Розглянуто тривимірну задачу про напружено-деформований стан плити, у якої характеристики пружності матеріалу змінюються за товщиною та описуються довільними інтегровними функціями. Показано, що точні, в сенсі Сен-Венана, аналітичні розв'язки крайової задачі можна отримати, якщо розподіл навантажень задовольняє двовимірному полігармонічному рівнянню. Сформульовано наближені теорії для інженерних розрахунків згину неоднорідних пластин

Ключові слова: теорія пружності, ізотропні тіла, неоднорідні матеріали, згин плит, напруження та деформації

Рассмотрена трехмерная задача о напряженно-деформированном состоянии плиты, в которой характеристики упругости материала изменяются по толщине и описываются произвольными интегрируемыми функциями. Показано, что точные, в смысле Сен-Венана, аналитические решения краевой задачи можно получить, если распределение нагрузок удовлетворяет двумерному полигармоническому уравнению. Сформулированы приближенные теории для инженерных расчетов изгиба неоднородных пластин

Ключевые слова: теория упругости, изотропные тела, неоднородные материалы, изгиб плит, напряжения и деформации

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ANALYTICAL STUDY OF THE BENDING OF ISOTROPIC PLATES, INHOMOGENEOUS IN THICKNESS

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1. Introduction

The heterogeneity is a characteristic feature of virtually all materials used in engineering and construction. It is due to a number of factors, which conditionally can be divided into three groups:

1. Action of environment (temperature fields, radioactive radiation, uneven humidity, etc.).
2. Manufacturing techniques peculiarities (rolling, forging, hardening of casting and concrete etc.).
3. Project implementation plan (availability of the reinforcement, layers of other materials, etc.), which is an important source of reduction in weight, size and cost of the projected designs.

The improvement of calculations of machine elements, constructions and structures is associated with taking into account the impact of heterogeneity of actual materials on the stressed and deformed states of elastic bodies. Therefore, a relevant task is the search for the methods of determining the displacements and stresses in three-dimensional bodies of a relatively simple form – slabs and plates made of inhomogeneous materials.

2. Analysis of scientific literature and the problem setting

Nowadays most of the studies in this direction consider a fairly limited number of the types of heterogeneities, for which analytical solutions to the problems of elasticity theory were obtained [1, 2]. This is due to the fact that the difficulties faced during consideration of specific problems are relatively more complex than the similar problems of the classical theory of elasticity, since variable coefficients appear in its fundamental equations. These characteristic features offer explanation to the fact that analytical solutions to a limited number of tasks for the bodies of the simplest geometric shapes have been obtained until now: rectangular [1, 5, 6] and circular plate [2], structure [3] and dimensions [4, 6] and elementary dependencies of the characteristics of materials elasticity on the coordinates of the points of the body [5].

The analysis of the above mentioned publications demonstrated that the exact solutions to variable thickness of inhomogeneous rectangular plates were considered, the influence of heterogeneity degree, the ratios of sides [1] and the heterogeneous boundary conditions [4] were taken into account. As result of the research, it was found that the influence of elasticity module on the plate of (FG) and (EG) materials rested on the elastic foundation is produced according to the exponential law [5, 6], and the influence of material properties and the strength of the elastic foundation on mechanical properties of the plate was studied.

The bodies of the type of sphere, half-space and the layered system were most often studied. The bodies with exponential [7, 8] or degree [9, 10] Young's law of the module change and stable [4], and in some cases even the variable

Poisson coefficient [11, 12] were considered. It was proven that such heterogeneities usually significantly affect both stressed and deformed state of bodies. The works [13, 14] are devoted to more complex dependencies.

Therefore, one can state that only quite a limited number of types of heterogeneities have been studied until now, for which analytical solutions to the problems of the theory of elasticity were obtained. At the same time, there are no exact, in the sense of Saint Venant, solutions to the boundary problems for the plates, inhomogeneous in thickness, exposed to the actions of the mass forces and the surface loads in a general case. Therefore, various approximate methods of solving the problems of mechanics of a deformed solid body have been developed over the recent decades, which allow receiving numeric solutions relatively easy.

But the vast majority of researchers feel the lack of a sufficient number of analytical solutions to the problems, which might serve as reliable test examples for approximate methods. Therefore, this research is devoted to searching for the new exact, in the sense of Saint Venant, solutions to the boundary problems for the plates, inhomogeneous in thickness, as well as the ways of building up approximate solutions that have the set degrees of accuracy.

3. The aim and the tasks of the research

The purpose of the research is to build up analytical solutions to the boundary problems for the plates, inhomogeneous in thickness, exposed to the action of mass forces and surface loads.

To achieve this aim, the following tasks were set:

– to find the criteria, to which the distribution of the forces inside and on the body surface should correspond, if the material's heterogeneity is described by arbitrary integrated functions of one Cartesian coordinate;

– to obtain the exact, in the sense of Saint Venant, solutions of boundary problems for the plates, inhomogeneous in thickness, that can be used in engineering calculations, as well as when testing the existing approximate theories;

– to develop methods for building up approximate solutions that have a specified degree of accuracy.

4. Analytical solution of the problem about the bending of a plate, inhomogeneous in thickness, and its discussion

Let us assume that the plate, inhomogeneous in thickness, is loaded with mass forces X, Y, Z and is limited from the sides by the cylindrical surface Γ (Fig. 1) (the load of the plate surface is considered as a special case).

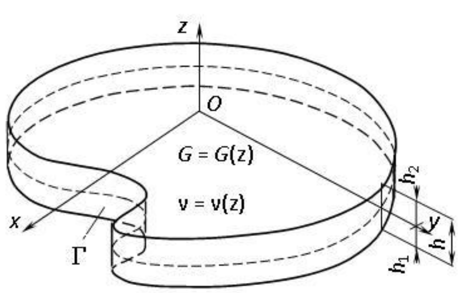

Fig. 1. Scheme of the plate, inhomogeneous in thickness (external forces are not shown)

We will try to set the conditions that the mass forces should meet for the problem on stressed and deformed states of a studied elastic body to have the solution, exact in the sense of Saint Venant, i. e. the plate thickness h will be considered a relatively small magnitude.

We will formulate the ways of building up so-called "technical" (approximate) theories of the bending of plates.

1. To solve the problem, we will connect the body with the Cartesian coordinate system, the O origin of which lies inside the plate and the axes Ox, Oy are directed by parallel bordering planes $z=-h_1$ and $z=h_2$, that is, the plate thickness $h=h_1+h_2$ (position of the point O by height will be chosen later from the condition of the simplest recording of final formulas). With this choice of axes, the material's heterogeneity will be described by the functions of only one coordinate z, which will be considered arbitrary and integrable.

As it is shown in the paper [14], the solution for the problem formulated above comes down to the search for the solutions of two linear differential equations with variable coefficients, one of which is of the fourth, and the other is of the second order. The equations have the form:

$$\frac{\partial^2}{\partial z^2}\left(\frac{1}{a_4}\frac{\partial^2 L}{\partial z^2}\right)+\Delta\left[\frac{\partial^2}{\partial z^2}\left(\frac{a_2}{a_4}L\right)-\frac{\partial}{\partial z}\left(a_1\frac{\partial L}{\partial z}\right)+\frac{a_2}{a_4}\frac{\partial^2 L}{\partial z^2}\right]+$$

$$+\left(\frac{a_2^2}{a_4}-a_3\right)\Delta^2 L=R, \tag{1}$$

$$\Delta N+G_1\frac{\partial}{\partial z}\left(\frac{1}{G}\frac{\partial N}{\partial z}\right)=-G\frac{\partial}{\partial z}\left(\frac{\Lambda_3}{G}\right), \tag{2}$$

where

$$R=-\frac{\partial^2}{\partial z^2}\left[\frac{1}{a_4}\left(\frac{\partial \Lambda_1}{\partial z}-\Lambda_2\right)\right]+\Delta\left[\frac{\partial}{\partial z}(a_1\Lambda_1)-\frac{a_2}{a_4}\left(\frac{\partial \Lambda_1}{\partial z}-\Lambda_2\right)\right],$$

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},$$

where $\Lambda_1,\ \Lambda_2,\ \Lambda_3$ are the special solutions for these problems:

$$\Delta^2\Lambda_1=Z,\quad \Delta\Lambda_2=\frac{\partial f_2}{\partial x}+\frac{\partial f_3}{\partial y},$$

$$\Delta\Lambda_3=\frac{\partial f_2}{\partial y}-\frac{\partial f_3}{\partial x},\ (\Delta f_2=X_1,\ \Delta f_3=Y), \tag{3}$$

In this case, $a_i=a_i(z)$ (i=1, 2, 3, 4) may be expressed through the material's elasticity parameters like this:

$$a_1=\frac{1}{G},\quad a_2=-\frac{\nu}{1-\nu},$$

$$a_3=\left(1-\frac{2\nu^2}{1-\nu}\right)\frac{1}{E},\quad a_4=-\frac{E}{1-\nu^2}, \tag{4}$$

where G=G(z) is the module of the plate material's displacement; E=E(z) is the Young module; ν=ν(z) is the Poisson ratio.

If the functions S_i(i=1, 2, 3, 4), T_j(j=1,2) are introduced for consideration with the help of the following dependencies:

$$S_1=-\frac{1}{a_4}\left[\frac{\partial}{\partial z}\left(\frac{\partial L}{\partial z}+\Lambda_1\right)+a_4\Delta L-\Lambda_2\right],$$

$$S_2=-a_1\Delta\left(\frac{\partial L}{\partial z}+\Lambda_1\right)-\frac{\partial S_1}{\partial z},$$

$$S_3=\Delta^2 L,\quad S_4=-\Delta\left(\frac{\partial L}{\partial z}+\Lambda_1\right),\tag{5}$$

$$T_1=-\frac{1}{G}\left(\frac{\partial N}{\partial z}+\Lambda_1\right),\quad T_2=\Delta N,$$

then for determining the displacements u_x, u_y, u_z and the stresses σ_x, σ_y,..., σ_{xy}, we will receive the following formulas:

$$u_x=\frac{\partial S_1}{\partial x}+\frac{\partial T_1}{\partial y},\quad u_y=\frac{\partial S_1}{\partial y}-\frac{\partial T_1}{\partial x},\quad u_z=S_2,$$

$$\sigma_x=-\left(a_4\Delta+2G\frac{\partial^2}{\partial y^2}\right)S_1-a_2S_3+2G\frac{\partial^2 T_1}{\partial x\partial y},$$

$$\sigma_y=-\left(a_4\Delta+2G\frac{\partial^2}{\partial x^2}\right)S_1-a_2S_3-2G\frac{\partial^2 T_1}{\partial x\partial y},$$

$$\sigma_z=S_3,\quad \tau_{zx}=\frac{\partial S_4}{\partial x}+\frac{\partial T_2}{\partial y},\quad \tau_{zy}=\frac{\partial S_4}{\partial y}+\frac{\partial T_2}{\partial x},$$

$$\tau_{xy}=2G\frac{\partial^2 S_1}{\partial x\partial y}-G\left(\frac{\partial^2 T_1}{\partial x^2}-\frac{\partial^2 T_1}{\partial y^2}\right).\tag{6}$$

Thus, when solving any problem of the theory of elasticity for bodies of one-dimensional heterogeneity, it is necessary in general case to find the solutions for the differential equations (1) and (2) by the corresponding boundary conditions. In some cases, it is enough to examine only one of them.

2. To find the solutions for the equations (1) and (2), let us first proceed to the dimensionless coordinates:

$$\xi=\frac{x}{l},\quad \eta=\frac{y}{l},\quad \zeta=\frac{z}{h}$$

and designate:

$$D_1=\frac{\partial}{\partial\xi},\quad D_2=\frac{\partial}{\partial\eta},\quad D^2=D_1^2+D_2^2,$$

$$\lambda_1=\frac{\Lambda_1}{l^4},\quad \lambda_2=\frac{\Lambda_2}{l^3},\quad \lambda_3=\frac{\Lambda_3}{l^3},$$

$$\varepsilon=\frac{h}{l},\quad \kappa_1=\frac{h_1}{h},\quad \kappa_2=\frac{h_2}{h},\quad \phi_2=\frac{f_2}{l^2},\quad \phi_3=\frac{f_3}{l^2}.$$

Here l is the characteristic plate dimension on the plan.

Now, the dependencies (3) take the form:

$$D^4\Lambda_1=Z,\ D^2\Lambda_2=\frac{\partial\phi_2}{\partial\xi}+\frac{\partial\phi_3}{\partial\eta},\ D^2\Lambda_3=\frac{\partial\phi_2}{\partial\eta}-\frac{\partial\phi_3}{\partial\xi},$$

$$(D^2\phi_2=X,\quad D^2\phi_3=У).$$

The formal solution for the equations (1) and (2) will be searched for in the form of the series, similar to how it was made in the paper [15]

$$L=hl^4\left[L_0+(\varepsilon D)^2L_1+(\varepsilon D)^4L_2+...\right],\tag{7}$$

$$N=hl^3\left[N_0+(\varepsilon D)^2N_1+(\varepsilon D)^4N_2+...\right],\tag{8}$$

where L_n, N_m(m, n=0,1,...) are the functions to be defined.

Let us substitute L and N to the equations (1) and (2) and collect similar members by the same degrees εD. Then, equating them to zero, we will obtain two recurrent sequences of ordinary differential equations:

$$\left(\frac{1}{a_4}L_0''\right)''=-\left[\frac{1}{a_4}\left(\lambda_1'-\varepsilon\lambda_2\right)\right]'',$$

....

$$\left(\frac{1}{a_4}L_n''\right)''=-\left(\frac{a_2}{a_4}L_{n-1}\right)''+\left[a_1\left(L_{n-1}'+\delta_{1n}\lambda_1\right)\right]'-\frac{a_2}{a_4}\times$$

$$\times\left[L_{n-1}''+\delta_{1n}\left(\lambda_1'-\varepsilon\lambda_2\right)\right]-\left(1-\delta_{1n}\right)\left(\frac{a_2^2}{a_4}-a_3\right)L_{n-2},\tag{9}$$

....

$$\left(\frac{1}{G}N_0'\right)'=-\left(\frac{\lambda_3}{G}\right)',$$

....

$$\left(\frac{1}{G}N_m'\right)'=-\frac{1}{G}N_{m-1}.\tag{10}$$

....

We will enter the symbol here:

$$\delta_{ij}=\begin{cases}0, & i\neq j,\\ 1, & i=j.\end{cases}$$

In these formulas, the differentiation by the coordinate ζ. is marked with a stroke.

The boundary conditions for the functions L_n, N_n follow from the boundary conditions of the problem.

We agree to designate the value of any function by $\zeta=-\kappa_1$, with the top index in the form of a degree, and the value $\zeta=\kappa_2$ is designated in the form of an asterisk.

First consider a separate case of the plate's load.

Let us assume that the lower and the upper surfaces of the plate $\zeta=-\kappa_1$ and $\zeta=\kappa_2$ are free from the action of forces, and only mass forces affect the body.

Then the boundary conditions on the planes $\zeta=-\kappa_1$ and $\zeta=\kappa_2$ will be written down as follows:

$$\sigma_z^o=S_3^o=0,\quad l\tau_{zx}^o=D_1S_4^o+D_2T_2^o=0,$$

$$l\tau_{zy}^o=D_2S_4^o-D_1T_2^o=0,\tag{11}$$

$$\sigma_z^*=S_3^*=0,\quad l\tau_{zx}^*=D_1S_4^*+D_2T_2^*=0,$$

$$l\tau_{zy}^*=D_2S_4^*-D_1T_2^*=0.$$

Thus, S_4^o and T_2^o, as well as S_4^* and T_2^*, are interrelated by the Cauchy-Riemann conditions and, therefore, are two-dimensional harmonic functions ω^o and ω^*, i. e.

$$S_4^\circ=D_1\omega^\circ,\quad T_2^\circ=D_2\omega^\circ,$$

$$S_4^*=D_1\omega^*,\quad T_2^*=D_2\omega^*\ (D^2\omega^{o,*}=0).$$

However, it is easy to prove that the influence of the functions ω^o and ω^* on the stressed and deformed states of the plate can be taken into account by using the arbitrary elements in the selection of the functions λ_1, λ_2 and λ_3. So we put $\omega^o=\omega^*=0$. Then the boundary conditions will take the form:

$$S_3^o=S_3^*=S_4^o=S_4^*=T_2^o=T_2^*=0.$$

They are automatically satisfied if the condition is set:

$$L_n^{\circ} = L_n^* = 0, \quad L'^{\circ}_n = -\delta_{on}\lambda_1^{\circ},$$

$$L'^*_n = -\delta_{on}\lambda_1^*, \quad N_m^{\circ} = N_m^* = 0 \quad (n, m = 0, 1, \ldots).$$

We will consider the following geometric characteristics of cross section of the plate:

$$F_1 = -\int_{-\kappa_1}^{\zeta} a_4 d\zeta; \quad S = \int_{-\kappa_1}^{\zeta} a_4 \zeta d\zeta;$$

$$I = \int_{-\kappa_1}^{\zeta} S d\zeta; \quad F_2 = \int_{-\kappa_1}^{\zeta} G d\zeta \tag{12}$$

and choose the position of the origin of the coordinate O from the condition $S^*=0$.

Then the solution of the first equation of the sequence (9) can be recorded as:

$$L_0 = \varepsilon \cdot Q_2 - Q_1 - \left(\varepsilon \cdot Q_2^* - Q_1^*\right) \cdot \psi_1 - \varepsilon \cdot Q_2'^* \cdot \psi_2, \tag{13}$$

where

$$Q_1 = \int_{-\kappa_1}^{\zeta} \lambda_1 d\zeta, \quad Q_2 = \int_{-\kappa_1}^{\zeta} d\zeta \int_{-\kappa_1}^{\zeta} \lambda_2 d\zeta,$$

$$\psi_1 = \frac{I}{I^*}, \quad \psi_2 = \frac{1}{F_1^*}\left(S + \zeta F_1\right) - \kappa_2 \psi_1. \tag{14}$$

Solutions of other equations are determined by the following recurrent dependencies:

$$L_n = M_n - M_n^* \psi_1 - M_n'^* \psi_2, \tag{15}$$

In this case,

$$M_n = -\int_{-\kappa_1}^{\zeta} d\zeta \int_{-\kappa_1}^{\zeta} \left[a_2 L_{n-1} + a_4\left(M_{1n} - M_{2n}\right)\right] d\zeta,$$

$$M_{1n} = \int_{-\kappa_1}^{\zeta} d\zeta \int_{-\kappa_1}^{\zeta} \left\{\frac{a_2}{a_4}\left[L''_{n-1} + \delta_{1n}\left(\lambda_1' - \varepsilon\lambda_2\right)\right] + \right.$$

$$\left. + \left(1 - \delta_{1n}\right)\left(\frac{a_2^2}{a_4} - a_3\right) L_{n-2}\right\} d\zeta,$$

$$M_{2n} = \int_{-\kappa_1}^{\zeta} a_1\left(L'_{n-1} + \delta_{1n}\lambda_1\right) d\zeta.$$

By solving equations of the sequence (10), we obtain:

$$N_0 = -Q_3 + Q_3^* \chi, \quad N_m = -R_m + R_m^* \chi,$$

$$Q_3 = \int_{-\kappa_1}^{\zeta} \lambda_3 d\zeta, \quad R_m = \int_{-\kappa_1}^{\zeta} G d\zeta \int_{-\kappa_1}^{\zeta} N_{m-1} \frac{d\zeta}{G}, \quad \chi = \frac{F_2}{F_2^*}. \tag{16}$$

If we substitute series (7), (8) to the formulas (5) for S_1, T_1, T_2, S_2, S_3, S_4, we will find:

$$S_1 = \frac{l^4}{h} \sum_{n=0}^{\infty} (\varepsilon D)^{2n} U_{1n}, \quad S_2 = \frac{l^4}{h^2} \sum_{n=0}^{\infty} (\varepsilon D)^{2n} W_n,$$

$$S_3 = \frac{l^4}{h^3} \sum_{n=0}^{\infty} (\varepsilon D)^{2n+4} L_n,$$

$$S_4 = -\frac{l^4}{h^2}\left[(\varepsilon D)^2 \lambda_1 + \sum_{n=0}^{\infty} (\varepsilon D)^{2n+2} L'_n\right], \tag{17}$$

$$T_1 = l^3 \sum_{m=0}^{\infty} (\varepsilon D)^{2m} U_{2m}, \quad T_2 = \frac{l^3}{h} \sum_{m=0}^{\infty} (\varepsilon D)^{2m+2} N_m.$$

Here,

$$U_{10} = -Q_1^* \frac{\zeta}{I^*} + \varepsilon\left[\left(Q_2^* - \kappa_2 Q_2'^*\right)\frac{\zeta}{I^*} - \frac{Q_2'^*}{I^*}\right],$$

$$U_{1n} = M_{1n} - M_{2n} - \frac{M_n'^*}{F_1^*} + \left(M_n^* - \kappa_2 M_n'^*\right)\frac{\zeta}{I^*},$$

$$W_0 = \left[Q_1^* - \varepsilon\left(Q_2^* - \kappa_2 Q_2'^*\right)\right]\frac{1}{I^*},$$

$$W_n = -M'_{1n} - \left(M_n^* - \kappa_2 M_n'^*\right)\frac{1}{I^*},$$

$$U_{20} = -\frac{Q_3^*}{F_2^*},$$

$$U_{2m} = \frac{R'_m}{G} - \frac{R_m^*}{F_2^*}.$$

Thus, the stress and deformations have the form of the series, the members of which grow by degrees of the operator D^2. Therefore, if the functions X, Y, Z are polyharmonic, i. e. satisfy a two-dimensional equation:

$$D^{2k}\left\| X(\xi, \eta, \zeta), \quad Y(\xi, \eta, \zeta), \quad Z(\xi, \eta, \zeta) \right\| = 0 \; (k = 1, 2, 3 \ldots)$$

(the variable ζ plays the role of a parameter), the functions L_n and N_n will be also polyharmonic. As a result, the series (7), (8) and (17) will break and by using the formulas (17) and (6) we will find the exact solution to the problem. Let us remind that in this case the boundary conditions on the planes $\zeta = -\kappa_1$ and $\zeta = \kappa_2$ will be satisfied exactly.

3. Let us focus in more detail on the possible forms of setting the functions X, Y and Z in the search for the exact solutions to the problem on the stressed and deformed states of the plate.

It is easy to prove that the following features are characteristic for the polyharmonic functions:

$$\text{if } D^{2m}\phi = 0, \text{ then } D^{2(m+1)}\left\| \xi, \eta, \rho^2 \right\| \phi = 0 \quad \left(\rho = \sqrt{\xi^2 + \eta^2}\right).$$

So, if there is any polyharmonic function ϕ with arbitrary m, then the new solution Φ of the polyharmonic equation with a larger m can be built up by using the formula:

$$\Phi = \phi P_k,$$

In this case

$$P_k = \sum_{i+j=0}^{k} \alpha_{ij}(\zeta) \xi^i \eta^j$$

or

$$P_k = \sum_{i=0}^{k} \alpha_i(\zeta) \rho^{2i}.$$

Here $\alpha_{ij}(\zeta)$ and $\alpha_i(\zeta)$ are the arbitrary integrable functions of the coordinate z.

We will receive additional solutions if we take:

a) linear combinations of the known solutions;

b) the derivative of an arbitrary order by the parameter S

$$\frac{\partial\Phi(\xi,\eta,\zeta,s)}{\partial s};$$

c) the integral by the parameter with the weight function dependent on it

$$\int \Phi(\xi,\eta,\zeta,s)f(s)ds.$$

As an example, here are a few more types of polyharmonic functions, by which it is convenient to approximate volumetric loads during the solution of the problems of the theory of elasticity in Cartesian coordinates:

$$\Phi=\left\|\begin{matrix}\sin n\xi\\ \cos n\xi\end{matrix}\right\| e^{\pm n\eta}\sum_{i=0}^{k}\alpha_i(\zeta)\eta^i,$$

$$\Phi=\left\|\begin{matrix}\sin n\eta\\ \cos n\eta\end{matrix}\right\| e^{\pm n\xi}\sum_{i=0}^{k}\alpha_i(\zeta)\xi^i.$$

4. The solution in the form of (17) is built in such a way that the boundary conditions on the planes $\zeta=-\kappa_1$ and $\zeta=\kappa_2$ are satisfied exactly. The conditions on the lateral area D have to be satisfied by using arbitrariness in choosing the functions λ_1, λ_2 and λ_3. The so-called "homogeneous solutions", which determine the stressed and deformed states of the plate with the load of the lateral side D, correspond to this arbitrariness.

Homogeneous solutions are easy to find if you put X=Y=Z=0. Then the above mentioned formulas imply that λ_1, λ_2 and λ_3 are biharmonic functions.

Now, if we find L_n, and N_m from the dependencies (15) and (16) and substitute them to the formulas (17), we will find functions S_i, and T_j:

$$\frac{S_1}{lh}=-\left[\zeta-(\varepsilon D)^2\left(\int_{-\kappa_1}^{\zeta}d\zeta\int_{-\kappa_1}^{\zeta}a_2\zeta d\zeta-\int_{-\kappa_1}^{\zeta}\frac{S}{G}d\zeta\right)\right]\Psi_1-$$
$$-\left[1-(\varepsilon D)^2\left(\int_{-\kappa_1}^{\zeta}\frac{F_1}{G}d\zeta+\int_{-\kappa_1}^{\zeta}d\zeta\int_{-\kappa_1}^{\zeta}a_2 d\zeta\right)\right]\frac{\Psi_2}{F_1^*},$$

$$\frac{S_2}{l}=\left[1-(\varepsilon D)^2\int_{-\kappa_1}^{\zeta}a_2\zeta d\zeta\right]\Psi_1-(\varepsilon D)^2\int_{-\kappa_1}^{\zeta}a_2 d\zeta\frac{\Psi_2}{F_1^*},$$

$$S_3=0,\ \frac{S_4}{l}=-(\varepsilon D)^2\left(S\Psi_1-\frac{F_1}{F_1^*}\Psi_2\right),\qquad(18)$$

$$\frac{T_1}{lh}=-\left[1-(\varepsilon D)^2\int_{-\kappa_1}^{\zeta}\frac{F_2}{G}d\zeta\right]\frac{\Psi_3}{F_2^*},\ \frac{T_2}{l}=\frac{F_2}{F_2^*}(\varepsilon D)^2\Psi_3.$$

Here symbols $\Psi_{1,2,3}$ designate the two-dimensional biharmonic functions. Between the two of them – Ψ_2 and Ψ_3, there is a link: $D^2\Psi_2$ and $D^2\Psi_3$ are adjoint harmonic functions.

Now using the dependencies (6), it is easy to find the formulas for determining the displacements and stresses in the plate. They, along with (18), allow receiving the solution to the problem on the balance of the plate, inhomogeneous in thickness, loaded with mass forces of polyharmonic type. Note that the boundary conditions may be satisfied exactly on the planes $\zeta=-\kappa_1$ and $\zeta=\kappa_2$, and on the lateral side Γ, they are "softened" in the sense of Saint Venant.

We will note that the problem of searching for the "homogeneous" solutions for a non-homogeneous plate, but from the electro elastic material, was studied earlier in [16].

5. The solution to the problem on the balance of the plate subject to the actions of the surface polyharmonic loads may be obtained from the solution given above.

Without limiting the studies generality, we will demonstrate this with the example of loading the upper surface $\zeta=\kappa_2$ with the normal efforts $\sigma_z^*=\sigma(\xi,\eta)$.

Put

$$Z=\delta(\zeta-\kappa_2)\sigma(\xi,\eta).\qquad(19)$$

In this case, $\delta(\zeta-\kappa_2)$ is the asymmetric impulse function [17], and we will first obtain the solution to the problem in the case when the plate is exposed to the mass forces Z in the form (19), that is, we might consider that the load is applied inside the plate under the surface $\zeta=\kappa_2$.

We have the equation for defining the functions λ_1, λ_2 and λ_3:

$$D^4\lambda_1=\delta(\zeta-\kappa_2)\sigma(\xi,\eta),\ \ D^2\lambda_2=D_1\phi_2+D_2\phi_3,$$

$$D^2\lambda_3=D_2\phi_2-D_1\phi_3,\ \ D^2\phi_2=0,\ \ D^2\phi_3=0.\qquad(20)$$

Obviously, it is sufficient to find any special solutions to these equations, whereas the general homogeneous solutions, corresponding to them, are included in the formulas (18). So we take:

$$\lambda_1=\delta(\zeta-\kappa_2)F(\xi,\eta),\ \ \lambda_2=\lambda_3=0\ \ (D^4F=\sigma).$$

From the formulas (13), (14) and (16) we have:

$$L_0=\left[\psi_1-e(\zeta-\kappa_2)\right]F(\xi,\eta),\ \ N_0=0,$$

where

$$e(\zeta-\kappa_2)=\begin{cases}0, & \zeta<\kappa_2,\\ 1, & \zeta=\kappa_2.\end{cases}$$

Hence, using the recurrent dependency (15), it is easy to find the required number of functions L_n (according to (16), N_m=0 for all m), and when substituting them to the formulas (5) and (6), one can find components of the displacement vector and the stress tensor.

In this case, the boundary conditions on the planes $z=-\kappa_1$ and $\zeta=\kappa_2$ of the plate will have the form (11). Now, to proceed to the problem of the influence of the surface forces, it is necessary to exclude the boundary area $\zeta=\kappa_2$ from the area belonging to the plate, and thereby, to "bring" the load $\sigma(\xi,\eta)$ onto the surface $\zeta=\kappa_2$.

In this case, for example, the function L_0 will take the form:

$$L_0=\psi_1 F(\xi,\eta).$$

If now we use the homogeneous solutions in the form of (18), then it is possible in principle to satisfy the "softened" boundary conditions on the surface Γ and by so doing, to solve exactly in the sense of Saint Venant the problem of balance of a non-homogeneous plate under the action of the surface load σ on it.

Hence, if the external forces are represented by the polyharmonic functions with the variables ξ, η, then the problem of balance of a non-homogeneous plate may be solved exactly in the sense of Saint Venant.

The thinner the plate, the more exact the solution is. All the series introduced for consideration, break, and the result contains the finite number of members.

6. It follows from the formulas (17) that the order of each successive summand is proportional to the value ε^2 relative to the previous one. Therefore, if ε is small, it is possible in the series (17) to be limited by taking into consideration only the first few members, while the others may be discarded; an excessive precision of formulas in the theory, which permits using the principle of Saint-Venant, is hardly appropriate.

If the external loads are not polyharmonic functions, they may be in a sense approximated by such functions and, thereby, a solution may be found. Moreover, if $\varepsilon<<1$, then the precision, sufficient for practical purposes, can be obtained by leaving only one summand in the obtained formulas for calculating stresses and displacements.

However, a similar result may be found when the loads are not approximated by the polyharmonic functions. It is only necessary to break the series in the formulas.

The elementary «technical» theory of the bending of non–homogeneous plates may be obtained by leaving only one summand in the series. So, if X=Y=0, it is easy to find

$$u_x=-\varepsilon\zeta D_1 w,\ u_y=-\varepsilon\zeta D_2 w,\ u_z=w,$$

$$\sigma_x=-\frac{2\varepsilon\zeta G}{l(1-\nu)}(D_1^2+\nu D_2^2)w,\ \ \sigma_y=-\frac{2\varepsilon\zeta G}{l(1-\nu)}(\nu D_1^2+D_2^2)w,$$

$$\sigma_z=h\left(\frac{I}{I^*}\int_{-\kappa_1}^{\kappa_2}Zd\zeta-\int_{-\kappa_1}^{\zeta}Zd\zeta\right),\ \tau_{xy}=-\frac{2\varepsilon}{l}\zeta GD_1D_2w,$$

$$\tau_{zx}=-\frac{\varepsilon^2}{l}SD_1D^2w,\ \tau_{zy}=-\frac{\varepsilon^2}{l}SD_2D^2w.$$

Here w is the solution to a biharmonic equation

$$D^4w=q,\ q=\frac{l^4}{h^2I^*}\int_{-\kappa_1}^{\kappa_2}Zd\zeta.$$

To simplify these dependencies, it makes sense to proceed from the dimensionless coordinates to the ordinary ones.

In a separate case when the material is homogeneous, the above given formulas will in fact coincide with those used in the theory of bending thin plates. The classical theory does not take into account only the stress σ_z.

If we leave only two summands in the series for u_x, u_y, u_z, we will obtain a more precise method of calculation of the bending of non-homogeneous plates.

$$u_x=-\varepsilon D_1\left[\zeta w-(\varepsilon D)^2U_{10}\frac{l^4}{h^2}\right],$$

$$u_y=-\varepsilon D_2\left[\zeta w-(\varepsilon D)^2U_{10}\frac{l^4}{h^2}\right],$$

$$u_z=w+(\varepsilon D)^2W_1\frac{l^4}{h^2}.$$

We will note that it follows from the obtained results that the results, exact in the Saint Venant approximation, may be obtained in the case when the functions describing the distribution of loads inside the plate and on its surface satisfy two-dimensional polyharmonic equations (harmonic, biharmonic, etc.). This somehow limits the class of functions, which the real loads in a plate can correspond to, but allows receiving analytical solutions, exact in the Saint Venant approximation, which makes it possible, for example, to test numerical methods of solving the problems of mechanics of a deformed solid body.

5. Conclusions

A three-dimensional problem of bending the plate, in which the parameters of elasticity of the material vary by thickness and are arbitrary integrable functions, was examined. And the plate itself is exposed to the action of mass forces while the action of the surface loads is studied as a separate case:

1. We obtained analytical solution to the boundary problem by the operator methods in a case when the boundary conditions are satisfied exactly on the flat surfaces of the plates, and on the lateral surface – in the Saint Venant approximation.

2. It was theoretically proved that the exact, in the sense of Saint Venant, analytical solutions may be obtained if the plate is exposed to the action of mass and surface forces, distributed on the plate and on its surface by the two-dimensional polyharmonic law. In this case, the thinner the plate, the more exact the solution will be, since the corresponding solutions represent the series that contain a finite number of members.

3. The obtained solutions allow using them as an approximate, "technical" theory for engineering calculations of the stressed and deformed state of non-homogeneous plates.

4. It was demonstrated that the obtained formulas for the calculation of the bending of thin plates in the case of homogeneous material, transfer to the classic formulas of the theory of bending thin plates.

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