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VERIFICATION OF REALIZABILITY OF BOOLEAN FUNCTIONS BY A NEURAL ELEMENT WITH A THRESHOLD ACTIVATION FUNCTION

F. Geche
Doctor of Technical Sciences, Professor, Head of Department*
E-mail: fedir.geche@uzhnu.edu.ua
O. Mulesa
PhD, Associate Professor*
E-mail: oksana.mulesa@uzhnu.edu.ua
V. Buchok
Postgraduate student*
E-mail: vityabuchok@gmail.com
*Department of cybernetics and applied mathematics
Uzhhorod National University
Narodna sq., 3, Uzhhorod, Ukraine, 88000

На основі властивостей ядер функцій алгебри логіки доведено критерій їх реалізованості одним нейронним елементом з пороговою функцією активації. Використовуючи зображення ядер булевих функцій матрицями толерантності отримано ряд необхідних і достатніх умов їх реалізованості одним нейронним елементом, які можуть бути ефективно застосовані при синтезі цілочислових нейронних елементів з великим числом входів

Ключові слова: матриця толерантності, опукла лінійна оболонка, вектор структури, функція активації

На основе свойств ядер функций алгебры логики доказан критерий их реализуемости одним нейронным элементом с пороговой функцией активации. Используя представление ядер булевых функций матрицами толерантности, получен ряд необходимых и достаточных условий их реализуемости одним нейронным элементом, которые могут быть эффективно применены при синтезе целочисленных нейронных элементов с большим числом входов

Ключевые слова: матрица толерантности, выпуклая линейная оболочка, вектор структуры, функция активации

1. Introduction

One may define recent years as a period of rapid development of technical means and information technologies with high performance efficiency that led to the creation and implementation of more effective methods of processing and analysis of data and new methods of solving complex applied problems. In this regard, there is a surge of theoretical and practical techniques in the field of neurocomputers and there is increased interest in neuro-like structures, which are widely applied in various areas of human activity – pattern recognition, forecasting, business, medicine, engineering.

Solving applied problems in neuro-basis would be possible if practically applicable methods of the synthesis of neural elements and the synthesis of logical circuits from them are developed.

Significant resources that are invested in creating software and hardware implementation of artificial neural networks, as well as widespread use of neuro-like structures, indicate that the problem of synthesis of neural elements with different activation functions and the construction of logical circuits from them is relevant and practically significant.

In practice, when recognizing discrete images, at the compression and transmission of discrete signals, it is necessary to be able to synthesize neural elements, that have a large number of inputs (≥100); in these cases, the classical methods of approximation of different orders and various iterative methods cannot be actually applied to the synthesis of neural elements for the realization of discrete functions.

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Therefore, the development of new methods of synthesis of neural elements that allow finding vectors of a neural elements structure with a large number of inputs, is a relevant and practical important task.

The results obtained in present work make it possible to synthesize neural elements with a large number of inputs for the implementation of Boolean functions under certain constraints on their kernels.

2. Literature review and problem statement

Artificial neural networks have been successfully used for the development of various components of intelligent systems. The main problem of applying these networks is to select required input data for the set problem, to form a test sample for training a neural network and to choose a learning algorithm.

The scope of practical applications of artificial neural networks is wide. They are effectively used to improve quality [1], segmentation [2], classification and recognition of images [3, 4]. Based on them, intelligent blocks of different systems for the chemical processes control are devised [5], for the classification of diseases [6] and diagnosis [7], to predict economic [8], biological [9] processes and forecasting the number of incidences of the disease under study [10]. As evidenced by research, neural network techniques are successfully used for the compression of signals and images [11–13], in the banking sector to assess credit risk [14].

It should be noted that the basis for constructing neural networks used in the aforementioned spheres of human activity is formed by various iterative methods that solve appropriate problems with certain accuracy. However, there are problems for which approximated solutions are not acceptable, for example, the problem on the realization of Boolean and multi-valued logical functions by one neural element with a threshold activation function or in the synthesis of combinational circuits of the specified neural elements. These combinational circuits can be successfully used when building functional blocks of logical devices to manage technological processes, for the compression of discrete signals, for the recognition of discrete images, etc. The shortcomings of iterative methods of training neural elements and neural networks for solving the problems on the Boolean and multi-valued functions realizability by one neural element (neural network) include:

– instead of precise solution, we receive approximated solution for the problem (for example, discrete function is realized by one neural element while iterative methods of relatively set accuracy reveal lack of its implementation (there is a problem of the choice of accuracy and process convergence relative to the assigned accuracy));

– the possibility of applying iterative methods for training artificial neurons with a small number of inputs (40, 50), whereas biological neurons may possess thousands of inputs [15].

The methods obtained in present work for the verification of realizability of Boolean functions by one neural element with a threshold activation function and the synthesis of relevant neural elements under certain constraints on their kernels can be applied as well in the case when the application of iterative methods is not expedient or practically impossible.

3. The aim and tasks of the study

The aim is to devise efficient methods for the synthesis of neural elements with a threshold activation function on the basis of which it is possible to synthesize neural networks for solving practically important problems in the field of compression and transmission of discrete signals, recognition of discrete images, diagnosis of technical devices.

To achieve the set aim, the following tasks are to be solved:

– to obtain realizability criteria of functions of the algebra of logic by one neural element with a threshold activation function;

– to establish necessary conditions for the realizability of Boolean functions by one neural element of the above described type;

– to propose sufficient conditions for the realizability of functions of the algebra of logic, which underlie the synthesis of integer neural elements with a large number of inputs.

4. Verification of the realizability of Boolean functions by one neural element with a threshold activation function and the method for synthesis of these elements

4. 1. Criteria and necessary conditions for the realizability of Boolean functions by one neural element with a threshold activation function

Assume $Z_2=\{0,1\}$ and Z_2^n is the nth Cartesian power of set Z_2. For Boolean function $f(x_1,\ldots,x_n)(f:Z_2^n\to Z_2)$, we shall define sets $f^{-1}(1)$, $f^{-1}(0)$:

$$f^{-1}(1)=\{\mathbf{x}\in Z_2^n\,|\,f(\mathbf{x})=1\},f^{-1}(0)=\{\mathbf{x}\in Z_2^n\,|\,f(\mathbf{x})=0\}. \quad (1)$$

By definition, neural element with a threshold activation function with a vector of structure $[\mathbf{w}=(\omega_1,\ldots,\omega_n);\omega_0]$ (n-dimensional real vector, called a weight vector, ω_0 is the real number (threshold)) implements Boolean function $f(x_1,\ldots,x_n)$, if condition is satisfied

$$\mathbf{x}\in f^{-1}(0)\Leftrightarrow(\mathbf{x},\mathbf{w})<\omega_0, \quad (2)$$

where $(\mathbf{x},\mathbf{w})$ is the scalar product of vectors $\mathbf{x}$ and $\mathbf{w}$.

If through

$$\left|f^{-1}(i)\right| \left(i\in\{0,1\}\right)$$

we denote the number of elements of set $f^{-1}(i)$, then, according to [16], kernel K(f) of Boolean function f is determined as:

$$K(f)=f^{-1}(1),$$

If $\left|f^{-1}(1)\right|\le\left|f^{-1}(0)\right|$, and $K(f)=f^{-1}(0)$ otherwise.

Assume $K(f)=\{\mathbf{a}_1,\ldots,\mathbf{a}_q\}$ is the kernel of Boolean function $f:Z_2^n\to Z_2$ and $K(f)^*=Z_2^n\setminus K(f)$. From the elements of kernel K(f) we shall construct matrix $K_\xi(f)$ in the following way: the first line of matrix $K_\xi(f)$ is vector

$$\mathbf{a}_{\xi(1)}=\left(\alpha_{\xi(1)1},\ldots,\alpha_{\xi(1)n}\right)$$

with K(f), the second line of the matrix is vector

$$\mathbf{a}_{\xi(2)}=\left(\alpha_{\xi(2)1},\ldots,\alpha_{\xi(2)n}\right),$$

the final line $K_\xi(f)$ is

$$\mathbf{a}_{\xi(q)} = \left(\alpha_{\xi(q)1},\ldots,\alpha_{\xi(q)n}\right),$$

where $\xi(i)$ is the substitution action $\xi \in S_q$ for i.

Matrix N, built from the first r lines of tolerance matrix $L \in E_n$ [16], is called tolerance prematrix and is denoted N=L(r).

Remark. If $K(f)=\varnothing$, then we consider that $K_\xi(f)=L(0)$, where L is the arbitrary matrix with E_n.

Let us determine convex linear shell convK(f) of kernel K(f) as follows:

$$\mathrm{conv}K(f)=\{\mathbf{x}\in[0,1]^n \,|\, \mathbf{x}=\sum_{i=1}^{q}\lambda_i\mathbf{a}_i,\ \sum_{i=1}^{q}\lambda_i=1,$$

$$\lambda_1\ge 0,\ldots,\lambda_q\ge 0;\mathbf{a}_1,\ldots,\mathbf{a}_q\in K(f)\}. \tag{3}$$

Theorem 1. Boolean function $f{:}Z_2^n\to Z_2$ is implemented by one neural element with a threshold activation function if, and only if, when $\mathrm{conv}K(f)\cap\mathrm{conv}K(f)^*=\varnothing$.

The proof. The necessity is to be proved by contradiction. Suppose that

$$\mathrm{conv}K(f)\cap\mathrm{conv}K(f)^*\ne\varnothing,\quad K(f)=f^{-1}(1)$$

and function f is implemented by one neural element with a threshold activation function. Then, as it is known [16], there is such $\xi\in S_q$ and $L=L_{\mathbf{w}}\in E_n$, that $K_\xi(f)=L_{\mathbf{w}}(q)$. It follows from the latter equality that for all $\mathbf{a}\in K(f)$ and for all $\mathbf{b}\in K(f)^*$

$$(\mathbf{a},\mathbf{w})>(\mathbf{b},\mathbf{w}). \tag{4}$$

Assume $\mathbf{d}\in\mathrm{conv}K(f)\cap\mathrm{conv}K(f)^*$. Then

$$\mathbf{d}=\sum_{i=1}^{q}\lambda_i\mathbf{a}_i;\ \sum_{i=1}^{q}\lambda_i=1;\lambda_1,\ldots\lambda_q\ge 0;\mathbf{a}_1,\ldots,\mathbf{a}_q\in K(f), \tag{5}$$

$$\mathbf{d}=\sum_{i=1}^{q'}\lambda'_i\mathbf{b}_i;\ \sum_{i=1}^{q'}\lambda'_i=1;\lambda'_1,\ldots\lambda'_{q'}\ge 0;\mathbf{b}_1,\ldots,\mathbf{b}_{q'}\in K(f)^*. \tag{6}$$

Assume

$$\omega_{min}=\min\{(\mathbf{a}_i,\mathbf{w})\big|\mathbf{a}_i\in K(f)\}$$

and

$$\omega'_{max}=\max\{(\mathbf{b}_j,\mathbf{w})\big|\mathbf{b}_j\in K(f)^*\}.$$

We obtain based on (4)–(6):

$$(\mathbf{d},\mathbf{w})=\sum_{i=1}^{q}\lambda_i(\mathbf{a}_i,\mathbf{w})\ge\left(\sum_{i=1}^{q}\lambda_i\right)\omega_{min}>\omega'_{max}=$$

$$=\left(\sum_{j=1}^{q'}\lambda'_j\right)\omega'_{max}\ge\sum_{j=1}^{q'}\lambda'_j(\mathbf{b}_j,\mathbf{w})=(\mathbf{d},\mathbf{w}). \tag{7}$$

The resulting inequality $(\mathbf{d},\mathbf{w})>(\mathbf{d},\mathbf{w})$ demonstrates that our assumption

$$\mathrm{conv}K(f)\cap\ \mathrm{conv}K(f)^*\ne\varnothing$$

at $K(f)=f^{-1}(1)$ is not valid. If $K(f)=f^{-1}(0)$, then we have $(\mathbf{d},\mathbf{w})<(\mathbf{d},\mathbf{w})$ and the necessity is proven.

Let us show that when $\mathrm{conv}K(f)\cap\mathrm{conv}K(f)^*=\varnothing$, then function f is realized by one neural element with a threshold activation function. Using convex shells convK(f) i convK(f)*, we shall build a set

$$D=\{\mathbf{d}=\mathbf{a}-\mathbf{b}\,|\,\mathbf{a}\in\mathrm{conv}K(f),\ \mathbf{b}\in\mathrm{conv}K(f)^*\}, \tag{8}$$

which is, obviously, convex and dows not contain zero vector **0** because

$$\mathrm{conv}K(f)\cap\mathrm{conv}K(f)^*=\varnothing.$$

Convex linear shells convK(f) and convK(f)* are compact [17], therefore, the set D is also compact and is therefore locked. Then, based on the separation theorem [18], one may argue that for D in the n-dimensional Euclidean space R^n there is such a hyperplane $\pi=\{\mathbf{x}\in R^n\,|\,(\mathbf{p},\mathbf{x})=p_0\}$ $(\mathbf{p}\ne\mathbf{0})$, $p_0\in R$, which satisfies conditions

$$p_0=(\mathbf{p},\mathbf{0})=0 \tag{9}$$

and for all $\mathbf{d}\in D$

$$(\mathbf{p},\mathbf{d})>p_0. \tag{10}$$

With regard to

$$\mathbf{d}=\mathbf{a}-\mathbf{b}\ (\mathbf{a}\in\mathrm{conv}K(f),\mathbf{b}\in\mathrm{conv}K(f)^*)$$

it follows from the latter inequlity that

$$(\mathbf{p},\mathbf{a})>(\mathbf{p},\mathbf{b}), \tag{11}$$

for any $\mathbf{a}\in\mathrm{conv}K(f)$ and for arbitrary $\mathbf{b}\in\mathrm{conv}K(f)^*$.

Hence inequality (11) holds for all $\mathbf{a}\in K(f)$ and for all $\mathbf{b}\in K(f)^*$.

Then, as shown in [19], there is such a vector $\mathbf{w}\in\Omega_n$ (Ω_n is the set of all such n-dimensional real vectors that $(\mathbf{x_1},\mathbf{w})\ne(\mathbf{x_2},\mathbf{w})$, if $(\mathbf{x_1}\ne\mathbf{x_2})$ and $\mathbf{x_1},\mathbf{x_2}\in Z_2^n$), which satisfies (11). This means that from the elements of kernel K(f) one may build such a matrix $K_\xi(f)$, that $K_\xi(f)=L_{\mathbf{w}}(q)$ $(L_{\mathbf{w}}\in E_n)$. Therefore, function f is implemented by one neural element with a threshold activation function and the theorem is proven.

Let us determine distance $\rho(\mathbf{a},\mathbf{b})$ between elements

$$\mathbf{a}=(\alpha_1,\ldots,\alpha_n)\ \text{and}\ \mathbf{b}=(\beta_1,\ldots,\beta_n)\in Z_2^n$$

as follows:

$$\rho(\mathbf{a},\mathbf{b})=\sum_{i=1}^{n}|\alpha_i-\beta_i|. \tag{12}$$

It is obvious, $\rho(\mathbf{a},\mathbf{b})$ is the number of coordinates in which vectors **a** and **b** are different.

Assume **a, b** are the arbitrary elements of kernel K(f) $(\mathbf{a}\ne\mathbf{b})$ of Boolean function $f:Z_2^n\to Z_2$ and $O(\mathbf{a},\mathbf{b})$ is the set of such unit vectors $\mathbf{e}_{i_1},\ldots,\mathbf{e}_{i_s}$, that

$$\mathbf{a}\oplus\mathbf{b}=\mathbf{e}_{i_1}+\mathbf{e}_{i_2}+\ldots+\mathbf{e}_{i_s},$$

where $\oplus$ is the coordinate-wise sum of vectors by module 2, $i_r\ne i_k$, if $r\ne k$. We shall denote through $H(\mathbf{a},\mathbf{b})$ a subgroup of group Z_2^n (Z_2^n forms a group relative to operation $\oplus$), which is generated by elements $O(\mathbf{a},\mathbf{b})$, that is,

$$H(\mathbf{a},\mathbf{b}) = \left\langle \mathbf{e}_{i_1},\ldots,\mathbf{e}_{i_s} \mid \mathbf{e}_{i_1},\ldots,\mathbf{e}_{i_s} \in O(\mathbf{a},\mathbf{b}) \right\rangle.$$

Assume

$$\mathbf{a} = (\alpha_1,\ldots,\alpha_n),\ \mathbf{b} = (\beta_1,\ldots,\beta_n) \in Z_2^n.$$

Coordinate-wise conjunction of vectors **a** and **b** will be denoted through

$$\mathbf{a} \,\&\, \mathbf{b} = (\alpha_1 \,\&\, \beta_1,\ldots,\alpha_n \,\&\, \beta_n)$$

and through $H(\mathbf{a} \,\&\, \mathbf{b})$ we shall denote adjacent class of group Z_2^n by subgroup H(**a, b**), which is determined by element $\mathbf{a} \,\&\, \mathbf{b}$, that is,

$$H(\mathbf{a} \,\&\, \mathbf{b}) = \mathbf{a} \,\&\, \mathbf{b} \oplus H(\mathbf{a},\mathbf{b}).$$

Theorem 2. If Boolean function $f{:}Z_2^n \to Z_2$ is implemented by one neural element with a threshold activation function, then for any two different elements a, b with K(f), for which

$$| H(\mathbf{a} \,\&\, \mathbf{b}) \cap K(f)^* | \geq 2$$

and for any two different elements g, h with $H(\mathbf{a} \,\&\, \mathbf{b}) \cap K(f)^*$, inequality $\rho(\mathbf{g},\mathbf{h}) < \rho(\mathbf{a},\mathbf{b})$ is true.

The proof. Let

$$\mathbf{a} = (\alpha_1,\ldots,\alpha_n),\ \mathbf{b} = (\beta_1,\ldots,\beta_n)$$

are the arbitrary different elements with K(f) $(\mathbf{a} \neq \mathbf{b})$,

$$\mathbf{g} = (\gamma_1,\ldots,\gamma_n),\ \mathbf{h} = (\delta_1,\ldots,\delta_n)$$

are the arbitrary different elements with $H(\mathbf{a} \,\&\, \mathbf{b}) \cap K(f)^*$ and $\rho = \rho(\mathbf{a},\mathbf{b})$. Without confining the generality of reasons, we shall assume that the first ρ coordinates of vectors **a** and **b** are different, while others are equal, that is, $\alpha_i \neq \beta_i$ for $i = 1,2,\ldots,\rho$ and $\alpha_i = \beta_i, i = \rho+1,\ldots,n$. It follows from theorem 1 and from the fact that function f is realized by one neural element with a threshold activation function:

$$\operatorname{conv} K(f) \cap \operatorname{conv} K(f)^* = \varnothing. \quad (13)$$

Therefore,

$$\lambda_1 \mathbf{a} + (1-\lambda_1)\mathbf{b} \neq \lambda_2 \mathbf{g} + (1-\lambda_2)\mathbf{h}, \quad (14)$$

for all $\lambda_1, \lambda_2 \in [0,1]$.

Given that points **a, b** $(\mathbf{a} \neq \mathbf{b})$, **g, h** $(\mathbf{g} \neq \mathbf{h})$ are the corner points of the corresponding sets convK(f), convK(f)* and $K(f) \cap K(f)^* = \varnothing$, inequality (14) can be replaced with inequality

$$\lambda_1(\mathbf{a}-\mathbf{b}) + \mathbf{b} \neq \lambda_2(\mathbf{g}-\mathbf{h}) + \mathbf{h} \quad (15)$$

provided

$$\lambda_1 \in (0,1) \text{ and } \lambda_2 \in (0,1). \quad (16)$$

It follows from (16) that there is such a number $r \in \{1,\ldots,\rho\}$, for which there is inequality

$$\lambda_1(\alpha_r - \beta_r) + \beta_r \neq \lambda_2(\gamma_r - \delta_r) + \delta_r. \quad (17)$$

Let us demonstrate that with (15), (16) and $\alpha_r \neq \beta_r$, then $\gamma_r = \delta_r$. This means that

$$\rho(\mathbf{g},\mathbf{h}) < \rho(\mathbf{a},\mathbf{b}). \quad (18)$$

Let us consider the following possible cases:

1. Assume $\alpha_r = 1$. Then $\beta_r = 0$ and from (17) we obtain $\lambda_1 \neq \lambda_2(\gamma_r - \delta_r) + \delta_r$. Hence

$$\gamma_r - \delta_r \neq \frac{1}{\lambda_2}(\lambda_1 - \delta_r). \quad (19)$$

The left part of inequality (19) takes the values from set {–1, 0, 1}, because $\gamma_r, \delta_r \in Z_2$.

The right side of inequality (19) based on (16) cannot be equal to 0 at any values of λ_1, λ_2. Therefore, inequality (19) is valid at arbitrary $\lambda_1, \lambda_2 \in (0,1)$ only when $\gamma_r - \delta_r = 0$. Thus, $\rho(\mathbf{g},\mathbf{h}) < \rho(\mathbf{a},\mathbf{b})$.

2. Assume $\alpha_r \neq 0$. Then $\beta_r = 1$ and it follows from (17) that

$$-\lambda_1 + 1 \neq \lambda_2(\gamma_r - \delta_r) + \delta_r, \quad (20)$$

or

$$(\gamma_r - \delta_r) \neq \frac{1}{\lambda_2}(1 - \lambda_1 - \delta_r). \quad (21)$$

Similar to the previous case, the latter inequality holds for all $\lambda_1, \lambda_2 \in (0,1)$ only when $\gamma_r - \delta_r = 0$. Thus, $\rho(\mathbf{g},\mathbf{h}) < \rho(\mathbf{a},\mathbf{b})$. The theorem is proved.

Assume

$$K(f) = \{\mathbf{a}_1,\ldots,\mathbf{a}_q\}$$

is the kernel of Boolean function

$$f: Z_2^n \to Z_2 \text{ and } K(f)_i = \{\mathbf{a}_i \oplus \mathbf{a}_1,\ldots,\mathbf{a}_i \oplus \mathbf{a}_q\}$$

is the reduced kernel [16] of function f relative to element $\mathbf{a}_i \in K(f)$.

We shall denote the set of all reduced kernels of Boolean function f through

$$T(f) = \left\{ K(f)_i = \mathbf{a}_i K(f) \,\middle|\, i = 1,2,\ldots,q \right\}.$$

They say that vector

$$\mathbf{a} = (\alpha_1,\ldots,\alpha_n) \in Z_2^n$$

precedes vector

$$\mathbf{b} = (\beta_1,\ldots,\beta_n) \in Z_2^n \ (\mathbf{a} \prec \mathbf{b}),$$

if $\alpha_i \leq \beta_i\ (i = 1,2,\ldots,n)$.

We shall denote through $M_{\mathbf{a}}$ the set of all such vectors from Z_2^n, which precedes vector **a**.

Assume $L = (\alpha_{ij})$ is the tolerance matrix over

$$Z_2 \left(j = 1,2,\ldots,n;\ i = 1,2,\ldots,2^{n-1} \right).$$

Let us build matrix $L^* = (\beta_{rj})$ in the following fashion:

$$\beta_{rj} = \overline{\alpha}_{2^{n-1}-i+1j},$$

where $\bar{\alpha}_{ij}$ is the negation of element α_{ij} and define operation ∇ over matrices L and L* as:

$$(L\nabla L^*)=\begin{pmatrix} L \\ L^* \end{pmatrix}.$$

Assume

$$\mathbf{w}=(\omega_1,\ldots,\omega_n)\in\Omega_n,$$

$$\mathbf{a}=(\alpha_1,\ldots,\alpha_n)\in Z_2^n \text{ and } \sigma\in S_n.$$

Let us define operations:

$$\mathbf{a}\mathbf{w}=\left((-1)^{\alpha_1}\omega_1,\ldots,(-1)^{\alpha_n}\omega_n\right) \text{ and } \mathbf{w}^\sigma=\left(\omega_{\sigma(1)},\ldots,\omega_{\sigma(n)}\right).$$

Theorem 3. If Boolean function $f:Z_2^n\to Z_2$ is implemented by one neural element with a threshold activation function, then in the set of reduced kernels T(f) there is such an element $K(f)_i$ that

$$\forall\mathbf{a}\in K(f)_i\Rightarrow M_\mathbf{a}\subset K(f)_i. \tag{22}$$

The proof. It follows from the fact that function f is realized by one neural element with a threshold activation function that there exist such $L\in E_n$ and $\xi\in S_q$ $\left(q=|K(f)|\right)$ [16] that satisfy condition

$$K_\xi(f)=L(q). \tag{23}$$

Assume $L=L_\mathbf{w}$, then [16]

$$(L\nabla L^*)\cdot\mathbf{w}^T=\mathbf{c}_\mathbf{w}^T, \tag{24}$$

where $\mathbf{c}_\mathbf{w}=(c_1,c_2\ldots,c_{2^n})$, $c_1>c_2>\ldots>c_{2^n}$.

Let us select by $\mathbf{a}_i$ the first line of matrix L and transform equality (23) as:

$$\mathbf{a}_iK_\xi(f)=\mathbf{a}_iL(q). \tag{25}$$

The coordinates of vector $\mathbf{w}_1=\mathbf{a}_i\mathbf{w}$ are negative, and matrix $L_{\mathbf{w}_1}=\mathbf{a}_iL$ satisfies condition [16]

$$(L_{\mathbf{w}_1}\nabla L^*_{\mathbf{w}_1})\cdot\mathbf{w}_1^T=\mathbf{c}_{\mathbf{w}_1}^T. \tag{26}$$

We shall position coordinates of vector w_1 in descending order, that is, construct vector $\mathbf{w}_2=\mathbf{w}_1^\sigma$, where $\sigma\in S_n$. Then, based on (26), we shall obtain:

$$(L_{\mathbf{w}_2}\nabla L^*_{\mathbf{w}_2})\cdot\mathbf{w}_2^T=\mathbf{c}_{\mathbf{w}_2}^T, \tag{27}$$

where $L_{\mathbf{w}_2}=L^\sigma_{\mathbf{w}_1}$. Assume **d** is the arbitrary line of matrix $\mathbf{a}_i^\sigma K_\xi^\sigma(f)=L_{\mathbf{w}_2}(q)$, except for the first one. If $\mathbf{b}\in Z_2^n$ is such that $M_\mathbf{b}\subset M_\mathbf{d}$ $(\mathbf{b}\neq\mathbf{d})$, then inequality $(\mathbf{d},\mathbf{w}_2)<(\mathbf{b},\mathbf{w}_2)$ is true. It follows from the latter inequality and (27) that the ordinal number of line **d** in matrix $L_{\mathbf{w}_2}(q)$ is larger than the ordinal number of line **b**. Thus, if $M_\mathbf{b}\subset M_\mathbf{d}$ $(\mathbf{b}\neq\mathbf{d})$, then

$$\mathbf{d}\in K_\xi^\sigma(f)_i\Rightarrow\mathbf{b}\in K_\xi^\sigma(f)_i\ (K_\xi^\sigma(f)_i=\mathbf{a}_i^\sigma K_\xi^\sigma(f)).$$

If we denote through **a** vector $\mathbf{b}^{\sigma^{-1}}$, then it directly follows from the latter ratio

$$\forall\mathbf{a}\in K(f)_i\Rightarrow M_\mathbf{a}\subset K(f)_i, \tag{28}$$

and the theorem is proved.

Let $T(f)=\left\{K(f)_i=\mathbf{a}_iK(f)\,|\,\mathbf{a}_i\in K(f)\right\}$ is the set of reduced kernels of Boolean function

$$f:Z_2^n\to Z_2,\ \ \mathbf{a}=(\alpha_1,\ldots,\alpha_n)\in Z_2^n,$$

$$\|\mathbf{a}\|=\sum_{i=1}^n\alpha_i,\quad k_i^*=\max\left\{\|\mathbf{a}\|\,|\,\mathbf{a}\in K(f)_i\right\}$$

and $k^*=\min\left\{k_i^*\,|\,i=1,2,\ldots,|K(f)|\right\}$.

It directly follows from theorem 3.

Corrolary. If Boolean function $f:Z_2^n\to Z_2$ is implemented by one neural element with a threshold activation function, then

1) in the set of reduced kernels T(f) there is such element $K(f)_i$, that, for any arbitrary $\mathbf{a}\in K(f)_i$ and for any inherent integer $r<\|\mathbf{a}\|$, inequality is true

$$\left|\left\{\mathbf{b}\in K(f)_i\,|\,\|\mathbf{b}\|=r\right\}\right|\geq C^r_{\|\mathbf{a}\|}, \tag{29}$$

where C_n^m is the binomial coefficient;

2) $|K(f)|\geq 2^{k^*}$.

4. 2. Sufficient conditions of realizability of Boolean functions by one neural element with a threshold activation function and its application for the synthesis of neural elements

Assume $A=\{\mathbf{a}_1,\ldots,\mathbf{a}_q\}\subseteq Z_2^n$ and $\{L_1,\ldots,L_n\}$ is the set of tolerance matrices whose elements are built by recurrent relation:

$$L_1=(0_1),\ L_2=\begin{pmatrix} L_1 & 0_1 \\ L_1^* & 0_1\end{pmatrix},\ \ldots,\ L_n=\begin{pmatrix} L_{n-1} & 0_{n-1} \\ L^*_{n-1} & 0_{n-1}\end{pmatrix}. \tag{30}$$

We shall denote through $p\left(\mathbf{a}_i^{\sigma_i}A^{\sigma_i}\right)$ the matrix whose lines are elements of maximum subsets of set $(\mathbf{a}_iA)^{\sigma_i}$, which satisfy condition

$$(L_{j_i}\underbrace{0_{j_i}\ldots0_{j_i}}_{n-j_i})\nabla(L^*_{j_i}(q_0^i)\underbrace{0\ldots0}_{n-(j_i+1)})\nabla\ldots\nabla(L^*_{j_i+r_i}(q^i_{r_i})\underbrace{0\ldots0}_{n-(j_i+r_i+1)}), \tag{31}$$

where $q_0^i\geq q_1^i\geq\ldots\geq q^i_{r_i}$.

Let $B=\left(\beta_{kj}\right)$ is the rectangular m×n matrix over Z_2 and s(j;B) is the number of unities of jth column of matrix B. Element $\sigma\in S_n$ relative to $\mathbf{a}=(\alpha_1,\ldots,\alpha_n)\in B$ will be determined so that

$$s(j-1;\mathbf{a}^\sigma B^\sigma)\geq s(j;\mathbf{a}^\sigma B^\sigma), \tag{32}$$

where $\mathbf{a}^\sigma B^\sigma=(\alpha_{\sigma(j)}\oplus\beta_{s\sigma(j)})$.

Assume,

$$p\left(\mathbf{a}_i^{\sigma_i}A^{\sigma_i}\right)=p_0\left(\mathbf{a}_i^{\sigma_i}A^{\sigma_i}\right)\nabla p_1\left(\mathbf{a}_i^{\sigma_i}A^{\sigma_i}\right)\nabla\ldots\nabla p_{r_i+1}\left(\mathbf{a}_i^{\sigma_i}A^{\sigma_i}\right), \tag{33}$$

where

$$p_0\left(\mathbf{a}_i^{\sigma_i}A^{\sigma_i}\right)=(L_{j_i}\underbrace{0_{j_i}\ldots0_{j_i}}_{n-j_i});$$

$$p_1\left(\mathbf{a}_i^{\sigma_i}A^{\sigma_i}\right)=(L^*_{j_i}(q_0^i)\underbrace{0\ldots0}_{n-(j_i+1)});$$

$$\ldots$$

$$p_{r_i+1}\left(\mathbf{a}_i^{\sigma_i}A^{\sigma_i}\right)=(L^*_{j_i+r_i}(q^i_{r_i})\underbrace{0\ldots0}_{n-(j_i+r_i+1)}) \tag{34}$$

and element $\sigma_i \in S_n$ is determined by condition (32).

Let us denote through

$$\left|p\left(\mathbf{a}_i^{\sigma_i} A^{\sigma_i}\right)\right|$$

the number of lines in matrix $p\left(\mathbf{a}_i^{\sigma_i} A^{\sigma_i}\right)$ and assume

$$\left|p\left(\mathbf{a}_m^{\sigma_m} A^{\sigma_m}\right)\right| = \max\left\{\left|p\left(\mathbf{a}_i^{\sigma_i} A^{\sigma_i}\right)\right| \mid i = 1,2,\ldots,q\right\}.$$

If set

$$\left\{\left|p\left(\mathbf{a}_i^{\sigma_i} A^{\sigma_i}\right)\right| \mid i = 1,2,\ldots,q\right\}$$

contains several maximal elements, then we shall denote one of them through

$$\left|p\left(\mathbf{a}_m^{\sigma_m} A^{\sigma_m}\right)\right|.$$

Threshold operator p with marks a_m and σ_m relative to set A will be determined as: $p(A) = p\left(\mathbf{a}_m^{\sigma_m} A^{\sigma_m}\right)$, that is,

$$p(A) = p_0(A) \nabla p_1(A) \nabla \ldots \nabla p_{t_0}(A), \qquad (35)$$

where

$$p_s(A) = p_s\left(\mathbf{a}_m^{\sigma_m} A^{\sigma_m}\right), \quad s = 0,1,\ldots,r_m + 1 \; (t_0 = r_m + 1).$$

The maximal subset of set A, from whose elements matrix (35) can be constructed, will be called the p-a subset of set A with marks a_m and σ_m and denoted through $A^{(1)}$.

We shall determine index j_0 of p-a subset $A^{(1)}$ as:

$$j_0 = \log_2 |p_0(A)| + 1,$$

where $|p_0(A)|$ is the number of lines in matrix $p_0(A)$.

Let us build the following system of sets:

$$A_{j_0}^{(1)} = \left\{\mathbf{a} = \left(\alpha_1,\ldots,\alpha_{j_0-1},1,\alpha_{j_0+1},\ldots,\alpha_n\right) \mid \mathbf{a} \in \left(\mathbf{a}_m A\right)^{\sigma_m}\right\},$$

$$A_{j_0+1}^{(1)} = \left\{\mathbf{a} = \left(\alpha_1,\ldots,\alpha_{j_0},1,\alpha_{j_0+2},\ldots,\alpha_n\right) \mid \mathbf{a} \in \left(\mathbf{a}_m A\right)^{\sigma_m}\right\},$$

$$\ldots$$

$$A_{j_0+t_0-1}^{(1)} = \left\{\mathbf{a} = \left(\alpha_1,\ldots,\alpha_{j_0+t_0-2},1,\alpha_{j_0+t_0},\ldots,\alpha_n\right) \mid \mathbf{a} \in \left(\mathbf{a}_m A\right)^{\sigma_m}\right\}, \qquad (36)$$

whose elements will be called one-index sets of p-expansion of set A. We shall apply threshold operator p to each of these sets in accordance with marks

$$\mathbf{e}_{j_0} = (0,\ldots,0,\underset{(j_0)}{1},0,\ldots,0),\ \sigma_m^{(1)},$$

$$\mathbf{e}_{j_0+1} = (0,\ldots,0,\underset{(j_0+1)}{1},0,\ldots,0),\ \sigma_m^{(1)},$$

$$\ldots$$

$$\mathbf{e}_{j_0+t_0-1} = (0,\ldots,0,\underset{(j_0+t_0-1)}{1},0,\ldots,0),\ \sigma_m^{(1)}, \qquad (37)$$

where $\sigma_m^{(1)}$ satisfies conditions:

1) $\forall j \in \{1,2,\ldots,j_0-1, j_0+t_0, j_0+t_0+1,\ldots,n\}$ $\sigma_m^{(1)}(j) = \sigma_m(j)$;

2) If $i,j \in \{j_0, j_0+1,\ldots,j_0+t_0-1\}$ $(i \neq j)$ and $\sigma_m^{(1)}(i) = \sigma_m(j)$, $\sigma_m^{(1)}(j) = \sigma_m(i)$, then it is possible only in the case when the sum of unities in columns i and j matrix $(\mathbf{a}_m A)_\xi^{\sigma_m}$ $(\xi \in S_q)$ coincide, that is,

$$s\left(i;(\mathbf{a}_m A)_\xi^{\sigma_m}\right) = s\left(j;(\mathbf{a}_m A)_\xi^{\sigma_m}\right);$$

3) if

$$i,j \in \{j_0, j_0+1,\ldots,j_0+t_0-1\}$$

and

$$s\left(i;(\mathbf{a}_m A)_\xi^{\sigma_m}\right) \neq s\left(j;(\mathbf{a}_m A)_\xi^{\sigma_m}\right),$$

then

$$\sigma_m^{(1)}(i) = \sigma_m(i),\ \sigma_m^{(1)}(j) = \sigma_m(j).$$

Assume

$$p\left(A_{j_0}^{(1)}\right) = p_0\left(A_{j_0}^{(1)}\right) \nabla p_1\left(A_{j_0}^{(1)}\right) \nabla \ldots \nabla p_{t_{j_0}}\left(A_{j_0}^{(1)}\right),$$

$$p\left(A_{j_0+1}^{(1)}\right) = p_0\left(A_{j_0+1}^{(1)}\right) \nabla p_1\left(A_{j_0+1}^{(1)}\right) \nabla \ldots \nabla p_{t_{j_0+1}}\left(A_{t_{j_0+1}}^{(1)}\right),$$

$$\ldots$$

$$p\left(A_{j_0+t_0-1}^{(1)}\right) = p_0\left(A_{j_0+t_0-1}^{(1)}\right) \nabla p_1\left(A_{j_0+t_0-1}^{(1)}\right) \nabla \ldots \nabla p_{t_{j_0+t_0-1}}\left(A_{t_{j_0+t_0-1}}^{(1)}\right), \qquad (38)$$

$$p^2(A) = p_0(A) \nabla \left(\mathop{\nabla}_{i_0=j_0}^{j_0+t_0-1} p\left(A_{i_0}^{(1)}\right)\right). \qquad (39)$$

We shall denote through $A^{(2)}$ the maximal subset of set A, from whose elements matrix $p^2(A)$, can be built.

Assume $f: Z_2^n \to Z_2$ is the Boolean function and its kernel is A=K(f).

Theorem 4. If $A = A^{(2)}$ and the blocks of matrix

$$p^2(A) = p_0(A) \nabla p_1^2(A) \nabla \ldots \nabla p_{t_0}^2(A) \qquad (40)$$

satisfy conditions:

1) $\left|p_0\left(A_{j_0+i}^{(1)}\right)\right| = \left|p_0\left(A_{j_0+i+1}^{(1)}\right)\right|$, $i = 0,1,\ldots,t_0-2$;

2) at each fixed

$$i \in \{0,1,\ldots,t_0-2\}\ t_{j_0+i} = t_{j_0+i+1}$$

and for each $k \in \left\{1,2,\ldots,t_{j_0+i+1}\right\}$

$$\left|p_k\left(A_{j_0+i}^{(1)}\right)\right| - \left|p_k\left(A_{j_0+i+1}^{(1)}\right)\right| = q_i \geq 0, \qquad (41)$$

then function f is implemented by one neural element with a threshold activation function.

The proof. It is given that $A = A^{(2)}$, that is, there are such elements $\mathbf{a} \in A$ and $\sigma \in S_n$, relative to which a matrix can be constructed from the elements of set $\mathbf{a}^\sigma A^\sigma$

$$p^2(A) = p_0(A) \nabla p_1^{(2)}(A) \nabla \ldots \nabla p_{t_0}^{(2)}(A).$$

Let us demonstrate that when blocks

$p_1^{(2)}(A), p_2^{(2)}(A), \ldots, p_{t_0}^{(2)}(A)$ of the matrix satisfy conditions 1, 2, then there is such n-dimensional vector **w,** for which there is inequality

$$\forall \mathbf{x} \in a^\sigma A^\sigma, \forall \mathbf{y} \in Z_2^n \setminus a^\sigma A^\sigma \ (\mathbf{x}, \mathbf{w}) > (\mathbf{y}, \mathbf{w}). \qquad (42)$$

Assume

$$j_1 = \log_2 \left| p_0\left(A_{j_0}^{(1)}\right) \right| + 1 \ (j_1 < j_0)$$

and

$$\omega_1 = -1, \omega_2 = \omega_1 - 1, \ldots, \omega_{j_1} = \sum_{i=1}^{j_1 - 1} \omega_i - 1. \qquad (43)$$

We shall denote through $\mathbf{z}_0, \ldots, \mathbf{z}_{t_{j_0}-1}$ the last lines of the respective matrices

$$p_1\left(A_{j_0}^{(1)}\right), \ldots, p_{t_{j_0}}\left(A_{j_0}^{(1)}\right) \ (t_0 \geq 2)$$

and coordinates ω_{j_1+s} are successively found from equalities:

$$(\mathbf{z}_s, (\omega_1, \omega_{j_1}, \ldots, \omega_{j_1+s}, 0, \ldots, 0)) =$$
$$= (\mathbf{z}_{s-1}, (\omega_1, \omega_{j_1}, \ldots, \omega_{j_1+s-1}, 0, \ldots, 0)), s = 1, 2, \ldots, t_{j_0} - 1. \qquad (44)$$

Parameters j_0, j_1 and $t_{j_0} - 1$ are related to one of the relations:

a) $j_0 - (j_1 + t_{j_0} - 1) > 1;$

b) $j_0 - (j_1 + t_{j_0} - 1) = 1.$

In the first case, the coordinates

$$\omega_{j_1 + t_{j_0}}, \omega_{j_1 + t_{j_0} + 1}, \ldots, \omega_{j_0 - 1}$$

of vector **w** will be found as

$$\omega_{j_1 + t_{j_0}} = \ldots = \omega_{j_0 - 1} = (\mathbf{z}_t, (\omega_1, \ldots, \omega_{j_1 + t}, 0, \ldots, 0)) - 1, \qquad (45)$$

where $t = t_{j_0} - 1.$

In the second case, coordinates $\omega_1, \ldots, \omega_{j_1}, \ldots, \omega_{j_0 - 1}$ of vector **w** have been already defined. Coordinate ω_{j_0} will be determined by formula:

$$\omega_{j_0} = \sum_{i=1}^{j_0 - 1} \omega_i - 1.$$

Then from 2nd condition of the theorem for coordinates $\omega_{j_0 + i}$ we obtain:

$$\omega_{j_0 + i} = \omega_{j_0 + i - 1} - q_{i-1} \ (i = 1, 2, \ldots, t_0 - 1).$$

Coordinates $\omega_{j_0 + t_0}, \omega_{j_0 + t_0 + 1} \ldots, \omega_n$ will be defined by formula:

$$\omega_{j_0 + t_0} = \ldots = \omega_n = (\mathbf{z}_0, (\omega_1, \ldots, \omega_{j_1}, 0, \ldots, 0)) + \omega_{j_0} - 1. \qquad (46)$$

Then vector $\mathbf{w} = (\omega_1, \ldots, \omega_n)$ satisfies condition (42) and for vector $\mathbf{w}_1 = \mathbf{a}\mathbf{w}^{\sigma^{-1}}$ we have:

$$\forall \mathbf{x} \in A, \forall \mathbf{y} \in Z_2^n \setminus A \quad (\mathbf{x}, \mathbf{w}_1) > (\mathbf{y}, \mathbf{w}_1). \qquad (47)$$

It follows from the latter inequality and [19] that there exists such vector $\mathbf{v} \in \Omega_n$, which satisfies (47), too. Thus, it is possible to build from elements of set A tolerance prematrix $L_{\mathbf{v}}(q) (q = |A|)$ and in this case the theorem is proved.

If $t_0 = 0$, or $t_0 = 1$, then $A^{(2)} = A^{(1)}$ and vector **w** is built according to [20]. Hence, the theorem is proved.

Let $\mathbf{a} = (\alpha_1, \ldots, \alpha_j, \ldots, \alpha_{j+s}, \ldots, \alpha_n)$ is the arbitrary vector of set $s \in \{0, 1, \ldots, n - j\}$ and $j \geq 2$. On the set of coordinates $\{\alpha_1, \ldots, \alpha_n\}$ of vector $\mathbf{a} \in Z_2^n$ for fixed s and j we shall determine function $\varepsilon_j^k \ (k \in \{0, \ldots, s\})$ as follows:

$$\varepsilon_j^k(\alpha_i) = \begin{cases} \alpha_i, & \text{if } i \leq j - 1; \\ \alpha_i (j - r_k), & \text{if } i = j + k; \\ \alpha_i j, & \text{if } i > j + s, \end{cases} \qquad (48)$$

where

$$r_0, r_1, \ldots, r_s \in \{1, 2, \ldots, j - 1\}.$$

Through functions $\varepsilon_j^k \ (k = 0, \ldots, s)$ at fixed $s \in \{0, \ldots, n - j\}$ and j we shall assign mapping of

$$\varepsilon_j^s : Z_2^n \to Z_{j+1}^n \quad \left(Z_{j+1} = \{0, 1, \ldots, j\}, 2 \leq j \leq n\right)$$

as follows:

$$\varepsilon_j^s(\mathbf{a}) =$$
$$= \left(\varepsilon_j^s(\alpha_1), \ldots, \varepsilon_j^s(\alpha_{j-1}), \varepsilon_j^0(\alpha_j), \varepsilon_j^1(\alpha_{j+1}), \ldots,\right.$$
$$\left.\varepsilon_j^s(\alpha_{j+s}), \varepsilon_j^s(\alpha_{j+s+1}), \ldots, \varepsilon_j^s(\alpha_n)\right) \qquad (49)$$

and define functional v_j^s on set Z_2^n by formula:

$$\forall \mathbf{a} \in Z_2^n \ v_j^s(\mathbf{a}) = \sum_{i \in I_s(j)} \varepsilon_j^s(\alpha_i) + \sum_{i=0}^{s} \varepsilon_j^i(\alpha_{j+i}), \qquad (50)$$

where $I_s(j) = \{1, 2, \ldots, n\} \setminus \{j, j+1, \ldots, j+s\}$. Using functional v_j^s for each

$$k \in \{0, 1, \ldots, s\},$$

we shall construct a set of Boolean vectors $F_{j+k}^{(r_k, s)}$ as:

$$F_{j+k}^{(r_k, s)} = \left\{\mathbf{a} \in m\left(L_{j+k}^* 0 \ldots 0\right) \mid v_j^s(\mathbf{a}) \leq j - 1\right\}, \qquad (51)$$

where $m\left(L_{j+k}^* 0 \ldots 0\right)$ is the set of Boolean vectors, built of the lines of matrix $\left(L_{j+k}^* 0 \ldots 0\right)$, and

$$r_0, r_1, \ldots, r_s \in \{1, \ldots, j - 1\}.$$

Theorem 5. If in kernel K(f) of Boolean function $f \ (f : Z_2^n \to Z_2)$, in group S_n there accordingly exist such elements **a**, $\boldsymbol{\sigma}$ and such integers

$$r_0 \geq r_1 \geq \ldots \geq r_s > 0 \quad (r_0 \leq j - 1)$$

that

$$\mathbf{a}^\sigma K(f)^\sigma = m(L_j \underbrace{0 \ldots 0}_{n-j}) \cup \left(\bigcup_{i=0}^{s} F_{j+i}^{(r_i, s)}\right), \qquad (52)$$

then function f is implemented by one neural element with a threshold activation function.

The proof. We shall assign n-dimensional vector $\mathbf{w} = (\omega_1, \ldots, \omega_n)$ as follows:

$$\omega_1 = \ldots = \omega_{j-1} = -1,$$

$$\omega_j = r_0 - j, \omega_{j+1} = r_1 - j, \ldots, \omega_{j+s} = r_s - j, \omega_{j+s+1} = \ldots = \omega_n = -j.$$

Upon construction of vector **w**, we obtain:

$$\min\left\{(\mathbf{x},\mathbf{w}) \mid \mathbf{x} \in m(L_j 0\ldots0)\right\} = 1-j,$$
$$\forall k \in \{0,1,\ldots,s\} \min\left\{(\mathbf{x},\mathbf{w}) \mid \mathbf{x} \in F_{j+k}^{(r_k,k)}\right\} = 1-j >$$
$$> \max\left\{(\mathbf{x},\mathbf{w}) \mid \mathbf{x} \in m\left(L_{j+k}^* 0\ldots0\right) \setminus F_{j+k}^{(r_k,k)}\right\} = -j,$$
$$\forall t \in \{s+1,\ldots,n\} \max\left\{(\mathbf{x},\mathbf{w}) \mid \mathbf{x} \in m\left(L_t^* 0\ldots0\right)\right\} = -j. \quad (53)$$

Then it follows directly from (52):

$$\forall \mathbf{x} \in \mathbf{a}^\sigma K(f)^\sigma, \forall \mathbf{y} \in Z_2^n \setminus \mathbf{a}^\sigma K(f)^\sigma \;\; (\mathbf{x},\mathbf{w}) > (\mathbf{y},\mathbf{w}). \quad (54)$$

Thus, there is such vector $\mathbf{v} \in \Omega_n^-$ [19], which also satisfies inequality (54). This means that there is such element

$$\xi \in S_q \; (q = |K(f)|),$$

that

$$\mathbf{a}^\sigma K_\xi(f)^\sigma = L_v(q),$$

where $L_v \in E_n^-$. Then, from the elements of kernel K(f) one can build tolerance prematrix

$$L_1(q)\left(L_1 = \mathbf{a} L_v^{\sigma^{-1}}\right).$$

The theorem is proved.

5. Discussion of results of the study

5. 1. An application of necessary conditions for verifying the realizability of Boolean functions by one neural element with a threshold activation function

The efficiency of applying necessary conditions, obtained in present paper, for the realizability of functions of the algebra of logic by one neural element with a threshold activation function will be demonstrated using the following problem: is the given Boolean function $f: Z_2^{100} \to Z_2$ realized by one neural element if

$$f^{-1}(1) = \left\{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_{100}}\right\},$$

where $\mathbf{e}_i$ is the unit vector, whose *i*th coordinate equals 1.

In this case,

$$K(f) = f^{-1}(1).$$

Let us build a set of reduced kernels

$$T(f) = \left\{K(f)_1, K(f)_2, \ldots, K(f)_{100}\right\}.$$

By the definition of a reduced kernel for arbitrary fixed i we have

$$K(f)_i = \left\{\mathbf{e_i} \oplus \mathbf{e_j} \,\middle|\, j = 1,2,\ldots,100\right\}.$$

If $j \neq i$, then for element

$$\mathbf{a} = \mathbf{e_i} \oplus \mathbf{e_j} \in K(f)_i,$$

condition $M_\mathbf{a} \subset K(f)_i$ is not satisfied. Thus, function f is not implemented by one neural element with a threshold activation function.

If one applies iterative methods for the synthesis of neural elements with a threshold activation function, which have n inputs and structure vector

$$\mathbf{w} = (\omega_1, \omega_2, \ldots, \omega_n; \omega_0),$$

then for finding the value of the weighted sum

$$\mathbf{w}(a) = \omega_1\alpha_1 + \omega_2\alpha_2 + \ldots + \omega_n\alpha_n + \omega_0$$

in set $\mathbf{a} = (\alpha_1, \alpha_2, \ldots, \alpha_n; \alpha_0) \in Z_2^n$ it is required to perform n operations of multiplying and n adding operations, that is, 2n arithmetic operations. Hence, it directly follows that to perform one step of iteration (excluding those arithmetic operations that are required to find new structure vector $\mathbf{w}_1$ by recurrent formula) is equal to $2^{n+1}n$. Thus, the number of arithmetic operations needed for the verification of realizability of Boolean functions by one neural element with a threshold activation function at any accuracy of approximation by the iterative method is not less than number $2^{n+1}n$. In our case, not less than number $2^{101}100$. This indicates that the iterative method is practically not applicable when solving the set problem because of a large number of required arithmetic operations.

5. 2. Synthesis of integer neural elements with threshold activation functions

We shall demonstrate how theorems 4 and 5 are applied for the synthesis of integer neural elements with a threshold activation function.

Example 1. Assume n=10, $\mathbf{a}_i$=(1,1,1,1,1,0,0,0,0,0),

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 6 & 7 & 8 & 9 & 10 & 1 & 2 & 3 & 4 & 5 \end{pmatrix},$$

$$K^\sigma(f)_i = p^2(K(f)_i) = p_0(K(f)_i) \nabla p_1^2(K(f)_i) \nabla p_2^2(K(f)_i),$$

where

$$p_0(K(f)_i) = (L_7 000), p_1^2(K(f)_i) =$$
$$= p_0\left(K(f)_{i,7}^{(1)}\right) \nabla p_1\left(K(f)_{i,7}^{(1)}\right) \nabla p_2\left(K(f)_{i,7}^{(1)}\right) \nabla$$

$$\nabla p_3\left(K(f)_{i,7}^{(1)}\right) =$$
$$= (L_4 001000) \nabla (L_4^*(4)001000) \nabla (L_5^*(2)01000) \nabla (L_6^*(2)1000);$$

$$p_2^2(K(f)_i) = p_0\left(K(f)_{i,8}^{(1)}\right) \nabla p_1\left(K(f)_{i,8}^{(1)}\right) \nabla p_2\left(K(f)_{i,8}^{(1)}\right) \nabla p_3\left(K(f)_{i,8}^{(1)}\right) =$$
$$= (L_4 000100) \nabla (L_4^*(3)000100) \nabla (L_5^*(1)00100) \nabla (L_6^*(1)0100).$$

Let us find a structure vector of neural element that implements function $f(x_1, \ldots, x_{10})$. In line with theorem 4, we shall construct vector $\mathbf{w} = (\omega_1, \ldots, \omega_{10})$:

$$j_1 = \log_2\left|p_0\left(K(f)_{i,7}^{(1)}\right)\right| + 1 = 4,$$

then $\omega_1 = -1, \omega_2 = -2, \omega_3 = -4, \omega_4 = -8.$

Vectors

$$\mathbf{z}_0 = (1,1,0,1,0,0,1,0,0,0),$$

$$\mathbf{z}_1 = (1,0,0,0,1,0,1,0,0,0),$$

$$\mathbf{z}_2 = (1,0,0,0,0,1,1,0,0,0)$$

are built from the last lines of corresponding matrices

$$p_1\left(K(f)^{(1)}_{i,7}\right),\ p_2\left(K(f)^{(1)}_{i,7}\right),\ p_3\left(K(f)^{(1)}_{i,7}\right).$$

Coordinates ω_5, ω_6 of vector $\mathbf{w}$ are successively found from relations:

$$(\mathbf{z}_s,(\omega_1,\ldots,\omega_{j_1},\omega_{j_1+1},\ldots,\omega_{j_1+s}0,\ldots,0)) =$$
$$= (\mathbf{z}_{s-1},(\omega_1,\ldots,\omega_{j_1},\omega_{j_1+1},\ldots,\omega_{j_1+s-1}0,\ldots,0)),\ s = 1,2.$$

In this case, $\omega_5 = -10, \omega_6 = -10$. Parameters j_0=7, j_1=4, t_7=3 satisfy condition $j_0 - (j_1 + t_{j_0} - 1) = 1$.
Then

$$\omega_7 = \omega_1 + \ldots + \omega_6 - 1 = -36.$$

Parameter t_0=2, and by appropriate formula (theorem 4)

$$\omega_{j_0+i} = \omega_{j_0+i-1} - q_{i-1}\ (i=1,2,\ldots,t_0-1)$$

is determined $\omega_8 = \omega_7 - 1 = -37$. The last coordinates of vector $\mathbf{w}$ will be found from the following relation:

$$\omega_9 = \omega_{10} = (\mathbf{z}_0,(\omega_1,\ldots,\omega_4,0,0,0,0,0,0)) + \omega_7 - 1 = -48.$$

The built vector

$\mathbf{w}$=(–1, –2, –4, –8, –10, –10, –36, –37, –48, –48)

satisfies condition

$$\forall \mathbf{x} \in K^\sigma(f)_i, \forall \mathbf{y} \in Z_2^n \setminus K^\sigma(f)_i\ (\mathbf{x},\mathbf{w}) > (\mathbf{y},\mathbf{w}).$$

Thus, Boolean function $f(x_1,\ldots,x_{10})$ is realized by the neural element with weight vector

$$\mathbf{w}_1 = \mathbf{a}_i\mathbf{w}^{\sigma^{-1}} = (10,36,37,48,48,-1,-2,-4,-8,-10)$$

and threshold

$$\omega' = \left(\mathbf{w}_1, \mathbf{a}_i\mathbf{z}_0^{\sigma^{-1}}\right) = 132,$$

if K(f)=f^{-1}(1). In the opposite case (K(f)=f^{-1}(0)) function $f(x_1,\ldots,x_{10})$ is implemented by the neural element with structure vector $[-\mathbf{w}_1;\omega'']$, where $\omega'' = -131$.

Example 2. Assume n=10, $\mathbf{a}_i$=(0,0,0,0,0,1,1,1,1,1),

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix},$$

j=5, r_0= r_1=3, r_2=2, r_3=1, r_4= r_5=0.

j=5, r0= r1=3, r2=2, r3=1, r4= r5=0.

In this case s=3, since r_3>0 i r_4=0.
For arbitrary

$$\mathbf{a} = (\alpha_1,\ldots,\alpha_{10}) \in Z_2^{10}$$

we shall determine $\varepsilon_5^3(\mathbf{a})$:

j=5, r0= r1=3, r2=2, r3=1, r4= r5=0.

$$\varepsilon_5^3(\mathbf{a}) = (\varepsilon_5^3(\alpha_1), \varepsilon_5^3(\alpha_2), \varepsilon_5^3(\alpha_3), \varepsilon_5^3(\alpha_4), \varepsilon_5^0(\alpha_5),$$
$$\varepsilon_5^1(\alpha_6), \varepsilon_5^2(\alpha_7), \varepsilon_5^3(\alpha_8), \varepsilon_5^3(\alpha_9), \varepsilon_5^3(\alpha_{10})) =$$
$$= (\alpha_1,\alpha_2,\alpha_3,\alpha_4,2\alpha_5,2\alpha_6,3\alpha_7,4\alpha_8,5\alpha_9,5\alpha_{10}).$$

Let us successively build sets $F_5^{(3,3)}$, $F_6^{(3,3)}$, $F_7^{(2,3)}$, $F_8^{(1,3)}$ by rule:

$$F_{j+k}^{(r_k,s)} = \left\{\mathbf{a} \in m\left(L^*_{j+k}0\ldots0\right) \mid v_j^s(\mathbf{a}) \le j-1\right\},$$

where $m\left(L^*_{j+k}0\ldots0\right)$ is the set of Boolean vectors, constructed of the lines of matrix L^*_{j+k};

$$F_5^{(3,3)} = \{(0,0,0,0,1,0,0,0,0,0),(1,0,0,0,1,0,0,0,0,0),$$
$$(0,1,0,0,1,0,0,0,0,0),\ (1,1,0,0,1,0,0,0,0,0),$$
$$(0,0,1,0,1,0,0,0,0,0),(1,0,1,0,1,0,0,0,0,0),$$
$$(0,1,1,0,1,0,0,0,0,0),\ \ (0,0,1,1,1,0,0,0,0,0,0)\};$$

$$F_6^{(3,3)} = \{(0,0,0,0,0,1,0,0,0,0),(1,0,0,0,1,0,0,0,0,0),$$
$$(0,1,0,0,0,1,0,0,0,0),(1,1,0,0,0,1,0,0,0,0),$$
$$(0,0,1,0,0,1,0,0,0,0),(1,0,1,0,0,1,0,0,0,0),$$
$$(0,1,1,0,0,1,0,0,0,0),(0,0,1,1,0,1,0,0,0,0),$$
$$(0,0,0,0,1,1,0,0,0,0)\};$$

$$F_7^{(2,3)} = \{(0,0,0,0,0,0,1,0,0,0),(1,0,0,0,0,0,1,0,0,0),$$
$$(0,1,0,0,0,0,1,0,0,0),(0,0,1,0,0,0,1,0,0,0),$$
$$(0,0,0,1,0,0,1,0,0,0)\};$$

$$F_8^{(1,3)} = \{(0,0,0,0,0,0,0,1,0,0)\}$$

and let us consider the Boolean function whose reduced kernel allows for the following representation:

$$K^\sigma(f)_i = m(L_5 00000) \cup F_5^{(3,3)} \cup F_6^{(3,3)} \cup F_7^{(2,3)} \cup F_8^{(1,3)}.$$

Then by theorem 5

$$\omega_1 = \omega_2 = \omega_3 = \omega_4 = -1, \omega_5 = 3 - 5 = -2,$$

$$\omega_6 = -2, \omega_7 = -3,$$

$$\omega_8 = -4, \omega_9 = \omega_{10} = -5.$$

Neural element with weight vector

$$\mathbf{w}_1 = \mathbf{a}_i\mathbf{w}^{\sigma^{-1}} = (0,0,0,0,0,1,1,1,1,1)\times$$
$$\times(-5,-5,-4,-3,-2,-2,-1,-1,-1,-1) =$$
$$= (-5,-5,-4,-3,-2,2,1,1,1,1)$$

by threshold

$$\omega_0 = (\mathbf{a}_i\mathbf{x}^{\sigma^{-1}}, \mathbf{w}_1) = 2,$$

where

$$\mathbf{x} = (\mathbf{z},0,0,0,0,0),$$

z is the last line in tolerance matrix L_5^*, implements function $f(x_1,\ldots,x_{10})$, if K(f)=f^{-1}(1). In the opposite case (K(f)=f^{-1}(0)) function $f(x_1,\ldots,x_{10})$ is realized by one neural element with structure vector

$$[\mathbf{w}_2 = (5,5,4,3,2,-2,-1,-1,-1,-1); \omega_0' = -1].$$

The examples presented in the article demonstrate that, based on theorems 4 and 5, it is possible to construct effective algorithms for the synthesis of integer neural elements with a threshold activation function with a large number of inputs.

6. Conclusions

For a wide use of neural elements with threshold activation functions for solving applied problems, it is necessary to possess efficient methods for the verification of realizability of functions of the algebra of logic on such elements and the methods for their synthesis with a large number of inputs. These tasks may include: compression and transmission of discrete signals, classification and recognition of discrete images, coding of information, selecting fragments in discrete images.

Based on the results, described in present article, on the structure of kernels and reduced kernels of Boolean functions and properties of tolerance matrices, we obtained:

– criteria for the realizability of Boolean functions by one neural element with a threshold activation function;

– effective necessary conditions for the validation of realizability of Boolean functions by one neural element with a threshold activation function;

– sufficient conditions for the realizability of functions of the algebra of logic by one neural element with a threshold activation function, using which it is possible to synthesize neural elements with integer structure vectors with a large number of inputs.

Obtained results might be successfully applied when developing methods for the synthesis of neural network circuits from integer neural elements with a large number of inputs. These neural network circuits can be effectively used for the encoding and compression, for the classification and recognition of discrete signals and images.

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Розглянуто використання факторного аналізу для вивчення факторів ризику виникнення кризи в сімейних відносинах, які приводять до дисциркуляторної енцефалопатії. За допомогою факторного аналізу обґрунтовано розбиття системи показників на змістовні блоки. Виявлені фактори дозволяють визначити мішені психокорекції, які включають особистісні якості й фактори, умовно віднесені до блоку сімейної кризи

Ключові слова: сімейна криза, дисциркуляторна енцефалопатія, факторний аналіз, тіснота зв'язку, когнітивні та емоційні розлади

Рассмотрено применение факторного анализа для изучения факторов риска возникновения кризиса в семейных отношениях, которые могут привести к дисциркуляторной энцефалопатии. С помощью факторного анализа обосновано разбиение системы показателей на содержательные блоки. Выявленные факторы позволяют определить мишени психокорекции, которые включают личностные качества и факторы, условно отнесенные к блоку семейного кризиса

Ключевые слова: семейный кризис, дисциркуляторная энцефалопатия, факторный анализ, теснота связи, когнитивные и эмоциональные расстройства

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FACTOR ANALYSIS OF CRISIS EMERGENCE IN FAMILY RELATIONS, CONTRIBUTING TO THE DEVELOPMENT OF DYSCIRCULATORY ENCEPHALOPATHY

L. Rysovana
Assistant
Department of Medical and Biological Physics and Medical Informatics*
E-mail: rluba@mail.ru

O. Vysotska
Doctor of technical Sciences, Professor
Department of biomedical engineering
Kharkiv National University of Radio Electronics
Nauky ave., 14, Kharkiv, Ukraine, 61166
E-mail: olena.vysotska@nure.ua

H. Falyova
PhD, Associate Professor
Department of scientific bases of management and psychology
Kharkiv National Pedagogical University after G. S. Skovoroda
Valentynivska str., 2, Kharkiv, Ukraine, 61000
E-mail: lena_f5@list.ru

M. Georgiyants
MD, Professor, Head of Department
Department of Anesthesiology and Intensive Care
Kharkiv Medical Academy of Postgraduate Education
Amosova str., 58, Kharkiv, Ukraine, 61176

V. Klymenko
MD, Professor
Department of Fundamentals of Pediatrics № 2*
E-mail: klymenkoviktoriia@gmail.com
*Kharkiv National Medical University
Nauky ave., 4, Kharkiv, Ukraine, 61022

1. Introduction

Today in Ukraine we may observe a considerable increase in negative phenomena in the sphere of marriage and family, namely a reduction in the quantity of new marriages, an increase in divorce rate, weakening family bonds and others [1].

The institute of family in the contemporary society, including Ukrainian, is subjected to significant changes. Its value-normative space is changing, the new types of families and family relations appear, and functional relations between the family and the society are transformed. The transitive nature of society development could not but affect

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