Розглянуто задачу визначення невідомих пластових тисків та фільтраційних коефіцієнтів двох газоносних пластів, що сумісно дренуються однією свердловиною, за даними вимірів на стаціонарних режимах тиску та сумарного дебіту на гирлі свердловини. Задачу зведено до розв'язку складної системи трьох нелінійних рівнянь. Створено алгоритм її чисельного розв'язку та його комп'ютерну реалізацію для одного з можливих варіантів. Наведено приклад чисельного розв'язку

Ключові слова: фільтрація газу, надстисливість, зв'язані пласти, дебіт свердловини і пластів, точки повороту, метод Ньютона

Рассмотрена задача определения неизвестных пластовых давлений и фильтрационных коэффициентов двух газоносных пластов, разрабатываемых совместно одной скважиной, по данным замеров на стационарных режимах давления и суммарного дебита на устье скважины. Задача сводится к решению сложной системы трех нелинейных уравнений. Построен алгоритм ее численного решения и его компьютерная реализация для одного из возможных вариантов. Приведен пример численного решения

Ключевые слова: фильтрация газа, сверхсжимаемость, связанные пласты, дебит скважины и пластов, точки поворота, метод Ньютона

UDK 533.6.011.51
DOI: 10.15587/1729-4061.2016.86177

ON IDENTIFYING THE FORMATION PRESSURE AND FILTRATION COEFFICIENTS OF TWO GAZ-BEARING FORMATIONS

A. Markovskii
PhD, Associate Professor, Senior Researcher
Institute of Applied Mathematics and Mechanics of NAS of Ukraine
Generala Batiuka str., 19, Slaviansk, Ukraine, 84100
E-mail anatolymarkowski@gmail.com

1. Introduction

During gas field exploitation, there are cases when a well opens several gas-bearing formations simultaneously. Moreover, there may occur gas cross-flows between the formations, which is considered to be economically undesirable. Given that a wellhead pressure p is the only parameter by changing which it is possible to control the production rate, it is important to know how production rates of exploited formations functionally depend on such a pressure. The pressure problem on two formations has already been considered in the author's previous studies. By this it is necessary to know the formation pressures and filtration coefficients. However, this condition often doesn't hold in practice. The present study considers the problem of determining unknown formation pressures and filtration coefficients of two gas-bearing formations opened by a single well on the basis of stationary wellhead pressure conditions and the total production rate measurements data. The problem is reduced to solving a complicated system of three nonlinear equations. The research involves an algorithm and an example of a numerical solution, with software processing of one of the possible options.

2. Literature review and problem statement

For clarity and convenience in understanding the research focus, the primary intention is to give a direct problem statement. It will allow introducing all required terminology. Next, there will be an overview of publications connected with the stated problem.

In [1, 2], attention is given to a mathematical model of a high-pressure gas flow in a gas-dynamic system consisting of a D-diameter vertical well and two horizontal gas-saturated porous formations of thicknesses h_j, opened at depths L_j, j=1.2 to the surface. The gas-bearing formations are isolated from each other by enclosing rocks, so they affect each other only through perforations (holes) vented in the well along the formation thickness. It is assumed that the gas flow in the formation is steady, isothermal, and radially symmetrical about the well axis. It is also assumed that h_j thicknesses are sufficiently small, so $h_j/L_j<<1$, and thus the through-thickness pressure variation can be neglected. The diameter D is significantly smaller than the depth L_j; therefore, it is reasonable to assume that the gas flow in the well is one-dimensional at a resistance force per unit mass being equal to $\lambda u^2/2D$, where u is an axial velocity and λ is a resistance coefficient. The latter assertionis explained in [3].

The pressure usually reaches tens and hundreds of atmospheres, therefore, the gas state equation is taken to be

$$s = z(s)\rho RT,$$

where s is a current pressure, ρ is a gas density, R is a gas constant, T is an average well bore temperature, and z(s) is a real gas factor.

Like in [1, 2], below we use representation

$$z(s) = e^{-\alpha s} + \beta s \ [4].$$

Here,

$$\alpha = -\ln \kappa/p_C, \ \beta = 0.1/p_C \ \text{and} \ \kappa = 0.73 + 0.17371\ln T/T_C,$$

where p_c and T_c are the critical pressure and temperature. Gas filtration in the j-formation complies with the nonlinear law of flow – the Darcy-Forchheimer law:

$$\frac{ds_j}{dr} = -\left(\alpha_j + \beta_j\rho_j\left|w_j\right|\right)w_j, j = 1,2.$$

The substantiation of this law is provided in [5] and [6]. In the present article, like in [1] and [2], the nonlinear law of flow (the Darcy-Forchheimer law) is used to account for a homogeneous porous medium in a radially symmetric gas flow. In [7], this law is investigated for a homogeneous porous medium in a 3D-dimension case. The Darcy-Forchheimer law is applied in [8] to study heterogeneous porous media.

Here, r is the current radius measured from the well axis, $s_j = s_j(r)$ is a pressure in the j-formation, $\rho_j = \rho_j(r)$ is a gas density, w_j is a filtration velocity, and α_j, β_j are the constants characterizing the formation filtration capacity and the gas viscosity. In this notation, $U = s_1(D/2)$ and $V = s_2(D/2)$ are wellbore pressures at the depths L_1 and L_2, $X = s_1(r_1)$ and $Y = s_2(r_2)$ are pressures in the upper and lower formations, r_j is a supply contour radius of the j-formation, j=1, 2. The production rate q_j of the j-formation is the gas volume under normal conditions produced from the formation per unit of time. The well production rate is $Q = q_1 + q_2$, under the assumptions made that $q_j = q_j(p)$ and $Q = Q(p)$. Here, p is a wellhead pressure. The filtration coefficients of the formations $a_j, b_j, j = 1,2$ are assumed to be determined as

$$a_j = \frac{\alpha_j s_a T_*}{\pi h_j}\ln\frac{2r_j}{D} > 0,\ b_j = \frac{\beta_j s_a \rho_a T_a}{2\pi^2 h_j^2}\left(\frac{2}{D} - \frac{1}{r_j}\right) > 0, \tag{1}$$

where s_a is an atmospheric pressure, ρ_a is a gas density under atmospheric pressure, $T_* = T/T_n$, $T_n = 293$ K. These values are derived from the filtration law. For instance, let us assume that j=1. The gas mass flowing along the well through the sidewall of the cylinder of radius $r > \frac{D}{2}$, and a height h_1 per unit of time equals $-2\pi r h_1 \rho_1(r) w_1(r)$. Under the formation conditions, this mass has a density $\rho_1(r)$, a temperature T, and it is subjected to a pressure $s_1(r)$. Under steady-state filtration conditions, this mass does not change, though under normal conditions it equals $s_a q_1 / RT_n$ – according to the Ideal Gas Law.

Taking into consideration that $s_1 = z(s_1)\rho_1 RT$ and comparing the two expressions for the mass, we shall obtain

$$w_1 = -\frac{q_1 s_a z(s_1) T_*}{2\pi r h_1 s_1}.$$

By substitutiing the obtained expression in the filtration law and integrating it with respect to r from D/2 to r_1, we shall obtain the following equation of steady filtration from the upper formation:

$$b_1 q_1^2 + a_1 q_1 = \int_U^X \frac{2sds}{z(s)}, q_1 > 0.$$

Along with this, b_1 and a_1 are calculated using formula (1) at j=1.

Let us define that g is a gravity acceleration: $\sigma_j = \frac{gL_j}{RT}$, $j = 1, 2,$ and $\sigma_3 = \sigma_2 - \sigma_1$. Let X_1, Y_1, X_2, Y_2 be the solutions of the following equations:

$$\int_{X_1}^{X} \frac{z(s)ds}{s} = \sigma_1,\ \int_{Y_1}^{Y} \frac{z(s)ds}{s} = \sigma_2,$$
$$\int_{X}^{X_2} \frac{z(s)ds}{s} = \sigma_3,\ \int_{Y_2}^{Y} \frac{z(s)ds}{s} = \sigma_3. \tag{2}$$

These values are uniquely determined and are effectively calculable using Newton's method [9].

Now we are ready to make an overview of previous publications on this topic.

The stationary production rate of a well draining several gas-bearing strata is treated in [10]. There it is assumed that all strata pressures are neighbouring, a gas cross-flow between the formations is absent, and a real gas factor is a constant. These assumptions are used to obtain the approximation formula for the well production rate. The problem of a simultaneous exploitation of two formations with identical pressures is studied in [11] under an assumption that the strata summary rate is a constant and the gas cross-flow is absent. The results of [10] become specified in [12] for a case when the difference of two squares of formation and bore pressures is identical for all formations. In works [10–12]:

1) gas cross-flows in a running well bore are not taken into account;

2) the real gas factor is assumed to be a constant.

The present article is based on [1, 2] in the following respects:

1) the described limitations are overcome;

2) the turning points concept is essentially applied.

The notion of a turning point is of significant importance for further research.

The definition. The turning point of the total well production rate Q(p) is a value of a wellhead pressure $p=P^*$, so that $Q(P^*)=0$. The turning point of the production rate $q_j(p)$ of a j-formation is such a value of the well-head pressure $p = P_j^*$ that $q_j(P_j^*) = 0$.

An assertion. There always exists a P^*. If $Y>X_2$, there also exists a P_2^*, with the inequalities $X_1 < P^* < Y_1 < P_2^*$ being satisfied. May $q_2>0$ be determined from the solution (V, q_2) of a system of equations, with $a_2>0$, $b_2>0$, and $\mu>0$ being specified as follows:

$$\int_X^V \frac{sz(s)ds}{s^2 + \mu q_2^2 z(s)} = \sigma_3, b_2 q_2^2 + a_2 q_2 = \int_V^Y \frac{2sds}{z(s)}. \tag{3}$$

For a P_1^* to exist, it is necessary and sufficient that the next inequality is satisfied in the following way:

$$\int_1^X \frac{sz(s)ds}{s^2 + \mu q_2^2 z^2(s)} \geq \sigma_1. \tag{4}$$

Under (4), $P_1^* < X_1$. If q_2 is so large that inequality (4) does not hold, then P_1^* does not exist, and for all p>1 there will be $q_1(p)<0$, i. e. even for a fairly small p there occurs a gas cross-flow from the lower formation into the upper one.

We argue that the existence of a P_1^* is independent of (a_1, b_1). It should also be noted that, according to the engineering system of units, L_j are in metres, D are in centimetres, q is a $10^3 m^3$/24-hour period, T is a Kelvin temperature. Therefore, when the value of μ is introduced into (3), (4) is calculable using the formula:

$$\mu = \frac{1{,}3761\lambda T^2}{D^5}. \tag{5}$$

By virtue of [1], system (3) is uniquely solvable. The systems of equations from which P^*, P_2^*, P_1^* are calculated by Newton's method are also taken into account. As shown in [1], if a P_1^* exists, and $1 \le p < P_1^*$, the production rates $q_1(p)$, $q_2(p)$ and $Q(p)$ and the well bore pressures U and V are connected with the formation pressures X and Y as well as the filtration coefficients (a_1, b_1) and (a_2, b_2) by the system of equations:

$$\int_p^U \frac{sz(s)ds}{s^2 + \mu Q^2 z^2(s)} = \sigma_1, \tag{6}$$

$$b_1 q_1^2 + a_1 q_1 = \int_U^X \frac{2sds}{z(s)}, \; q_1 > 0, \tag{7}$$

$$\int_U^V \frac{sz(s)ds}{s^2 + \mu q_2^2 z^2(s)} = \sigma_3, \tag{8}$$

$$b_2 q_2^2 + a_2 q_2 = \int_V^U \frac{2sds}{z(s)}, \; q_2 > 0, \tag{9}$$

$$q_1 + q_2 = Q. \tag{10}$$

If the condition $P_1^* < p < P^*$ is satisfied or a P_1^* does not exist, equations (6) and (8)–(10) are preserved, and (7) is replaced by the following equation:

$$b_1 q_1^2 - a_1 q_1 = \int_X^U \frac{2sds}{z(s)}, \; q_1 < 0, \; U > X. \tag{11}$$

The relations of type (6) for the first time are received and investigated in [13].

By virtue of [1], system (6)–(10) is uniquely solvable, and the description of its numerical solution algorithm is given as follows: at known X, Y and a_1, b_1, a_2, b_2, it is possible to find $q_j=q_j(p)$, U(p), V(p) and Q=Q(p) as functions of p. Hence, a solution of the direct problem is obtained. It is noteworthy that there is no closed-form evaluation of the integrals in formulae (6)–(9); thus, it is not yet the exact solution of the system. In [2], analogous results are obtained in the case of $Y>X_2$. The difficulty of the formulation of the direct problem is that the formation pressures X and Y and the filtration coefficients are unknown in practice. Hence, this raises the problem of calculating the six parameters – X, Y and a_1, b_1, a_2, b_2 – with respect to six-time stationary pressure measurements of p_i and a corresponding total production rate Q_i, where i=1,…,6. Thus, an inverse problem arises.

3. The purpose and objectives of the study

The purpose of this study is to show that the formation pressures X and Y and the filtration coefficients (a_1, b_1, a_2, b_2) may be calculated by the data of (p_i, Q_i) $(i = 1, 2, \ldots, 6)$ from the results of the direct problem solution.

To achieve the outlined aim, the following tasks are formulated and solved.

1. A three nonlinear equations system with unknowns $(\xi, \theta_3, \theta_5)$ is receive on the basis of the corresponding (12) system of equations of the direct problem.
2. We use a fundamental fact: the required (X, Y, a_1, b_1, a_2, b_2) may be expressed in terms of $(\xi, \theta_3, \theta_5)$; and show the parameters $(\xi, \theta_3, \theta_5)$ are changed in the bounded domain while the primary unknowns (X, Y, a_1, b_1, a_2, b_2) are a priori unbounded.
3. The geometry of the domain of the parameters $(\xi, \theta_3, \theta_5)$ is investigated.
4. The calculation method for solving the received three-equations system and its software realization are developed.
5. The test calculation for checking the efficiency and precision of the suggested method is carried out.

4. Formulation of the inverse problem

If the formation pressures and filtration coefficients are unknown, then P_1^*, P_2^* and even their existence are unknown. However, P^* can be measured experimentally: it is a stabilized shut-in wellhead pressure, so $Q(P^*) = 0$. Therefore, the measurements of p_i are conducted so that

$$p_1 < p_2 < \ldots < p_6 < P^*.$$

As shown in [1, 2], the production rates $q_j(p)$ are monotone decreasing functions of p as well as Q(p), just as U and V are monotone increasing functions of p. Hence,

$$Q_1 > Q_2 > \ldots > Q_6 > 0, \quad U_1 < U_2 < \ldots < U_6,$$

and

$$V_1 < V_2 < \ldots < V_6$$

are satisfied for $Q_i = Q(p_i)$. Considering that there is no information concerning the turning points P_1^*, P_2^* and the relations between X_2 and Y, it is a cause for viewing various options of the gauge p_i position with respect to the turning points P_1^* and P_2^* as well if one of them does not exist. It is easy to calculate that there are 13 such options, and each of them requires detailed consideration. In [14], attention is given to an option when $X_2<Y$, P_1^* exists, and the inequalities $p_1 < \ldots < p_6 < P_1^*$ are satisfied. In [15], the considered option is when

$$p_1 < .. < p_4 < P_1^* < p_5 < p_6 < P^*.$$

The present study deals with a more complicated case:

$$p_1 < p_2 < P_1^* < p_3 < .. < p_6 < P^* \tag{12}$$

under the same assumptions made concerning X_2, Y, P_1^*.

Let us assume $x_i = q_1(p_i)$, $y_i = q_2(p_i)$, $i = 1, \ldots, 6$ as unknown formation production rates, whereas U_i and V_i are unknown well bore pressures at $p=p_i$. Instead of the unknown formation pressures X and Y, let us introduce the parameters c_1 and c_2 according to the formulae:

$$c_1 = \int_1^X \frac{2sds}{z(s)}, c_2 = \int_1^Y \frac{2sds}{z(s)}. \tag{13}$$

Moreover, let us also suggest that

$$g_i = \int_1^{U_i} \frac{2sds}{z(s)}, d_i = \int_1^{V_i} \frac{2sds}{z(s)}. \tag{14}$$

From the monotony of the production rates and from (12), it follows that

$$x_1 > x_2 > 0 > x_3 > ... > x_6, \qquad (15)$$

$$y_1 > y_2 > ... > y_6 > 0. \qquad (16)$$

Now, from (6)–(14), it follows that the unknown production rates x_i and y_i and the filtration coefficients of the formations (a_1,b_1) and (a_2,b_2), as well as the parameters c_1 and c_2, and the well bore pressures U_i and V_i (i=1,...,6) (30 unknown variables in total) are connected with the measured wellhead pressures p_i and the total production rates Q_i, (i=1,...,6) by the following system of 30 equations:

$$\int_{p_i}^{U_i} \frac{sz(s)ds}{s^2+\mu Q_i^2 z^2(s)} = \sigma_1, \quad (i=1,...,6), \qquad (17)$$

$$b_1 x_i^2 + a_1 x_i - c_1 = -g_i,\ x_i > 0, \quad (i=1,2), \qquad (18)$$

$$b_1 x_i{}^2 - a_1 x_i + c_1 = g_i, \quad x_i < 0, \quad (i=3,...,6), \qquad (19)$$

$$\int_{U_i}^{V_i} \frac{sz(s)ds}{s^2+\mu y_i^2 z^2(s)} = \sigma_3, \quad (i=1,...,6), \qquad (20)$$

$$b_2 y_i^2 + a_2 y_i - c_2 = -d_i,\ y_i > 0, \quad (i=1,...,6), \qquad (21)$$

$$x_i + y_i = Q_i, \quad (i=1,...,6). \qquad (22)$$

Furthermore, from physical considerations as well as from (11), (15) and (16), it follows that the following inequalities are to be satisfied:

$$a_1 > 0, b_1 > 0, a_2 > 0, b_2 > 0, \qquad (23)$$

$$g_2 < c_1 < g_3, \qquad (24)$$

$$c_2 > d_6\,. \qquad (25)$$

5. Reduction of the system (17)–(22)

In equations (17), at each i=1,...,6, the values of p_i and Q_i are known, so U_i can be uniquely determined from this equation. There is an efficient U_i calculation algorithm based on Newton's method. It follows that U_i is a known parameter; thus g_i is known, too. The remaining system (18)–(22) contains 24 equations with the unknowns a_1, b_1, a_2, b_2, c_1, c_2, x_i, y_i and V_i (i=1,...,6). Let us set x_1, x_3 and x_5 as the principal unknowns and express the rest through them using relations (18)–(22). Let us consider the equations (18) at i=1 and (19) at i=3,5 as the system of the three linear equations for (b_1, a_1, c_1) pre-setting the following:

$$x_1 = x, \quad x_3/x = \theta_3, \quad x_5/x = \theta_5, \quad g_i/g_j = \lambda_{ij},$$

$$x_1 = x, \quad x_3/x = \theta_3, \quad x_5/x = \theta_5, \quad g_i/g_j = \lambda_{ij}, \ (1 \le i < j \le 6).$$

Then we shall obtain

$$b_1 = \frac{g_5}{x^2}\frac{[1-\lambda_{35}+(\lambda_{15}-1)\theta_3+(\lambda_{35}-\lambda_{15})\theta_5]}{(\theta_5-\theta_3)(1+\theta_3+\theta_5-\theta_3\theta_5)} =$$
$$= \frac{g_5}{x^2}\psi_1(\theta_3,\theta_5), \qquad (26)$$

$$a_1 = \frac{g_5}{x}\frac{[\lambda_{35}-1+(\lambda_{15}-1)\theta_3^2+(\lambda_{35}-\lambda_{15})\theta_5^2]}{(\theta_5-\theta_3)(1+\theta_3+\theta_5-\theta_3\theta_5)} =$$
$$= \frac{g_5}{x}\psi_2(\theta_3,\theta_5), \qquad (27)$$

$$c_1 = g_5\frac{[-\theta_3^2-\theta_3+\lambda_{15}\theta_3\theta_5(\theta_3-\theta_5)+\lambda_{35}(\theta_5^2+\theta_5)]}{(\theta_5-\theta_3)(1+\theta_3+\theta_5-\theta_3\theta_5)} =$$
$$= g_5\psi_3(\theta_3,\theta_5). \qquad (28)$$

On account of (12),

$$x_1 > x_2 > 0 > x_3 > x_4 > x_5 > x_6, \qquad (29)$$

by virtue of the monotony of U_i and V_i,

$$g_i < g_j,\ d_i < d_j,\ (1 \le i < j \le 6), \qquad (30)$$

so $0<\lambda_{ij}<1$. From (23), (24) and (26)–(30), it follows that the below-listed inequalities are to be satisfied:

$$\psi_1(\theta_3,\theta_5) > 0, \psi_2(\theta_3,\theta_5) > 0, \lambda_{25} < \psi_3(\theta_3,\theta_5) < \lambda_{35}, \qquad (31)$$

as well as the inequalities

$$\theta_5 < \theta_3 < 0. \qquad (32)$$

We also note that $\psi_3 = \lambda_{15} + \psi_1 + \psi_2$. In what follows from (18) at i=2 and from (19) at i=4,6, as well as using expressions (26)–(28) for b_1, a_1 and c_1, we shall obtain

$$x_i = xm_i(\theta_3,\theta_5), (i=2,4,6), \qquad (33)$$

where

$$m_2(\theta_3,\theta_5) =$$
$$= \frac{[\psi_2^2(\theta_3,\theta_5)+4\psi_1(\theta_3,\theta_5)(\psi_3(\theta_3,\theta_5)-\lambda_{25})]^{1/2}-\psi_2(\theta_3,\theta_5)}{2\psi_1(\theta_3,\theta_5)}, \qquad (34)$$

$$m_i(\theta_3,\theta_5) =$$
$$= \frac{\psi_2(\theta_3,\theta_5)-[\psi_2^2(\theta_3,\theta_5)+4\psi_1(\theta_3,\theta_5)(\lambda_{i5}-\psi_3(\theta_1,\theta_3))]^{1/2}}{2\psi_1(\theta_3,\theta_5)},$$
$$(i=4,6). \qquad (35)$$

Let us also establish that

$$\xi = \frac{x}{Q_1},\ t_i = \frac{Q_i}{Q_1}, \quad (i=2,4,6),$$

then from (22) and (33) it follows that

$$y_i = Q_1(t_i - \xi m_i(\theta_3,\theta_5)),\ (i=2,4,6). \qquad (36)$$

Similarly, from (21) at i=2, 4, 6, we shall obtain

$$b_2 = \frac{(d_4-d_6)y_2+(d_6-d_2)y_4+(d_2-d_4)y_6}{(y_2-y_4)(y_2-y_6)(y_4-y_6)}, \qquad (37)$$

$$a_2 = \frac{(d_6-d_4)y_2^2+(d_2-d_6)y_4^2+(d_4-d_2)y_6^2}{(y_2-y_4)(y_2-y_6)(y_4-y_6)}, \qquad (38)$$

$$c_2 = \frac{y_2y_4(y_2-y_4)d_6+y_2y_6(y_6-y_2)d_4+y_4y_6(y_4-y_6)d_2}{(y_2-y_4)(y_2-y_6)(y_4-y_6)}. \qquad (39)$$

According to (14), d_i is dependent on V_i, which is calculated through (20) at a known U_i and is thus implicitly dependent on $\gamma = \mu y_i^2$. By virtue of the z(s) function structure, there is no closed-form evaluation of the integral in formula (20); thus, it is not to be spoken of the exact evaluation of V_i. It should be noted, though, that V_i is analytically dependent on the parameter γ at a point γ=0. So it is natural to seek a solution V_i of the equation

$$\int_{U_i}^{V_i} \frac{sz(s)ds}{s^2+\gamma z^2(s)} = \sigma_3$$

in the form of

$$V_i = V_{i0} + \gamma V_{i1} + \frac{1}{2}\gamma^2 V_{i2} + \ldots, \qquad (40)$$

thus restricting ourselves to the first three addends. Alongside this, V_{i0} is determined from the equation

$$\int_{U_i}^{V_{i0}} \frac{z(s)ds}{s} = \sigma_3$$

and is calculated by Newton's method. V_{i1} and V_{i2} are determined successively in terms of onefold and twofold differentiations of the identity

$$\int_{U_i}^{V_i(\gamma)} \frac{sz(s)ds}{s^2+\gamma z^2(s)} = \sigma_3$$

with respect to the parameter γ; then let γ=0. Hence, we shall obtain

$$V_{i1} = \frac{V_{i0}}{z(V_{i0})} I_3(i),$$

$$V_{i2} = \frac{2z(V_{i0})}{V_{i0}} I_3(i) + \frac{V_{i0}(1+\alpha V_{i0})e^{-\alpha V_{i0}}}{z^3(V_{i0})} I_3^2(i) - \frac{2V_{i0}}{z(V_{i0})} I_5(i),$$

here

$$I_3(i) = \int_{U_i}^{V_{i0}} \frac{z^3(s)ds}{s^3}$$

and

$$I_5(i) = \int_{U_i}^{V_{i0}} \frac{z^5(s)ds}{s^5}.$$

Using (40), we assume

$$d_i = \int_1^{V_i(\gamma)} \frac{2sds}{z(s)}$$

in the form of

$$d_i(\gamma) = d_i(0) + \gamma d_{i1} + \frac{1}{2}\gamma^2 d_{i2} + \ldots,$$

also keeping the first three addends. Then we shall obtain

$$d_{i0} = \int_1^{V_{i0}} \frac{2sds}{z(s)}, \quad d_{i1} = \frac{2V_{i0}V_{i1}}{z(V_{i0})},$$

$$d_{i2} = 2\left(\frac{e^{-\alpha V_{i0}}(1+\alpha V_{i0})V_{i1}^2}{z^2(V_{i0})} + \frac{V_{i0}V_{i2}}{z(V_{i0})}\right).$$

If we set

$$\gamma_{i0} = d_{i0}, \gamma_{i1} = \mu d_{i1}, \gamma_{i2} = \frac{1}{2}\mu^2 d_{i2},$$

then for d_i we shall obtain the following approximation:

$$d_i = \gamma_{i0} + \gamma_{i1} y_i^2 + \gamma_{i2} y_i^4, \ (i=1,\ldots,6). \qquad (41)$$

At i=2, 4, 6, let us process the expression for y_i from (36) through (41) and then process the obtained expressions through formulae (37)–(39). Then we shall obtain the expressions for a_2, b_2 and c_2 of the form

$$b_2 = \frac{h(\xi)}{\eta(\xi)}, \ a_2 = \frac{k(\xi)}{\eta(\xi)}, \ c_2 = \frac{l(\xi)}{\eta(\xi)}. \qquad (42)$$

Here, h(ξ), k(ξ) and l(ξ) are the polynomials in ξ as to degrees 5, 6, 7, and η(ξ) is a third-degree polynomial in ξ. The coefficients of η(ξ) are polynomials in $m_j(\theta_3,\theta_5)$, j=2, 4, 6, and are not cited because of their complicatedness.

We note that 0<ξ<1. By virtue of the monotony of the production rates, we shall obtain

$$y_2 > y_4 > y_6. \qquad (43)$$

We also note that

$$t_{ij} = t_i - t_j, \ m_{ij}(\theta_3,\theta_5) = m_i(\theta_3,\theta_5) - m_j(\theta_3,\theta_5).$$

From (36) and (43), it follows that

$$0 < \xi < \min\left(1, {}^{t_{24}}\!/\!_{m_{24}}, {}^{t_{26}}\!/\!_{m_{26}}, {}^{t_{46}}\!/\!_{m_{46}}\right) = \xi_r. \qquad (44)$$

From (22) at i=1, 3, 5, we shall obtain

$$y_1 = Q_1(1-\xi), y_3 = Q_1(t_3 - \theta_3\xi), \ \ y_5 = Q_1(t_5 - \theta_5\xi). \qquad (45)$$

Let us now finally turn to the formulae (21) at i=1, 3, 5 by processing expressions (42) and (45) through them. Then, it is easily seen that the following relations are satisfied:

$$M_1(\xi) \equiv h(\xi)(1-\xi)^2 + Q_1^{-1}k(\xi)(1-\xi) - \\ -Q_1^{-2}l(\xi) + \eta(\xi)(Q_1^{-2}\gamma_{10} + \gamma_{11}(1-\xi\ )^2 + Q_1^2\gamma_{12}(1-\xi)^4) = 0, \quad (46)$$

$$M_2(\xi) \equiv h(\xi)(t_3-\theta_3\xi)^2 + Q_1^{-1}k(\xi)(t_3-\theta_3\xi) - Q_1^{-2}l(\xi) + \\ +\eta(\xi)(Q_1^{-2}\gamma_{30} + \gamma_{31}(t_3-\theta_3\xi)^2 + Q_1^2\gamma_{32}(t_3-\theta_3\xi)^4) = 0, \qquad (47)$$

$$M_3(\xi) \equiv h(\xi)(t_5-\theta_5\xi)^2 + Q_1^{-1}k(\xi)(t_5-\theta_5\xi) - Q_1^{-2}l(\xi) + \\ +\eta(\xi)(Q_1^{-2}\gamma_{50} + \gamma_{51}(t_5-\theta_5\xi)^2 + Q_1^2\gamma_{52}(t_5-\theta_5\xi)^4) = 0. \qquad (48)$$

The left-hand sides of these relations are represented by the seven-degree polynomials in ξ with the coefficients that are polynomially dependent on θ_3, θ_5 and on the algebraic functions of $m_j(\theta_3, \theta_5)$, j=2, 4, 6 as well as on p_i, Q_i, i=1,...,6. Thus, it is shown that if the system of equations (17)–(22) is satisfied, then the triple (ξ, θ_3, θ_5) satisfies the system of three equations (46)–(48) that are polynomial in the variable ξ. It is to be recalled that equalities (40) and (41) provide an approximation for V_i and d_i. Along with this, it turns out that the filtration coefficients (a_1, b_1), (a_2, b_2) and the coefficients c_1, c_2, which are connected with the formation pressures X and Y by formulae (13) are expressed through the solution (ξ, θ_3, θ_5) of the system (46)–(48) using formulae (26)–(28) and (37)–(39). At known c_1 and c_2, the formation pressures are evaluated by Newton's method. Thus, the solution of the inverse problem reduces to the solution of the three equation system (46)–(48). The advantage of choosing x_1, x_3, x_5 as the principal unknowns lies in the fact that, as it will be clear from the subsequent, the solution (ξ, θ_3, θ_5) lies within a bounded domain while the required parameters X, Y, a_1, ..., b_2 are a priori unbounded. Along with that, the number of the required parameters can be half. In part 6, we will show that inequalities (23)–(25) set bounds to the domain of variables $(\theta_3,\theta_5)\in\Omega_0$, with the geometry of the bounded domain Ω_0 being dependent on the inverse problem data.

6. The numerical solution of the inverse problem

Equations (46)–(48) mean that ξ is the common root of the three polynomials

$$M_j(\xi,\theta_3,\theta_5),\ j=1,2,3$$

at a point

$$(\theta_3,\theta_5)\in\Omega_0.$$

For approximate evaluation of (ξ,θ_3,θ_5), the domain Ω_0 is covered by the uniformly spaced node grid

$$(\theta_3(k),\ \theta_5(l)),$$

where

$$1\le k\le N,\ 1\le l\le N_0$$

are all integer numbers. In each node $(\theta_3(k),\ \theta_5(l))$, the coefficients of the polynomials $M_j(\xi)$ are determined as well as

$$\xi_r(k,l)=\xi_r(\theta_3(k),\theta_5(l)).$$

Such a node as $(\theta_3^*(k),\ \theta_5^*(l))$ is considered to be an approximate solution of the problem in which each of the M_j-polynomials has at least one root ξ_j^*, (j=1, 2, 3), and the distance between these roots on the grid of the nodes is minimal. Then

$$\frac{\xi_1^*+\xi_2^*+\xi_3^*}{3}$$

is taken as a corresponding approximate solution. Along with this, the satisfiability of the inequalities

$$b_2\left(\xi^*,\ \theta_3(k),\ \theta_5(l)\right)>0,$$

$$a_2\left(\xi^*,\ \theta_3(k),\ \theta_5(l)\right)>0,$$

$$c_2\left(\xi^*,\ \theta_3(k),\ \theta_5(l)\right)>d_6$$

is rechecked in accordance with (23), (25), and (37)–(39). It may occur that the polynomial

$$M_j\left(\xi,\ \theta_3(k),\ \theta_5(l)\right)$$

has a set K_j of roots of power that is greater than one in the interval

$$\left(0,\ \xi_r\left(\theta_3(k),\ \theta_5(l)\right)\right).$$

Then we select one point out of the sets K_1, K_2 and K_3 and find the diameter of the set obtained. As $\xi_1^*, \xi_2^*, \xi_3^*$, we take such elements of the sets K_j(j=1, 2, 3) that implement the minimum with the diameters obtained. The roots of the polynomials M_j are calculated using Newton's method. The algorithm is processed through software.

7. The description of the range of (ξ, θ_3, θ_5)

Here we give the description of the domain Ω_0 at the corner

$$\Lambda=\left\{(\theta_3,\theta_5):\theta_5<\theta_3<0\right\}$$

in which the inequalities (31) are satisfied. The denominator $\Delta(\theta_3,\theta_5)$ in (26)–(28) becomes zero on a hyperbola h_0 with the equation

$$\theta_3=h_0(\theta_5)=1+\frac{2}{\theta_5-1}$$

and $\Delta(\theta_3,\theta_5)>0$ at the intersection Λ with the domain of points lying under the lower branch of a hyperbola h_0. This branch meets the bisector of the third quadrant at a point

$$A=(1-\sqrt{2},1-\sqrt{2}),$$

so $\Delta(\theta_3,\theta_5)<0$ is in a curvilinear triangle with the vertexes at the points $A,(0,-1),(0,0)$.

As we establish that

$$\nu=\frac{1-\lambda_{15}}{\lambda_{35}-\lambda_{15}},$$

then $\nu>1$ and

$$\frac{1-\lambda_{35}}{\lambda_{35}-\lambda_{15}}=\nu-1.$$

The equality $\psi_1(\theta_3,\theta_5)=0$ is satisfied on the line

$$\tau_1:\theta_3=\frac{\theta_5}{\nu}+1-\frac{1}{\nu}$$

that passes through the points

$$\left(\theta_3=1-\frac{1}{\nu},\ \theta_5=0\right)\ \text{and}\ (\theta_3=0,\ \theta_5=1-\nu),$$

although in (26) the numerator is >0 to the left of the line τ_1 and it is <0 to the right of τ_1.

7. 1. The case when $\nu<2$

Let $\nu<2$, then $1-\nu>-1$. The line τ_1 and the hyperbola h_0 have a common point

$$\left(\theta_3^0=1-\sqrt{\frac{2}{\nu}},\ 1-\sqrt{2\nu}\right)\ \text{at}\ \Lambda.$$

The inequality $\psi_1(\theta_3,\theta_5)>0$ is satisfied in the subdomain of the corner Λ that consists of two connected components, which are described by the inequality in accordance with

$$1)-\infty<\theta_5<\theta_5^0,\ \theta_5<\theta_3<\tau_1(\theta_5),$$

$$\theta_5^0<\theta_5<1-\sqrt{2},\ \theta_5<\theta_3<h_0(\theta_5),$$

$$2)-1<\theta_5<\theta_5^0,\ h_0(\theta_5)<\theta_3<0,$$

$$\theta_5^0<\theta_5<1-\sqrt{2},\ \tau_1(\theta_5)<\theta_3<0. \quad (49)$$

Now we turn to the inequality $\psi_2(\theta_3,\theta_5)>0$. The numerator in (27) becomes zero on the branch of the hyperbola

$$\theta_3=\tau_2(\theta_5)=-\sqrt{\frac{\theta_5^2-\nu+1}{\nu}},\ \theta_5<-\sqrt{\nu-1},$$

and it is positive at points Λ lying below τ_2. The line

$$\theta_3=\frac{\theta_5}{\sqrt{\nu}}$$

lying in Λ is an asymptote of this branch. It is directly checked that $(\theta_3^0,\theta_5^0)\in\tau_2$, so the line τ_1 and hyperbolas h_0 and τ_2 meet at the only point in which the numerators ψ_1 and ψ_2 and the denominator become 0. The inequalities

$$\psi_1(\theta_3,\theta_5)>0,\quad \psi_2(\theta_3,\theta_5)>0$$

are simultaneously satisfied in the subdomain Λ that consists of two connected components in accordance with

$$1)-\infty<\theta_5<\theta_5^0,\ \tau_1(\theta_5)<\theta_3<\tau_2(\theta_5),$$

$$2)\ \theta_5^0<\theta_5<-\sqrt{\nu-1},\ \tau_1(\theta_5)<\theta_3<\tau_2(\theta_5),$$

$$-\sqrt{\nu-1}<\theta_5<1-\nu,\tau_1(\theta_5)<\theta_3<0.$$

We note that at $\nu\geq 1$ the next inequality holds:

$$\sqrt{2\nu}-1\geq\sqrt{\nu-1}, \quad (50)$$

so $\theta_5^0<-\sqrt{\nu-1}$.

Let us consider the inequality

$$\psi_3(\theta_3,\theta_5)<\lambda_{35} \quad (51)$$

in the domain where $\Delta>0$. Then it is possible to conceive of (51) as

$$h_1(\theta_3,\theta_5)=$$
$$=(\theta_5-\theta_3-\nu+1)(\theta_5+\nu-1)+(\nu-1)(\nu-2)>0. \quad (52)$$

The equation $h_1(\theta_3,\theta_5)=0$ is the equation of a hyperbola with the asymptote $\theta_5-\theta_3=\nu-1$, $\theta_5=1-\nu$. It is directly checked that h_1 has a common point $(0,-\sqrt{\nu-1})$ with the hyperbola τ_2 and also passes through the point (θ_3^0,θ_5^0), so this point is common for the curves h_0, h_1, τ_1 and τ_2.

From what has been said, it follows that inequality (52) is satisfied in the subdomain Ω:

$$\Omega=\{(\theta_3,\theta_5):\theta_5<\theta_3<0,\ \Delta>0,$$
$$\psi_1(\theta_3,\theta_5)>0,\ \psi_2(\theta_3,\theta_5)>0\}.$$

Let us turn to the inequality

$$\lambda_{25}<\psi_3(\theta_3,\theta_5) \quad (53)$$

in the domain where $\Delta>0$. Let us establish that

$$\sigma=\frac{1-\lambda_{25}}{\lambda_{25}-\lambda_{15}},\zeta=\frac{\lambda_{35}-\lambda_{25}}{\lambda_{25}-\lambda_{15}}=\frac{\sigma+1-\nu}{\nu}>0. \quad (54)$$

Inequality (53) can be perceived as

$$F(\theta_3,\theta_5)=-(\theta_5+\sigma)\theta_3^2+(\theta_5^2-\sigma)\theta_3+\zeta\theta_5(\theta_5+1)>0. \quad (55)$$

From (54), it follows that $\sigma>\nu-1$. Our prime interest is in the set

$$\Omega_0=\Omega\cap\{(\theta_3,\theta_5)\in\Lambda,F(\theta_3,\theta_5)>0\}.$$

Taking (50) into account, it can be shown that at $\theta_5=-\sigma$, inequality (55) is satisfied in Ω if $1>\sigma>\sqrt{2\nu}-1$, and it is not satisfied if $\sigma<\sqrt{2\nu}-1$. At $\theta_5\neq\sigma$, we consider $F(\theta_3,\theta_5)$ as a quadratic trinomial in θ_3 the coefficients of which are dependent on the parameters θ_5,σ,ζ.

Let $\varphi_0(\theta_5)$ be its discriminant

$$\varphi_0(\theta_5)=(\theta_5^2-\sigma)^2+4\zeta\theta_5(\theta_5+\sigma)(\theta_5+1). \quad (56)$$

Considering that the polynomial

$$\varphi_1(\theta_5)=\theta_5(\theta_5+\sigma)(\theta_5+1)$$

is negative at

$$\theta_5\in(-\infty,-1)\cup(-\sigma,0)$$

and is positive at $\sigma\in(-1,-\sigma)$, it seems reasonable to say that $\varphi_0(\theta_5)>0$ at an open set $G\subset(-\infty,0)$ that contains the neighbourhoods of the points

$$\theta_5=-1,\ \theta_5=-\sigma,\ \theta_5=0.$$

So at G, the real θ_3 denotes the roots of the polynomial $F(\theta_3,\theta_5)$ that are determined as

$$r_1(\theta_5)=\frac{\theta_5^2-\sigma-\sqrt{\varphi_0(\theta_5)}}{2(\theta_5+\sigma)},\quad r_2(\theta_5)=\frac{\theta_5^2-\sigma+\sqrt{\varphi_0(\theta_5)}}{2(\theta_5+\sigma)}, \quad (57)$$

and the satisfiability of inequality (55) is dependent on a mutual arrangement of θ_3 and $r_1(\theta_5), r_2(\theta_5)$ at a fixed θ_5. Many different variants for the boundaries of the domain Ω_0 are possible here.

7. 1. 1. Let $1>\sigma>\sqrt{2\nu}-1$, so the following inequalities are satisfied

$$-1<-\sqrt{\sigma}<-\sigma<1-\sqrt{2\nu}=\theta_5^0<-\sqrt{\nu-1}<0. \quad (58)$$

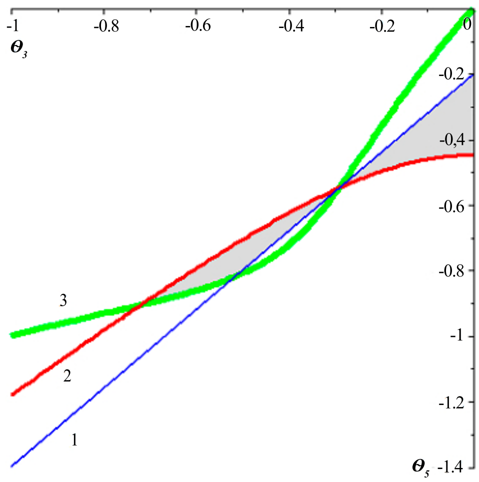
At $\theta_5\in(-1,-\sigma)$, it is evident that

$$r_1(\theta_5)>0,r_2(\theta_5)<0,$$

so inequality (55) is satisfied if $\theta_3<r_2(\theta_5)$. Fig. 1 shows the boundaries $1(\tau_1)$, $2(\tau_2)$ and $3(r_1)$ of the variables (θ_3,θ_5) and the view of a corresponding domain Ω_0 for this variant.

We also note that the following equalities hold:

$$\tau_2(-\sqrt{\sigma})=r_2(-\sqrt{\sigma})=-\sqrt{\zeta},\quad \tau_1(-\sigma)=\lim_{\theta_5\to-\sigma}r_2(\theta_5)=-\zeta.$$



$1 - \tau_1$, $2 - \tau_2$, $3 - r_1$

Fig. 1. A view of the domain Ω_0 with boundaries 1, 2, 3 for variant 7.1.1

Lemma 1. At $\theta_5 \in G \cap (-\sigma, 0)$, the following inequalities are true:

$$r_1(\theta_5) < -1, \; r_2(\theta_5) > -\zeta. \tag{59}$$

Proof. Since $\theta_5 + \sigma > 0$, from (57) it follows that

$$\operatorname{sign}\left(r_1(\theta_5)+1\right)=\operatorname{sign}\left(\theta_5^2+2\theta_5+\sigma-\sqrt{\varphi_0(\theta_5)}\right);$$

$$\operatorname{sign}\left(r_1(\theta_5)+1\right)=-1$$

for those values of θ_5 for which $\theta_5^2+2\theta_5+\sigma<0$. If not, then

$$\operatorname{sign}\left(r_1(\theta_5)+1\right)=\operatorname{sign}\left((\theta_5^2+2\theta_5+\sigma)^2-\varphi_0(\theta_5)\right)=$$
$$=\operatorname{sign}\left((1-\zeta)\varphi_1(\theta_5)\right)=-1,$$

since $\zeta<1$, $\operatorname{sign}\left(\varphi_1(\theta_5)\right)=-1$. The second inequality in (59) is proved in a similar way.

Consequence. Since $\zeta<1$, from the inequalities (59), it follows that $\varphi_0(\theta_5)>0$ at $(-\sigma, 0)$, so the inequalities (59) hold at $(-\sigma, 0)$.

Lemma 2. At $-\sigma<\theta_5<1-\sqrt{2\nu}$, the following inequality holds:

$$\tau_1(\theta_5)<r_2(\theta_5). \tag{60}$$

Proof. It is enough to show that

$$\operatorname{sign}[\nu(\sigma-\theta_5^2)+2(\theta_5+\nu-1)(\theta_5+\sigma)-\nu\sqrt{\varphi_0(\theta_5)}\;]=-1.$$

At the set

$$\Sigma_1=\left\{\theta_5\in(-\sigma,\theta_5^0):\nu(\sigma-\theta_5^2)+2(\theta_5+\nu-1)(\theta_5+\sigma)<0\right\},$$

it is evident, and at $(-\sigma,\theta_5^0)/\Sigma_1$ the required equality is equivalent to the equality

$$\operatorname{sign}[(\nu(\sigma-\theta_5^2)+2(\theta_5+\nu-1)(\theta_5+\sigma))^2-\nu^2\varphi_0(\theta_5)]=-1.$$

Here, the expression within the square brackets is equal to

$$4(\theta_5+\sigma)(\nu(\sigma-\theta_5^2)(\theta_5+\nu-1)+(\theta_5+\nu-1)^2(\theta_5+\sigma)-$$
$$-\nu^2\zeta\theta_5(\theta_5+1))=4(\theta_5+\sigma)\varphi_2(\theta_5).$$

However, it turns out that $\varphi_2(\theta_5)$ can be perceived as

$$\varphi_2(\theta_5)=(1-\nu)(\theta_5+\sigma)(\theta_5-1+\sqrt{2\nu}))(\theta_5-1-\sqrt{2\nu}),$$

whence inequality (60) follows.

Lemma 3. At $\theta_5\in(-\sqrt{\sigma},\theta_5^0)$, the following inequality holds:

$$\tau_2(\theta_5)<r_2(\theta_5). \tag{61}$$

Proof. The curves $\theta_3=\tau_2(\theta_5)$ and $\theta_3=r_2(\theta_5)$ are branches of the algebraic curves

$$\tau(\theta_3,\theta_5)\equiv\nu\theta_3^2-\theta_5^2+\nu-1=0$$

and

$$F(\theta_3,\theta_5)\equiv-(\theta_5+\sigma)\theta_3^2+(\theta_5^2-\sigma)\theta_3+$$
$$+\zeta\theta_5(\theta_5+1)=0.$$

We find the intersections of these curves, for which purpose we calculate the resultant $R(\tau, F, \theta_5)$ of these polynomials as the polynomials on θ_3. The direct calculation of the corresponding determinant gives us

$$R(\tau,F,\theta_5)=2\nu\zeta\theta_5(\theta_5+1)(\theta_5+\sigma)(-\theta_5^2+\nu-1)+$$
$$+\nu^2\zeta^2\theta_5^2(\theta_5+1)^2+\nu(\theta_5^2-\sigma)^2(-\theta_5^2+\nu-1)+$$
$$+(\theta_5+\sigma)^2(-\theta_5^2+\nu-1)^2. \tag{62}$$

Considering that $\nu\zeta=\sigma-\nu+1$, it can be shown that (62) has the form of

$$R(\tau,F,\theta_5)=-(\nu-1)(\theta_5^2-\sigma)^2((\theta_5-1)^2-2\nu). \tag{63}$$

However,

$$\tau(\theta_3,\theta_5) \text{ and } F(\theta_3,\theta_5)$$

have common points if and only if $R(\tau, F, \theta_5)=0$, i. e. only at the points $\theta_5=-\sqrt{\sigma}$ and $\theta_5=\theta_5^0=1-\sqrt{2\nu}$, (the solutions $\theta_5=\sqrt{\sigma}$ and $\theta_5=1+\sqrt{2\nu}$ do not belong to Λ and are thus of no interest). The corresponding values are

$$\theta_3=\tau_2(-\sqrt{\sigma})=-\sqrt{\zeta},\theta_3^0=1-\sqrt{\frac{2}{\nu}}.$$

From (57), it follows that $r_1(-\sqrt{\sigma})>0$. Since $-\sigma<\theta_5^0$, from the consequence of lemma 1 we shall obtain $r_1(\theta_5^0)<-1$. Consequently, the branch $r_1(\theta_5)$ has no common points with the curve $\tau_2(\theta_5)$. Hence

$$r_2(-\sqrt{\sigma})=-\sqrt{\zeta}=\tau_2(-\sqrt{\sigma}),\;\; r_2(\theta_5^0)=\theta_3^0=\tau_3(\theta_5^0),$$

and these branches do not have any other common points at $(-\sqrt{\sigma},\theta_5^0)$. So, in order to prove inequality (61), it is enough to show that $(r_2{}'-\tau_2{}')_{\theta_5=-\sqrt{\sigma}}>0$. However,

$$\tau_2{}'(-\sqrt{\sigma})=\frac{\sqrt{\sigma}}{\sqrt{\nu}\sqrt{\sigma+1-\nu}}.$$

It is easier to accomplish the calculation of $r_2{}'(-\sqrt{\sigma})$ while considering that there is an identity $F(r_2(\theta_5),\theta_5)\equiv0$. By differentiating it with respect to θ_5 and considering that $r_2(-\sqrt{\sigma})=-\sqrt{\zeta}$, we shall obtain

$$r_2'(-\sqrt{\sigma})=\frac{1-\sqrt{\zeta}}{1-\sqrt{\sigma}}=\frac{\sqrt{\nu}-\sqrt{\sigma+1-\nu}}{\sqrt{\nu}(1-\sqrt{\sigma})}.$$

Therefore,

$$\sqrt{\nu}(r_2'(-\sqrt{\sigma})-\tau_2'(-\sqrt{\sigma}))=\sqrt{\nu(\sigma+1-\nu)}+\nu-1-\sqrt{\sigma},$$

so

$$\operatorname{sign}(\sqrt{\nu(\sigma+1-\nu)}-(\sqrt{\sigma}-(\nu-1))=$$
$$=\operatorname{sign}[\nu(\sigma+1-\nu)-(\sqrt{\sigma}-(\nu-1))^2]=$$
$$\operatorname{sign}[(\nu-1)((\sqrt{\sigma}+1)^2-2\nu)]=\operatorname{sign}(\sqrt{\sigma}+1-\sqrt{2\nu})=1,$$

since $\nu>1, \sqrt{\sigma}>\sigma>\sqrt{2\nu}-1$.

Similar reasoning shows that the following lemma holds.

Lemma 4. At $\theta_5<-\sigma$, there holds the following inequality:

$$r_2(\theta_5)<\tau_1(\theta_5), \tag{64}$$

and at $-\sigma<\theta_5<\theta_5^0$ there holds the following inequality:

$$\tau_1(\theta_5)<r_2(\theta_5). \tag{65}$$

From the cited results, it follows that in the domain $\Delta>0$ inequality (31) is satisfied in the subdomain Ω_2, described by the inequalities

$$(-\sqrt{\sigma}<\theta_5<-\sigma, \tau_2(\theta_5)<\theta_3<r_2(\theta_5)),$$

$$(-\sigma<\theta_5<\theta_5^0, \tau_2(\theta_5)<\theta_3<\tau_1(\theta_5)),$$

so the domain Ω_1 is bounded.

We now turn to the situation in the subdomain Λ_0, in which $\Delta<0$. In this case,

$$\psi_1(\theta_3,\theta_5)>0,$$

if

$$\theta_3>1+\frac{\theta_5-1}{\nu}, \quad \psi_2(\theta_3,\theta_5)>0,$$

if

$$\theta_3<-\sqrt{\frac{\theta_5^2+1-\nu}{\nu}}, \quad \theta_5^0<\theta_5<-\sqrt{\nu-1}.$$

In this case, inequality (51) has the form of

$$h_1(\theta_3,\theta_5)<0,$$

and it can be easily seen that it is automatically satisfied in the domain where

$$\Delta<0, \ \psi_1(\theta_3,\theta_5)>0 \ \text{and} \ \psi_2(\theta_3,\theta_5)>0.$$

What is left to consider is inequality (53). In this situation, it has the form of

$$F(\theta_3,\theta_5)<0$$

and it is satisfied at

$$\theta_5^0<\theta_5 \ \text{and} \ r_2(\theta_5)<\theta_3.$$

From the estimates obtained, it follows that in the domain where $\Delta<0$,

$$\psi_1(\theta_3,\theta_5)>0, \ \psi_2(\theta_3,\theta_5)>0,$$

inequality (53) is automatically satisfied. Hence, in the case under consideration, the inequalities (31) are satisfied in a bounded domain $\Omega_0=\Omega_1\cup\Omega_2$, where Ω_2 is described by the inequalities

$$(\theta_5^0<\theta_5<-\sqrt{\nu-1}, \tau_1(\theta_5)<\theta_3<\tau_2(\theta_5))$$

and

$$(-\sqrt{\nu-1}<\theta_5<1-\nu, \tau_1(\theta_5)<\theta_3<0).$$

7. 1. 2. Let $1>\sqrt{\sigma}>\sqrt{2\nu}-1>\sigma>\sqrt{\nu-1}$

In this case, the inequalities

$$-\sqrt{\sigma}<1-\sqrt{2\nu}<-\sigma<-\sqrt{\nu-1}<1-\nu$$

are satisfied, and the domain Ω_0 is bounded and is described by the inequalities

$$(-\sqrt{\sigma}<\theta_5<\theta_5^0, \tau_2(\theta_5)<\theta_3<r_2(\theta_5)),$$

$$(\theta_5^0<\theta_5<-\sigma, r_2(\theta_5)<\theta_3<\tau_2(\theta_5)),$$

$$(-\sigma<\theta_5<-\sqrt{\nu-1}, \tau_1(\theta_5)<\theta_3<\tau_2(\theta_5)),$$

and

$$(-\sqrt{\nu-1}<\theta_5<1-\nu, \tau_1(\theta_5)<\theta_3<0).$$

7. 1. 3. Let $1>\sqrt{\sigma}>\sqrt{2\nu}-1>\sqrt{\nu-1}>\sigma>\nu-1$

Here the domain Ω_0 consists of two components:

$$\left(-\sqrt{\sigma}<\theta_5<\theta_5^0, \ \tau_2(\theta_5)<\theta_3<r_2(\theta_5)\right).$$

The second component is described by the inequalities

$$\left(\theta_5^0<\theta_5<-\sqrt{\nu-1}, \ r_2(\theta_5)<\theta_3<\tau_2(\theta_5)\right),$$

$$\left(-\sqrt{\nu-1}<\theta_5<-\sigma, \ r_2(\theta_5)<\theta_3<0\right),$$

and

$$\left(-\sigma<\theta_5<1-\nu, \ \tau_1(\theta_5)<\theta_3<0\right).$$

7. 1. 4. Let $\sqrt{2\nu}-1>\sqrt{\sigma}>\sqrt{\nu-1}>\sigma$

In this case, Ω_0 consists of one connected component described by the inequalities

$$(-\sqrt{\sigma}<\theta_5<-\sqrt{\nu-1}, r_2(\theta_5)<\theta_3<\tau_2(\theta_5)),$$

$$(-\sqrt{\nu-1}<\theta_5<-\sigma, r_2(\theta_5)<\theta_3<0),$$

and

$$(-\sigma<\theta_5<1-\nu, \tau_1(\theta_5)<\theta_3<0).$$

Fig. 2 shows boundaries $1(\tau_1)$, $2(\tau_2)$ and $3(r_1)$ of the variables (θ_3,θ_5) and a view of the domain Ω_0 for this variant.

7. 1. 5. Let $\sqrt{2\nu}-1>\sqrt{\sigma}>\sigma>\sqrt{\nu-1}$

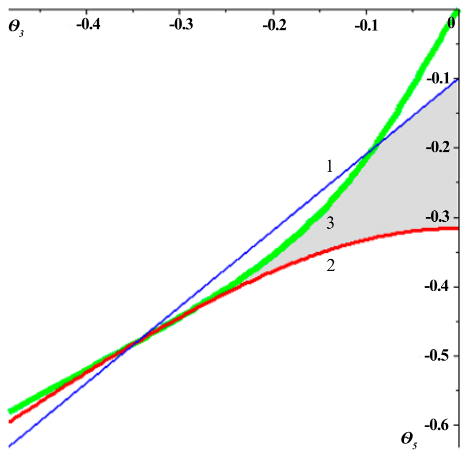
As in the previous case, Ω_0 consists of one component described by the inequalities

$$(-\sqrt{\sigma}<\theta_5<-\sigma, r_2(\theta_5)<\theta_3<\tau_2(\theta_5)),$$

$$(-\sigma<\theta_5<-\sqrt{\nu-1}, \tau_1(\theta_5)<\theta_3<\tau_2(\theta_5)),$$

and

$$(-\sqrt{\nu-1}<\theta_5<1-\nu, \tau_1(\theta_5)<\theta_3<0).$$



$1-\tau_1$, $2-\tau_2$, $3-r_1$

Fig. 2. A view of the domain Ω_0 with boundaries 1, 2, 3 for variant 7.1.4

7. 1. 6. Let $1<\sigma<2\nu-1$, **so** ζ**<1**

Now $\varphi_1(\theta_5)<0$ is at a set

$$(-1,0)\cup(-\infty,-\sigma), \text{ and } \varphi_1(\theta_5)>0$$

at $(-\sigma,-1)$. The reasoning as in 7. 1. 1 makes it possible to show that the domain Ω_0 consists of two connected components, which can be described by the inequalities in accordance with

$$(-\sigma<\theta_5<-\sqrt{\sigma}, r_1(\theta_5)<\theta_3<\tau_1(\theta_5)),$$

$$(-\sqrt{\sigma}<\theta_5<\theta_5^0, \tau_2(\theta_5)<\theta_3<\tau_1(\theta_5)),$$

and

$$(\theta_5^0<\theta_5<-\sqrt{\nu-1}, \tau_1(\theta_5)<\theta_3<\tau_2(\theta_5)),$$

$$(-\sqrt{\nu-1}<\theta_5<1-\nu, \tau_1(\theta_5)<\theta_3<0).$$

7. 1. 7. Let $\sigma>2\nu-1$, **so** ζ**>1**

Now lemma 1 is not applicable, and the question concerning the sign of a discriminant $\varphi_0(\theta_5)$ is solved by the following lemma.

Lemma 5. $\zeta>1$, then $\varphi_0(\theta_5)>0$ at $(-\infty,0)$.

Proof. Let us present $\varphi_0(\theta_5)$ in the following way:

$$\varphi_0(\theta_5)=(\theta_5^2+\sigma)^2+4\zeta(1+\sigma)\theta_5^2-4\sigma\theta_5^2+4\zeta\theta_5(\theta_5^2+\sigma)=$$
$$=(\theta_5^2+\sigma)^2+4\zeta\theta_5(\theta_5^2+\sigma)+4\zeta^2\theta_5^2-4\zeta^2\theta_5^2+4\zeta(1+\sigma)\theta_5^2-$$
$$-4\sigma\theta_5^2=(\theta_5^2+2\zeta\theta_5+\sigma)^2-4\theta_5^2(\zeta^2-\zeta(1+\sigma)+\sigma),$$

so what is left is to establish

$$\operatorname{sign}(\zeta^2-\zeta(1+\sigma)+\sigma).$$

Let

$$\delta=\frac{1}{\sigma}<\frac{1}{2\nu-1},$$

then

$$\zeta=\frac{\sigma(1-(\nu-1)\delta)}{\nu},$$

and

$$\operatorname{sign}(\zeta^2-\zeta(1+\sigma)+\sigma)=\operatorname{sign}((2\nu-1)\delta^2+2(\nu-1)\delta-1).$$

However, δ denotes the roots of the trinomial within the brackets, namely

$$\delta_+=\frac{1}{2\nu-1},\ \delta_-=-1,$$

so

$$\delta_-<\delta<\delta_+,$$

and

$$\operatorname{sign}\left((2\nu-1)\delta^2+2(\nu-1)\delta-1\right)=-1, \text{ Q.E.D.}$$

In this case, Ω_0 consists of two components described by the inequalities

$$(-\sigma<\theta_5<-\sqrt{\sigma}, r_1(\theta_5)<\theta_3<\tau_1(\theta_5)),$$

$$(-\sqrt{\sigma}<\theta_5<1-\sqrt{2\nu}, \tau_2(\theta_5)<\theta_3<\tau_1(\theta_5)),$$

$$(1-\sqrt{2\nu}<\theta_5<-\sqrt{\nu-1}, \tau_1(\theta_5)<\theta_3<\tau_2(\theta_5)),$$

and

$$(-\sqrt{\nu-1}<\theta_5<1-\nu,\ \tau_1(\theta_5)<\theta_3<0).$$

7. 2. The case when ν**>2**

Then

$$\sigma>1,\ \theta_3^0>0,\ \theta_5^0<0,$$

so the intersection point (θ_3^0,θ_5^0) of the hyperbolas τ_2, h_0, and h_1 and the line τ_1 is in the fourth quadrant (θ_3,θ_5), and it is thus of no interest. Now

$$1-\nu<-\sqrt{\nu-1},$$

and inequality (51) is satisfied in the subdomain Λ_0, in which

$$\psi_1(\theta_3,\theta_5)>0 \text{ and } \psi_2(\theta_3,\theta_5)>0.$$

We then turn to inequality (53), which is now considered in the domain where $\Delta>0$ and has the form of (55).

7. 2. 1. Let $\nu-1<\sqrt{\sigma}<\sigma<2\nu-1$, ζ**<1**

Inequality (31) is satisfied in the bounded domain Ω_0 that consists of one connected component according to the following inequalities

$$\left(-\sigma<\theta_5<-\sqrt{\sigma}, r_1(\theta_5)<\theta_3<\tau_1(\theta_5)\right),$$

$$\left(-\sqrt{\sigma}<\theta_5<1-\nu, \tau_2(\theta_5)<\theta_3<\tau_1(\theta_5)\right),$$

$$\left(1-\nu<\theta_5<-\sqrt{\sigma}, r_1(\theta_5)<\theta_3<0\right),$$

and

$$\left(1-\nu<\theta_5<-\sqrt{\nu-1}, \tau_2(\theta_5)<\theta_3<0\right).$$

7. 2. 2. Let $\sqrt{\sigma}<\nu-1<\sigma<2\nu-1$

In this case, Ω_0 is a bounded domain described by the inequalities

$$(-\sigma<\theta_5<1-\nu,\ r_1(\theta_5)<\theta_3<\tau_1(\theta_5)),$$

$$(1-\nu<\theta_5<-\sqrt{\sigma},\ r_1(\theta_5)<\theta_3<0),$$

and

$$(-\sqrt{\sigma}<\theta_5<-\sqrt{\nu-1},\ \tau_2(\theta_5)<\theta_3<0).$$

If $\sqrt{\sigma}>\sqrt{2\nu}-1$, then the domain is described by the same inequalities.

7. 2. 3. Let $\sqrt{\sigma}<\nu-1<2\nu-1<\sigma$

Now ζ>1, according to lemma 5, $\varphi_0(\theta_5)>0$ at $(-\infty,0)$.

The bounded domain Ω_0 consists of one component according to the following inequalities:

$$(-\sigma<\theta_5<1-\nu,\ r_1(\theta_5)<\theta_3<\tau_1(\theta_5)),$$

$$(1-\nu<\theta_5<-\sqrt{\sigma},\ r_1(\theta_5)<\theta_3<0),$$

and

$$(-\sqrt{\sigma}<\theta_5<-\sqrt{\nu-1},\tau_2(\theta_5)<\theta_3<0).$$

7. 2. 4. Let $\nu-1<\sqrt{\sigma}<2\nu-1<\sigma$

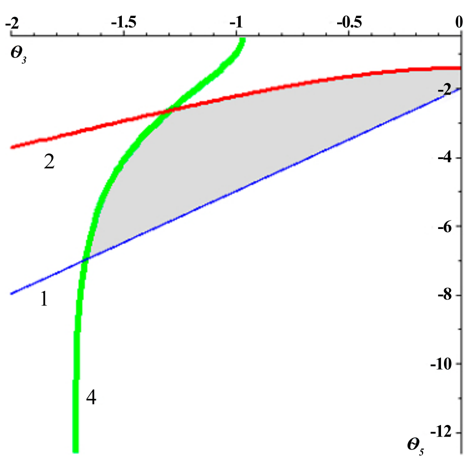
Here, there is also $\varphi_0(\theta_5)>0$ at $(-\infty,0)$, and Ω_0 is bounded and described by the inequalities

$$(-\sigma<\theta_5<-\sqrt{\sigma},\ r_1(\theta_5)<\theta_3<\tau_1(\theta_5)),$$

$$(-\sqrt{\sigma}<\theta_5<1-\nu,\ \tau_2(\theta_5)<\theta_3<\tau_1(\theta_5)),$$

and

$$(1-\nu<\theta_5<-\sqrt{\nu-1},\tau_2(\theta_5)<\theta_3<0).$$



$1-\tau_1$, $2-\tau_2$, $4-r_2$

Fig. 3. A view of the domain Ω_0 with boundaries 1, 2, 4 for variant 7.2.4

Fig. 3 shows the boundaries $1(\tau_1)$, $2(\tau_2)$ and $4(r_2)$ of the variables (θ_3,θ_5) and the view of a corresponding domain Ω_0 for this variant.

8. An example of the numerical solution of the inverse problem

Calculations were performed for the following values of the geophysical parameters: gas-methane formations where L_1=1,000 m and L_2=1,500 m; a wellhead temperature T_y=291 K and a bottom temperature T_3=310 K; a well diameter D=21.6 cm, and a resistance coefficient λ=0.023. By virtue of [16], the critical pressure and the critical temperature are taken as P_c=46.95 at T_c=190.55 K. In this case,

$$\alpha=0.0451,\quad \beta=0.00213,$$

$$\sigma_1=0.06368,\quad \sigma_3=0.03184,\ \text{and}\ \ \mu=0.00060745.$$

The selected values of X, Y, a_1, b_1 and a_2, b_2 are consistent with the data of [17]:

$$X=90\ \text{at},\ \ a_1=0.413,\ \ b_1=0.00062,$$

$$Y=100\ \text{at},\ \ a_2=0.863,\ \ b_2=0.001. \tag{66}$$

In this case,

$$P_1^*=83.3226\ \text{at}\ \ \text{and}\ \ P^*=85.7644\ \text{at}.$$

The wellhead pressures are taken as follows:

$$p_1=73\ \text{at},\ \ p_2=79\ \text{at},\ \ p_3=83.6\ \text{at},$$

$$p_4=84\ \text{at},\ \ p_5=85\ \text{at},\ \ p_6=85.5\ \text{at}. \tag{67}$$

By successively solving the direct problem with these data, we shall obtain the following total well production rate values:

$$Q_1=2{,}717.88377,\ \ Q_2=1{,}902.357766,$$

$$Q_3=730.937964,\ \ Q_4=578.428906,$$

$$Q_5=240.671,\ \ Q_6=81.983676. \tag{68}$$

Let us now take the data from (67) and (68) as the principal values in the solution of the inverse problem with the grid step of 0.002. As a result of solving the inverse problem, we shall obtain the following values of the unknowns X, a_1, b_1 and Y, a_2, b_2:

$$X=89.9972\ \text{at},\quad a_1=0.41218,\ \ b_1=0.00061,$$

$$Y=99.95035,\ \ a_2=0.77835,\ \ b_2=0.00108. \tag{69}$$

In the variant under consideration, ν=1.168724 and σ==1.13689, so case 1.6 occurs. As can be seen, a proper agreement of the parameters (69) with the assumed (66) is obtained.

9. Conclusion

The research results are the following:

– System (17)–(22) of 30 equations with 30 unknowns appears in a primary formulation of the inverse problem. It is reduced to a system of three equations about ξ, θ_3, θ_5, which are connected with the unknown rates q_1, q_3, q_5 of an upper stratum.

– It happens to be essential that these equations are polynomial of the seventh degree over the first unknown ξ, and their coefficients are expressed as some algebraic functions on θ_3 and θ_5.

– Another essential moment is the proof of an unbounded domain where the parameters (ξ, θ_3, θ_5) are changed and described in the domain geometry. Per contra, the desired formation pressures and filtration coefficients (X, a_1, b_1) and (Y, a_2, b_2) are a priori unbounded.

– These considerations allow applying the combination of a discretization technique to θ_3 and θ_5, with Newton's method being used to solve the polynomial equation of the seventh degree, and so the system (46)–(48).

– The algorithm of the numerical solution and its software realization were ready-built;

– The test calculation shows the efficiency and precision of the suggested method.

This article has described a step to constructing a full algorithm for solving the problem of identifying the pressures and filtration coefficients of two gas-bearing strata opened by a single well on the basis of the wellhead pressure and production rate measurements.

References

1. Markovskii, A. I. Stationary production rate of a well draining two gas-bearing formations [Text] / A. I. Markovskii // Fluid Dynamics. – 2007. – Vol. 42, Issue 4. – P. 673–680. doi: 10.1134/s0015462807040175
2. Markovskii, A. I. Stacionarnyj debit skvazhiny, vskryvajushhej dva gazonosnyh plasta [Text] / A. I. Markovskii // Sibirskii Zhurnal Industrialnoi Matematiki. – 2010. – Vol. 13, Issue 3. – P. 101–112.
3. Loicianskii, L. G. Mehanika zhidkosti i gaza [Text] / L. G. Loicianskii. – Moscow, 1957. – 379 p.
4. Latonov, V. V. Calculation of the compressibility coefficient of natural gases [Text] / V. V. Latonov, G. R. Gurevich // Gazovaya promishlennost. – 1969. – Issue 2. – P. 7–9.
5. Chekaliuk, E. B. Osnovy pjezometrii zalezhei nefti i gaza [Text] / E. B. Chekaliuk. – Kyiv: Gos.izdat. tehn. lit., 1961. – 288 p.
6. Pykhachev, G. B. Podzemnaja gidravlika [Text] / G. B. Pykhachev, R. G. Isaev. – Moscow, 1973. – 360 p.
7. Auriault, J.-L. Upscaling Forchheimer law [Text] / J.-L. Auriault, C. Geindreau, L. Orgeas // Transport in Porous Media. – 2007. – Vol. 70, Issue 2. – P. 213–229. doi: 10.1007/s11242-006-9096-x
8. Karimi-Fard, M. An Efficient Discrete-Fracture Model Applicable for General-Purpose Reservoir Simulators [Text] / M. Karimi-Fard, L. J. Durlofsky, K. Aziz // SPE Journal. – 2004. – Vol. 9, Issue 02. – P. 227–236. doi: 10.2523/79699-ms
9. Bellman, R. Quasilinearization and nonlinear boundary-value problems [Text] / R. Bellman, R. Kalaba. – American Elsevier publishing company, INC, New York, 1965.
10. Minskiy, E. M. Approximate calculation of gas discharge to a hole draining simultaneously several gas-bearing strata [Text] / E. M. Minskiy, M. L. Burshtein // Trudy VNIIneft – Gostoptehizdat. – 1956. – P. 263–278.
11. Fish (Burstein), M. L. On the simultaneous exploitation of two gas-bearing formations by one hole [Text] / M. L. Fish (Burshtein) // Trudy VNIIneft – Gostoptehizdat. – 1960. – Issue 9/17. – P. 99–111.
12. Korotaev, I. P. About a form of indicator curves of a hole draining several gas-bearing formations [Text] / I. P. Korotaev, G. A. Zotov // Trudy VNIIneft – Gostoptehizdat. – 1963. – Issue 18/26. – P. 97–104.
13. Markovskii, A. I. Steady-state flows in a vertical well at high pressures [Text] / A. I. Markovskii // Scientific bulletin of Ivano-Frankovsk National technical oil and gas university (IFNTUNG). – 2004. – Issue 1 (7). – P. 17–22.
14. Markovskii, A. I. Identification of formation pressures and flow coefficients of two gas-bearing formations opened by a single well from wellhead pressure and production rate measurements [Text] / A. I. Markovskii // Ukrayinskyi matematychnyi kongres. – 2010. – P. 142–157.
15. Markovskii, A. I. Identification of flow coefficients and formation pressures of two gas-bearing formations opened by a single well based on wellhead pressure and production rate measurements [Text] / A. I. Markovskii // Sibirskii zhurnal industrialnoi matematiki. – 2015. – Vol. 18, Issue 1 (61). – P. 95–109.
16. Directions for the complex investigation of gas and gascondensate for-mations and wells [Text] / G. A. Zotov, Z. S. Aliev (Eds.). – Moscow, 1980. – 301 p.
17. Kashparov, M. M. Certain features of the development of a cenomanian formation of the Yamburg field [Text] / M. M. Kashparov, G. M. Geresh // Problems of methods and novel technologies of natural gas field development. – Moscow, 1998. – P. 143–152.