonlinear heat transfer effects. Our analytical solutions can adequately describe the physical process by fulfilling the specified boundary conditions and values of the thermophysical parameters of the materials. With an increase in the number of inclusions or as a result of a significant complication of the geometric structure of the medium, the effectiveness of the analytical approach may decrease.

The disadvantages of the study include the use of simplified mathematical models that do not take into account nonlinear heat transfer effects, in particular, temperature dependences of thermophysical parameters, as well as possible anisotropic properties of media materials. In addition, the work lacks experimental verification of the results, which is associated with the complexity of reproducing localized thermal processes in the designs of technical devices with small-sized thermally active elements. The mutual influence of several foreign inclusions is also not taken into account and the behavior of temperature fields in non-stationary modes is not investigated. In the future, these shortcomings can be eliminated by constructing nonlinear mathematical models, taking into account the thermal sensitivity of structural materials, building models for multi-component structures, and performing experimental studies to verify the adequacy of theoretical results.

Further studies may be associated with the construction of mathematical models of heat transfer for 3D semi-infinite media with several foreign inclusions, local internal heat sources for the unsteady process of heat conduction. A promising direction is the development of models taking into account nonlinear effects, temperature dependence of thermophysical parameters and thermomechanical interaction of structural elements. Our analytical solutions could be used to develop effective computational algorithms and software tools for modeling thermal regimes in technical devices. However, the implementation of such research will be accompanied by significant difficulties of a mathematical and computational nature, associated with the need to solve nonlinear differential equations with singular coefficients, ensuring the stability of algorithms and high accuracy of numerical procedures. Certain difficulties may arise from experimental verification of models related to the complexity of measuring local temperature gradients in inhomogeneous media containing inhomogeneities with small geometric parameters.

7. Conclusions

1. A mathematical model of heat exchange in an isotropic semi-infinite 3D medium containing a foreign heat-active inclusion with small geometric parameters, in which internal heat sources are uniformly concentrated, has been built. The solution to the corresponding boundary value problem is an analytic expression containing an improper integral with an infinite upper bound. Numerical integration according to the 3/8 Newton method was used, which ensured high accuracy in determining numerical values of temperature. Although the local heterogeneity of the semi-infinite medium, described by the relation containing the Dirac delta function, is formally concentrated at the origin of the coordinate system, it is characterized by finite dimensions, which are associated with the reduced volume of the foreign inclusion. As a result, the finite dimensions of the inclusion are effectively taken into account.

2. A mathematical model of heat transfer in an isotropic semi-infinite 3D medium containing a foreign inclusion with small geometric parameters and subjected to a heat flux concentrated in a rectangular region on the boundary surface of the medium has been built. An analytical solution has been obtained to the corresponding boundary value problem of thermal conductivity, on the basis of which the temperature values for selected geometric and thermophysical parameters have been determined by numerical integration of the improper integral. On the basis of this, the behavior of the temperature field has been determined in the vicinity of the foreign inclusion. This indicates a feature of the Dirac delta function, which describes this inhomogeneity.

Conflicts of interest

The authors declare that they have no conflicts of interest in relation to the current study, including financial, personal, authorship, or any other, that could affect the study and the results reported in this paper.

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Data availability

All data are available in the main text of the manuscript.

Use of artificial intelligence

The authors confirm that they did not use artificial intelligence technologies when creating the current work.

Authors' contributions

Vasyl Havrysh: Conceptualization, Methodology, Formal analysis, Writing – original draft; **Svitlana Yatsyshyn**: Software, Visualization, Writing – review & editing; **Mykhailo Stepaniak**: Investigation, Resources, Validation; **Mykhailo Klymiuk**: Investigation, Resources, Validation; **Galyna Klym**: Conceptualization, Methodology.

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