Розробляються та аналізуються лінійні та нелінійні математичні моделі процесу теплопровідності для елементів електронних систем, які описано шаром і кусково-однорідним шаром із наскрізним чужорідним циліндричним включенням, на одній із межових поверхонь яких зосереджено тепловий потік. Із використанням узагальнених функцій, введених лінеаризуючих функцій, кусково-лінійної апроксимації температури на поверхнях спряження неоднорідних елементів шару та інтегрального перетворення Генкеля знайдено аналітично-числові розв'язки лінійних і нелінійних крайових задач теплопровідності

Ключові слова: теплопровідність, ізотропні шар і кусково-однорідний шар, чужорідне наскрізне включення, термочутливість, тепловий потік

Разрабатываются и анализируются линейные и нелинейные математические модели процесса теплопроводности для элементов электронных систем, описанные слоем и кусочно-однородным слоем со сквозным инородным цилиндрическим включением, на одной из граничных поверхностей которых сосредоточен тепловой поток. С использованием обобщенных функций, введенных линеаризирующих функций, кусочно-линейной аппроксимации температуры на поверхностях сопряжения неоднородных элементов слоя и интегрального преобразования Хенкеля построены численно-аналитические решения линейных и нелинейных краевых задач теплопроводности

Ключевые слова: теплопроводность, изотропные слой и кусочно-однородный слой, инородное сквозное включения, термочувствительность, тепловой поток

UDC 536.24
DOI: 10.15587/1729-4061.2017.92551

DEVELOPMENT AND ANALYSIS OF MATHEMATICAL MODELS FOR THE PROCESS OF THERMAL CONDUCTIVITY FOR PIECEWISE UNIFORM ELEMENTS OF ELECTRONIC SYSTEMS

V. Havrysh
Doctor of Technical Sciences, Professor*
E-mail: ikni.pz@gmail.com
I. Ovchar
PhD, Associate Professor**
Department of Mathematics
Ivano-Frankivsk National Technical University of Oil and Gas
Karpatska str., 15, Ivano-Frankivsk, Ukraine, 76019
E-mail: iovchar@hotmail.com
Ya. Baranetskij
PhD, Associate Professor**
E-mail: vm.imfn@gmail.com
Ya. Pelekh
PhD, Senior Researcher
Department of Computational Mathematics and Programming***
E-mail: omp.imfn@gmail.com
P. Serduik
PhD, Associate Professor*
E-mail: pavlo.serdyuk@gmail.com
H. Ivasyk
PhD**
E-mail: Ivasyk-G@yandex.ua
*Software Department***
Department of Mathematics*
***National University "Lviv Polytechnic"
S. Bandery str., 12, Lviv, Ukraine, 79013

1. Introduction

Of particular importance in the production of electronic devices are composite materials, development of which is one of the leading challenges of modern materials science. The emergence of new composite materials with improved operational physical-mechanical properties will contribute to the creation of new technologies in aviation, space, shipbuilding, energy, electronic industries, machine building and transport. Among the composite materials, important place is occupied by the structures with foreign inclusions, which are widely used in the designs of sophisticated electronic systems, in particular, in the integrated sensors for monitoring temperature and humidity, light-emitting elements for dynamic light emitting diode lightening, selective optical filters, etc. Since the indicated structures are in a wide temperature range, then their high operational parameters predetermine the need for the examination and solution of nonlinear problems, due to the dependence of thermal-physical parameters of materials on the temperature of structures and conditions of heat exchange, on the temperature of their surfaces, because calculations of temperature fields, based on the linear mathematical models of processes of thermal conductivity, do not always yield satisfactory results [1]. Therefore, in order to devise a mathematical model, the most adequate to the real process, it is necessary to take into account dependence of thermal-physical parameters of materials on temperature, density of surface flows and intensity of internal heat sources, change in body shape and possible phase and structural transformations [2, 3].

2. Literature review and problem statement

Determining thermal state of both uniform and non-uniform structures attracts attention of many researchers.

Paper [1] developed a mathematical model for calculating the quasi-stationary temperature field in solid cylinder of rotation from composite material with nonlinear boundary conditions, which took into account dependence of the thermal-physical parameters of materials on temperature. Analytic expressions obtained for determining temperature fields make it possible to select the composition of composite materials for the parts of cylindrical type for a purpose to extend their operational lifecycle.

One-dimensional (flat, cylindrical-symmetric and spherically-symmetric) nonlinear problems on thermal conductivity for a given heat flux in the origin of coordinates in the form of a power function dependent on time were explored. Approximate solutions were obtained for the indicated problems with their convergence analyzed [2].

Analytical-numerical solution for a nonlinear problem on thermal conductivity using the integral method of thermal balance was found [3]. In order to improve the accuracy of solution, temperature function is approximated by the polynomials of high degrees. To determine coefficients of polynomials, additional boundary conditions are introduced. It is demonstrated that such an approach as early as in the second approximation leads to a significant improvement in the accuracy of solving a problem.

In paper [4], analytical-numerical solution for a non-stationary problem on thermal conductivity for a hollow ball was received, the thermal-physical parameters of whose material are dependent on temperature. In a particular case, a solution for a solid ball was obtained.

A variation approach was employed for the development of a nonlinear mathematical model for the thermal conductivity process for a two-dimensional medium with a thin inclusion. For the linearization of the stated problem, they applied a Newton-Raphson method. Discretization by the time variable was performed by the intermediate point method [5].

Article [6] solved a non-stationary problem on thermal conductivity and thermoelasticity for functional-gradient thick-wall spheres. Thermal-physical and thermoelastic parameters of materials, except for Poisson coefficient, are arbitrary functions of the radial coordinate.

Axisymmetric stationary problem on thermal conductivity and thermoelasticity of the hollow functionally gradient areas relative to the heat source was considered. The solutions are obtained as functions from spatial coordinates for temperature, the displacement component vector and stress tensor by using boundary conditions for radial and angular coordinates [7].

An overview of basic literary sources that address development and research into mathematical models for the thermal conductivity process demonstrated that the models, which remain insufficiently examined and underdeveloped, are those that would consider the piecewise uniform structure design and their thermal sensitivity (dependence of thermal-physical parameters on temperature). Since the structures are exposed to temperature influences, then, in certain intervals of temperatures, an impact of thermal sensitivity on the results of calculation of temperature fields manifests itself vividly. This leads to the development of nonlinear models for the process of thermal conductivity and for their analysis, because the solutions of boundary problems that correspond to these models are more accurate than the solutions for the corresponding linear boundary problems. Calculations of temperature fields in such systems are used subsequently for designing electronic devices to provide for their thermal stability. The accuracy of these calculations will affect effectiveness of the methods that will be employed in this case.

3. The aim and tasks of the study

The aim of present work is to create linear and nonlinear mathematical models for the process of thermal conductivity for the elements of complex electronic systems, which are described by an isotropic layer and a piecewise uniform layer with a through-inclusion, which are heated by concentrated heat flow in the local area of their boundary surfaces.

To achieve the set aim, the following tasks are to be solved:

– to obtain original equations of thermal conductivity with discontinuous and singular coefficients and boundary conditions and their analytical-numerical solutions, which would allow expressing thermal field in arbitrary point of structure “layer – inclusion” and “piecewise uniform layer – inclusion”;

– using the introduced linearizing functions, to linearize original nonlinear boundary problems on thermal conductivity, to obtain relations to determine these functions and, for a linear-variable coefficient of thermal conductivity, to receive calculation formulas that express thermal field in arbitrary point of the thermosensitive structures “layer – inclusion” and “piecewise uniform layer – inclusion”.

4. Basic results of examining the process of thermal conductivity for piecewise uniform elements of electronic systems

Let us state the boundary linear and nonlinear problems on thermal conductivity, present a technique for solving them and obtain analytical-numerical solutions that determine thermal field in the elements of electronic systems, which are geometrically described by a layer and a piecewise uniform layer with a through inclusion of cylindrical shape.

4. 1. Isotropic layer with a through inclusion

Object of study and its mathematical model. Let us consider a layer, isotropic relative to thermal parameters, that contains a foreign through cylindrical inclusion with radius R, assigned to cylindrical coordinate system (Or z) with the origin in the center of inclusion. In region

$$\Omega_0=\{(r,\varphi,-l):r\le R,0\le\varphi\le 2\pi\}$$

boundary surfaces

$$L_-=\{(r,\varphi,-l):0\le r<\infty,0\le\varphi\le 2\pi\}$$

of layer of the system is heated by concentrated heat flow whose surface density is q_0=const, and the other part of this surface of the layer and surface

$$L_+=\{(r,\varphi,l):0\le r<\infty,0\le\varphi\le 2\pi\}$$

are thermally insulated. At the boundary surface of inclusion

$$K_R=\{(R,\varphi,z):0\le\varphi\le 2\pi,|z|\le l\}$$

there is an ideal thermal contact

$$t_0=t_1,\ \ \lambda_0\frac{\partial t_0}{\partial r}=\lambda_1\frac{\partial t_1}{\partial r}$$

for r=R (0 – for inclusion, 1 – for layer) (Fig. 1).

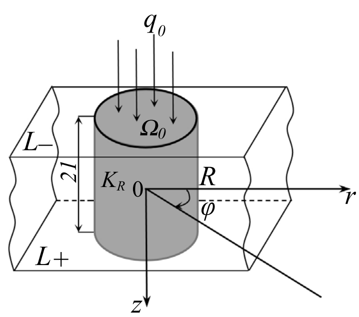

Fig. 1. Isotropic layer with a through inclusion

In the given structure, it is necessary to define temperature axisymmetric distribution t(r, z) by spatial coordinates, which we obtain upon solving the equation of thermal conductivity [8, 9]

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\lambda(r)\frac{\partial t}{\partial r}\right]+\lambda(r)\frac{\partial^2 t}{\partial z^2}=0 \tag{1}$$

with boundary conditions

$$t|_{r\to\infty}=t_c,\ \ \left.\frac{\partial t}{\partial r}\right|_{r\to\infty}=0,$$
$$\left.\frac{\partial t}{\partial z}\right|_{z=l}=0,\ \ \lambda_0\left.\frac{\partial t}{\partial z}\right|_{z=0}=-q_0 S_-(R-z), \tag{2}$$

where

$$\lambda(r)=\lambda_1+(\lambda_0-\lambda_1)S_-(R-r) \tag{3}$$

is the coefficient of thermal conductivity of a non-uniform layer; λ_1, λ_0 are the coefficients of thermal conductivity of materials of layer and inclusion, respectively; t_c is the ambient temperature

$$S_\pm(\zeta)=\begin{cases}1, & \zeta>0\\ 0{,}5\mp 0{,}5, & \zeta=0\\ 0, & \zeta<0\end{cases}$$

asymmetric single functions [10]. Let us introduce function [11]

$$T(r,z)=\lambda(r)\theta(r,z) \tag{4}$$

and differentiate it by variable r, with regard to the expression of coefficient of thermal conductivity λ(r) (3). As a result, we obtain:

$$\lambda(r)\frac{\partial\theta}{\partial r}=\frac{\partial T}{\partial r}+(\lambda_0-\lambda_1)\theta|_{r=R}\,\delta_+(r-R). \tag{5}$$

Here

$$\theta(r,z)=t(r,z)-t_c$$

is the excess temperature;

$$\delta_\pm(\varsigma)=\frac{dS_\pm(\varsigma)}{d\xi}$$

is the asymmetrical Dirac delta functions [10].

Substituting expression (5) in relation (1), we arrive at a differential equation with partial derivatives with singular coefficients

$$\Delta T+\frac{R}{r}(\lambda_0-\lambda_1)\theta|_{r-R}\,\delta_+'(r-R)=0, \tag{6}$$

where

$$\Delta=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{\partial^2}{\partial z^2}$$

is the Laplace operator in cylindrical coordinate system.

Therefore, the required thermal field in the indicated system is entirely determined by equation (6) with boundary conditions (2).

Analytical-numerical solution. We approximate function t(R, z) by variable z (Fig. 2) by expression [11, 12]

$$t(R,z)=t_1+\sum_{i=1}^{n-1}(t_{i+1}-t_i)S_-(z-z_i), \tag{7}$$

where $z_i\in(-l;\,l)$; $z_1\le z_2\le\ldots\le z_n$; n is the number of partitions of interval (–l; l); $t_i\ (i=\overline{1,n})$ are the unknown approximated temperature values.

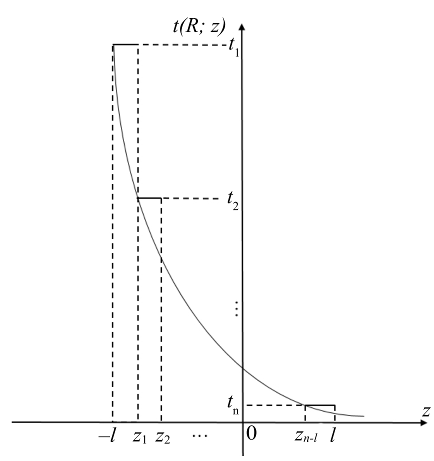

Fig. 2. Approximation of function t(R, z) by z coordinate

Substituting expression (7) in equation (6), we shall obtain:

$$\Delta T = -\frac{R}{r}(\lambda_0 - \lambda_1)\times$$
$$\times\left[\theta_1 + \sum_{i=1}^{n-1}(\theta_{i+1} - \theta_i)S_-(z - z_i)\right]\delta_+(r - R). \qquad (8)$$

Using the integral Hankel transform by coordinate r to equation (8) and boundary conditions (2), taking into consideration relation (4), we arrive at the ordinary differential equation with constant coefficients

$$\frac{d^2\overline{T}}{dz^2} - \xi^2\overline{T} =$$
$$= -R\xi J_1(R\xi)(\lambda_0 - \lambda_1)\left[\theta_1 + \sum_{i=1}^{n-1}(\theta_{i+1} - \theta_i)S_-(z - z_i)\right] \qquad (9)$$

and boundary conditions

$$\left.\frac{d\overline{T}}{dz}\right|_{z=l} = 0,\quad \left.\frac{d\overline{T}}{dz}\right|_{z=-l} = -\frac{q_0 R}{\xi}J_1(R\xi), \qquad (10)$$

where

$$\overline{T}(\xi,z) = \int_0^\infty rJ_0(r\xi)T(r,z)dr$$

is the transformant of function T(r, z); ξ is the parameter of the integral Hankel transform; $J_\nu(\zeta)$ is the Bessel function of first kind of the ν-th order.

A general solution for equation (9) will be found by the method of variation of constants in the form

$$\overline{T} = c_1e^{\xi z} + c_2e^{-\xi z} +$$
$$+\frac{R}{\xi}(\lambda_0 - \lambda_1)J_1(R\xi)\left[\theta_1 + \sum_{i=1}^{n-1}(\theta_{i+1} - \theta_i)(1 - \mathrm{ch}\,\xi(z - z_i))S_-(z - z_i)\right].$$

Here c_1, c_2 are the integration constants.

Using boundary conditions (10), we obtain a solution for problem (9), (10) in the form

$$\overline{T} = \frac{R}{\xi}J_1(R\xi)(\lambda_0 - \lambda_1)\times$$
$$\times\left[\theta_1 + \sum_{i=1}^{n-1}(\theta_{i+1} - \theta_i)(1 - \mathrm{ch}\,\xi(z - z_i))S_-(z - z_i)\right] +$$
$$+\frac{\mathrm{ch}\,\xi(l + z)}{\mathrm{sh}\,2\xi l}\mathrm{sh}\,2\xi(l - z_i) + \frac{q_0\mathrm{ch}\,\xi(l - z)}{\xi\,\mathrm{sh}\,2\xi l}. \qquad (11)$$

Applying the inverse integral Hankel transform to relationship (11), we shall receive

$$T(r,z) = R\int_0^\infty J_0(r\xi)J_1(R\xi)\times$$
$$\times\left\{(\lambda_0 - \lambda_1)\left[\theta_1 + \sum_{i=1}^{n-1}(\theta_{i+1} - \theta_i)\left(\frac{q_0\mathrm{ch}\,\xi(l + z)}{\xi\,\mathrm{sh}\,2\xi l}\right)\mathrm{sh}\,\xi(l - z_i) + \right.\right.$$
$$\left.\left. +(1 - \mathrm{ch}\,\xi(z - z_i))S_-(z - z_i)\right] + \frac{q_0\mathrm{ch}\,\xi(z - l)}{\xi\,\mathrm{sh}\,2\xi l}\right\}d\xi. \qquad (12)$$

Unknown approximated values $\theta_i\,(i = 1, n)$ of the excess temperature will be found by resolving system n of linear algebraic equations obtained from expression (12).

Therefore, the desired thermal field in the layer with a through cylindrical inclusion, induced by heat flow, concentrated on the boundary surface of the layer, expressed by formula (12), from which we obtain temperature value in arbitrary point of the construction "layer – inclusion".

4. 2. Isotropic piecewise uniform layer with a through inclusion

Object of study and its mathematical model. Let us consider a piecewise uniform layer, isotropic relative to the thermophysical parameters, which consists of *n* elements that differ in geometric and thermophysical parameters. The layer is related to cylindrical coordinate system (Orφz) with the origin in one of its boundary surfaces and contains a through foreign inclusion with radius R. At the conjugation surfaces

$$K_i = \{(r,\varphi,z_i) : r > R,\ 0 \le \varphi \le 2\pi,\ i = \overline{1, n-1}\},$$
$$K_R = \{(R,\varphi,z) : 0 \le \varphi \le 2\pi, 0 \le z \le z_n\}$$

there is an ideal thermal contact

$$t_i = t_{i+1},\quad \lambda_i\frac{\partial t_i}{\partial z} = \lambda_{i+1}\frac{\partial t_{i+1}}{\partial z}$$

for

$$z{=}z_i\ (i = \overline{1, n-1});\quad t_0 = t_i,\quad \lambda_0\frac{\partial t_0}{\partial r} = \lambda_i\frac{\partial t_i}{\partial r}\quad (i = \overline{1, n})$$

for r=R (0 – for inclusion, i – for the i-th element of layer). In region

$$\Omega_0 = \{(r,\varphi,0) : r \le R, 0 \le \varphi \le 2\pi\}$$

boundary conditions

$$K_0 = \{(r,\varphi,0) : r < \infty, 0 \le \varphi \le 2\pi\}$$

of layer, the system is heated by the concentrated heat flow whose surface density equals q_0, and the other part of this surface of the layer and surface

$$K_n = \{(r,\varphi,z_n) : r < \infty, 0 \le \varphi \le 2\pi\}$$

are thermally insulated (Fig. 3). In the indicated structure, it is necessary to determine temperature distribution t(r, z) by spatial coordinates, which we obtain upon solving equation of thermal conductivity [8, 9].

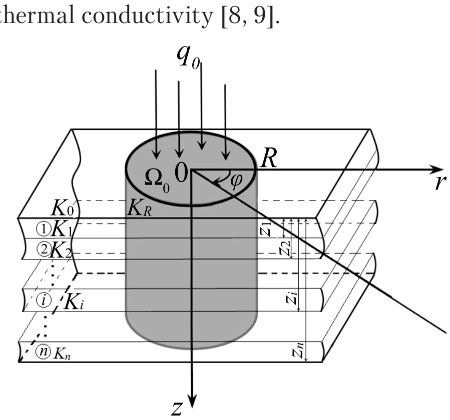

Fig. 3. Isotropic piecewise uniform layer with a through inclusion

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\lambda(r,z)\frac{\partial\theta}{\partial r}\right]+\frac{\partial}{\partial z}\left[\lambda(r,z)\frac{\partial\theta}{\partial z}\right]=0, \quad (13)$$

with boundary conditions

$$\theta\Big|_{r\to\infty}=0,\ \ \frac{\partial\theta}{\partial r}\Big|_{r\to\infty}=\frac{\partial\theta}{\partial z}\Big|_{z=z_n}=0,$$

$$\lambda_0\frac{\partial\theta}{\partial z}\Big|_{z=0}=-q_0S_-(R-r), \quad (14)$$

$$\lambda(r,z)=\sum_{i=1}^{n}\left[\lambda_i+(\lambda_0-\lambda_i)S_-(R-r)\right]N(z,z_{i-1}) \quad (15)$$

coefficient of thermal conductivity of piecewise uniform layer; λ_i, λ_0 are the coefficients of thermal conductivity of materials of the i-th element of the layer and inclusion, respectively;

$$z_0=0;\ \ N(z,z_{i-1})=S_-(z-z_{i-1})-S_+(z-z_i).$$

Let us introduce function [11]

$$T(r,z)=\lambda(r,z)\theta(r,z) \quad (16)$$

and differentiate it by variables r and z, taking into account the expression for coefficient of thermal conductivity $\lambda(r, z)$ (15). As a result, we shall obtain:

$$\lambda(r,z)\frac{\partial\theta}{\partial r}=\frac{\partial T}{\partial r}+\theta\Big|_{r=R}\delta_+(r-R)\sum_{i=1}^{n}(\lambda_0-\lambda_i)N(z,z_{i-1});$$

$$\lambda(r,z)\frac{\partial\theta}{\partial z}=\frac{\partial T}{\partial z}-$$
$$\sum_{i=1}^{n-1}\left[\lambda_i+(\lambda_0-\lambda_i)S_-(R-r)\right]\times$$
$$\times\left[\theta\Big|_{z=z_i-1}\delta_-(z-z_{i-1})-\theta\Big|_{z=z_i}\delta_+(z-z_{i-1})\right]. \quad (17)$$

Substituting expressions (17) in relationship (13), we arrive at a differential equations with partial derivatives with discontinuous and singular coefficients

$$\Delta T+\frac{R}{r}\theta\Big|_{r=R}\delta'_+(r-R)\times$$
$$\times\sum_{i=1}^{n}(\lambda_0-\lambda_i)N(z,z_{i-1},z_i)-$$
$$-\sum_{i=1}^{n-1}\left[\lambda_i+(\lambda_0-\lambda_i)S_-(R-r)\right]\times$$
$$\times\left[\theta\Big|_{z=z_i-1}\delta'_-(z-z_{i-1})-\theta\Big|_{z=z_i}\delta'_+(z-z_{i-1})\right]=0, \quad (18)$$

Analytical-numerical solution. We shall approximate function $\theta(R, z)$, $\theta(R, z_i)$ [11] in the form

$$t(R,z)=t_1+\sum_{i=1}^{n-1}(t_{i+1}-t_i)S_-(z-z_i),$$

$$\theta(r,z_i)=\theta_1^{(z_i)}+\sum_{l=1}^{p-1}(\theta_{l+1}^{(z_i)}-\theta_l^{(z_i)})S_-(r-r_l),$$

$$\theta(r,z_i)=\theta_1^{(z_i)}+\sum_{l=1}^{t-1}(\theta_{l+1}^{(z_i)}-\theta_l^{(z_i)})S_-(r-r_l),$$

$$r\in[0;R], \quad (19)$$

where

$$z_k^{(i)*}\in\,]z_{i-1};z_i[,\ \ z_1^{(i)*}\le z_2^{(i)*}\le\ldots\le z_{m-1}^{(i)*};$$

$$r_l\in\,]0;r^*[,\ r_1\le r_2\le\ldots\le r_{p-1},$$

$$r_1\le r_2\le\ldots\le r_{t-1};$$

m, p, t is the number of partitions in intervals

$$]z_{i-1};z_i[\ ,\ \ ]R;r^*[\ ,\ [0;R]$$

accordingly;

$$\theta_k^{(iR)}(k=\overline{1,m}),\ \ \theta_l^{(z_i)}(l=\overline{1,p+t}),$$

$i=\overline{1,n}$ are the unknown approximated value of temperature; r* is the value of radial coordinate, in which temperature t(r, z) is almost equal to t_c.

Substituting expressions (19) in equation (18), we shall obtain:

$$\Delta T=-\frac{R}{r}\delta'_+(r-R)\sum_{i=1}^{n}(\lambda_0-\lambda_i)\times$$
$$\times\left[\theta_1^{(iR)}N(z,z_{i-1},z_i)+\sum_{k=1}^{m-1}(\theta_{k+1}^{(iR)}-\theta_k^{(iR)})N(z,z_k^{(i)^*},z_i)\right]+$$
$$+\sum_{i=1}^{n}\Bigg\{\Bigg[\lambda_i\left(\theta_1^{(z_{i-1})}+\sum_{l=1}^{p-1}\left(\theta_{l+1}^{(z_{i-1})}-\theta_l^{(z_i)}\right)S_-(r-r_l)\right)+$$
$$+(\lambda_0-\lambda_i)\Bigg(\theta_1^{(z_{i-1})}S_-(R-r)+$$
$$+\sum_{l=1}^{t-1}\left(\theta_{l+1}^{(z_{i-1})}-\theta_l^{(z_{i-1})}\right)S_-(r-r_l)\Bigg)\Bigg]\delta'_-(z-z_{i-1})-$$
$$-\Bigg[\lambda_i\left(\theta_1^{(z_i)}+\sum_{l=1}^{p-1}\left(\theta_{l+1}^{(z_i)}-\theta_l^{(z_i)}\right)S_-(r-r_l)\right)+$$
$$+(\lambda_0-\lambda_i)\left(\theta_1^{(z_i)}S_-(R-r)+\sum_{l=1}^{t-1}\left(\theta_{l+1}^{(z_i)}-\theta_l^{(z_i)}\right)S_-(r-r_l)\right)\Bigg]\delta'_+(z-z_{i-1})\Bigg\}. \quad (20)$$

$$N(z,z_k^{(i)*},z_i)=S_-(z-z_k^{(i)*})-S_+(z-z_i).$$

Applying the integral Hankel transform by coordinate r to equation (20) and boundary conditions (14) with regard to relation (17), we arrive at the ordinary differential equation with constant coefficients

$$\frac{d^2\overline{T}}{dz^2}-\xi^2\overline{T}=\frac{1}{\xi}\sum_{i=1}^{n}\Bigg\{\Bigg[\lambda_i\left(\theta_1^{(z_{i-1})}\delta_+(\xi)+\sum_{l=1}^{p-1}\left(\theta_{l+1}^{(z_{i-1})}-\theta_l^{(z_{i-1})}\right)\left(\delta_+(\xi)-r_lJ_1(r_l\xi)\right)\right)+$$
$$+(\lambda_0-\lambda_i)\left(\theta_1^{(z_{i-1})}RJ_1(R\xi)+\sum_{l=1}^{t-1}\left(\theta_{l+1}^{(z_{i-1})}-\theta_l^{(z_{i-1})}\right)\left(\delta_+(\xi)-r_lJ_1(r_l\xi)\right)\right)\Bigg]\delta'_-(z-z_{i-1})-$$
$$-\Bigg[\lambda_i\left(\theta_1^{(z_i)}\delta_+(\xi)+\sum_{l=1}^{p-1}\left(\theta_{l+1}^{(z_i)}-\theta_l^{(z_i)}\right)\left(\delta_+(\xi)-r_lJ_1(r_l\xi)\right)\right)+$$
$$+(\lambda_0-\lambda_i)\left(\theta_1^{(z_i)}RJ_1(R\xi)+\sum_{l=1}^{t-1}\left(\theta_{l+1}^{(z_i)}-\theta_l^{(z_i)}\right)\left(\delta_+(\xi)-r_lJ_1(r_l\xi)\right)\right)\Bigg]\delta'_+(z-z_i)\Bigg\}-$$
$$-R\xi J_1(R\xi)\sum_{i=1}^{n}(\lambda_0-\lambda_i)\left[\theta_1^{(iR)}N(z,z_{i-1},z_i)+\sum_{k=1}^{m-1}\left(\theta_{k+1}^{(iR)}-\theta_k^{(iR)}\right)N\left(z,z_k^{(i)^*},z_i\right)\right] \quad (21)$$

and boundary conditions

$$\left.\frac{d\overline{T}}{dz}\right|_{z=0}=-\frac{q_0R}{\xi}J_1(R\xi),\quad \left.\frac{d\overline{T}}{dz}\right|_{z=z_n}=0, \qquad (22)$$

Upon solving equation (21) by the method of variation of constants, we shall receive its total solution

$$\begin{aligned}\overline{T}&=c_1e^{\xi z}+c_2e^{-\xi z}+\\&+\frac{1}{\xi}\sum_{i=1}^{n}\left\{\left[\lambda_i\left(\theta_1^{(z_{i-1})}\delta_+(\xi)+\sum_{l=1}^{p-1}\left(\theta_{l+1}^{(z_{i-1})}-\theta_l^{(z_{i-1})}\right)\left(\delta_+(\xi)-r_lJ_1(r_l\xi)\right)\right)+\right.\right.\\&\left.+(\lambda_0-\lambda_i)\left(\theta_1^{(z_{i-1})}RJ_1(R\xi)+\sum_{i=1}^{t-1}\left(\theta_{l+1}^{(z_{i-1})}-\theta_l^{(z_{i-1})}\right)\left(\delta_+(\xi)-r_lJ_1(r_l\xi)\right)\right)\right]\times\\&\times ch\xi(z-z_{i-1})S_-(z-z_{i-1})-\left[\lambda_i\left(\theta_1^{(z_i)}\delta_+(\xi)+\sum_{l=1}^{p-1}\left(\theta_{l+1}^{(z_i)}-\theta_l^{(z_i)}\right)\left(\delta_+(\xi)-r_lJ_1(r_l\xi)\right)\right)+\right.\\&\left.+(\lambda_0-\lambda_i)\left(\theta_1^{(z_i)}RJ_1(R\xi)+\sum_{l=1}^{t-1}\left(\theta_{l+1}^{(z_i)}-\theta_l^{(z_i)}\right)\left(\delta_+(\xi)-r_lJ_1(r_l\xi)\right)\right)\right]\times\\&\times ch\xi(z-z_i)S_+(z-z_i)\Big\}-\frac{R}{\xi}J_1(R\xi)\sum_{i=1}^{n}(\lambda_0-\lambda_i)\Big\{\theta_1^{(iR)}\big[ch\xi(z-z_{i-1})S_-(z-z_{i-1})-\\&-ch\xi(z-z_i)S_+(z-z_i)-N(z,z_{i-1},z_i)\big]+\\&+\sum_{k=1}^{m-1}\left(\theta_{k+1}^{(iR)}-\theta_k^{(iR)}\right)\left[ch\xi(z-z_k^{(i)^*})S_-(z-z_k^{(i)^*})-\right.\\&\left.\left.-ch\xi(z-z_i)S_+(z-z_i)-N(z,z_k^{(i)^*},z_i)\right]\right\}. \end{aligned} \qquad (22)$$

With regard to boundary conditions (22), upon finding integration constants c_1, c_2, a solution for problem (21), (22) will take the following form:

$$\begin{aligned}\overline{T}&=\frac{1}{\xi}\sum_{i=1}^{n}\left\{\left[\lambda_i\left(\theta_1^{(z_{i-1})}\delta_+(\xi)+\sum_{l=1}^{p-1}\left(\theta_{l+1}^{(z_{i-1})}-\theta_l^{(z_{i-1})}\right)\left(\delta_+(\xi)-r_lJ_1(r_l\xi)\right)\right)+\right.\right.\\&\left.+(\lambda_0-\lambda_i)\left(\theta_1^{(z_{i-1})}RJ_1(R\xi)+\sum_{i=1}^{t-1}\left(\theta_{l+1}^{(z_{i-1})}-\theta_l^{(z_{i-1})}\right)\left(\delta_+(\xi)-r_lJ_1(r_l\xi)\right)\right)\right]\times\\&\times\left[ch\xi(z-z_{i-1})S_-(z-z_{i-1})-\frac{ch\xi z}{sh\xi z_n}sh\xi(z_n-z_{i-1})\right]-\\&-\left[\lambda_i\left(\theta_1^{(z_i)}\delta_+(\xi)+\sum_{l=1}^{p-1}\left(\theta_{l+1}^{(z_i)}-\theta_l^{(z_i)}\right)\left(\delta_+(\xi)-r_lJ_1(r_l\xi)\right)\right)-\right.\\&\left.-(\lambda_0-\lambda_i)\left(\theta_1^{(z_i)}RJ_1(R\xi)+\sum_{l=1}^{t-1}\left(\theta_{l+1}^{(z_i)}-\theta_l^{(z_i)}\right)\left(\delta_+(\xi)-r_lJ_1(r_l\xi)\right)\right)\right]\times\\&\left.\times\left[ch\xi(z-z_i)S_+(z-z_i)-\frac{ch\xi z}{sh\xi z_n}sh\xi(z_n-z_i)\right]\right\}-\\&-\frac{R}{\xi}J_1(R\xi)\left\{\sum_{i=1}^{n}(\lambda_0-\lambda_i)\Big[\theta_1^{(iR)}\Big(\big(ch\xi(z-z_{i-1})S_-(z-z_{i-1})-\right.\\&-ch\xi(z-z_i)S_+(z-z_i)-N(z,z_{i-1},z_i)\big)-\frac{ch\xi z}{sh\xi z_n}\big(sh\xi(z_n-z_i)-sh\xi(z_n-z_{i-1})\big)\Big)+\\&+\sum_{k=1}^{m-1}\left(\theta_{k+1}^{(iR)}-\theta_k^{(iR)}\right)\Big(\Big(ch\xi(z-z_k^{(i)^*})S_-(z-z_k^{(i)^*})-\\&-ch\xi(z-z_i)S_+(z-z_i)-N(z,z_k^{(i)^*},z_i)\Big)-\\&\left.-\frac{ch\xi z}{sh\xi z_n}\Big(sh\xi(z_n-z_i)-sh\xi(z_n-z_k^{(i)^*})\Big)\Big)\Big]-\frac{q_0}{\xi}\frac{ch\xi(z-z_n)}{sh\xi z_n}\right\}.\end{aligned} \qquad (23)$$

Applying to expression (23) the inverse integral Hankel transform, we shall obtain a solution for boundary problem (13), (14) in the form:

$$T(r,z)=\int_0^\infty\xi\overline{T}(\xi,z)J_0(r\xi)d\xi. \qquad (24)$$

Unknown approximated temperature values

$$\theta_k^{(iR)}\ (k=\overline{1,m}),$$

$$\theta_l^{(z_i)}(l=\overline{1,p+t}),\quad i=\overline{1,n},$$

will be found by solving system n(m+p+t) of linear algebraic equations, obtained from expression (24).

Therefore, the desired thermal field in a piecewise uniform layer with a through inclusion of cylindrical shape is caused by the heat flow, concentrated at the boundary surface of the layer, expressed by formula (24), from which we obtain the value of temperature in arbitrary point of the structure "piecewise uniform layer – inclusion".

4. 3. Thermosensitive isotropic layer with a through inclusion

Object of study and its mathematical model. Let us consider a thermosensitive (thermophysical parameters depend on temperature) isotropic layer, relative to the thermophysical parameters, with a through inclusion of cylindrical shape (Fig. 1). With regard to thermal sensitivity of the system at the boundary surface of inclusion

$$K_R=\{(R,\varphi,z):0\le\varphi\le2\pi,|z|\le l\},$$

conditions of perfect thermal contact will be written down in the form

$$t_0=t_1,\quad \lambda_0(t)\frac{\partial t_0}{\partial r}=\lambda_1(t)\frac{\partial t_1}{\partial r}$$

for r=R.

Axisymmetric distribution of temperature t(r, z) by spatial coordinates, taking into account thermal sensitivity, we shall obtain upon solving a nonlinear equation of thermal conductivity [8, 9]

$$\frac{1}{r}\frac{\partial}{\partial r}[r\lambda(r,t)\frac{\partial t}{\partial r}]+ +\frac{\partial}{\partial z}\left[\lambda(r,t)\frac{\partial t}{\partial z}\right]=0 \qquad (25)$$

with boundary conditions

$$t|_{r\to\infty}=0,\quad \left.\frac{\partial t}{\partial r}\right|_{r\to\infty}=0,\quad \left.\frac{\partial t}{\partial z}\right|_{z=l}=0,$$

$$\lambda_0(t)\left.\frac{\partial t}{\partial z}\right|_{z=-l}=-q_0S_-(R-r), \qquad (26)$$

$$\lambda(r,t)=\lambda_1(t)+[\lambda_0(t)-\lambda_1(t)]\,S_-(R-r),$$

coefficient of thermal conductivity of non-uniform thermosensitive layer; $\lambda_0(t),\lambda_1(t)$ are the coefficients of thermal conductivity of the materials of inclusion and layer, respectively.

We shall introduce a linearizing function [13, 14]

$$\vartheta(r,z)=\int_0^{t(r,z)}\lambda_1(\zeta)d\zeta+S_-(R-r)\int_{t(R,z)}^{t(r,z)}\left[\lambda_0(\zeta)-\lambda_1(\zeta)\right]d\zeta, \quad (27)$$

upon differentiating which by variables r and z, we shall obtain

$$\lambda(t,r)\frac{\partial t}{\partial r}=\frac{\partial\vartheta}{\partial r},$$

$$\lambda(t,r)\frac{\partial t}{\partial z}=\frac{\partial\vartheta}{\partial z}+\left\{\left[\lambda_0(t)-\lambda_1(t)\right]\frac{\partial t}{\partial z}\right\}\Big|_{r=R}S_-(R-r). \quad (28)$$

Taking into consideration (28), the original equation (25) takes the following form:

$$\Delta\vartheta+\frac{\partial}{\partial z}\left\{\left[\lambda_0(t)-\lambda_1(t)\right]\frac{\partial t}{\partial z}\right\}\Big|_{r=R}S_-(R-r)=0. \quad (29)$$

Boundary conditions (26), using ratio (27), will be written down as:

$$\frac{\partial\vartheta}{\partial z}\Big|_{z=l}=0,\ \vartheta\big|_{r\to\infty}=0,\ \frac{\partial\vartheta}{\partial r}\Big|_{r\to\infty}=0, \quad (30)$$

$$\frac{\partial\vartheta}{\partial z}\Big|_{z=-l}=-\left\{q_0+\left[\left(\lambda_0(t)-\lambda_1(t)\right)\frac{\partial t}{\partial z}\right]\Big|_{\substack{r=R\\ z=-l}}\right\}S_-(R-r). \quad (31)$$

Linearizing function (27) allowed us to reduce a non-linear boundary problem (25), (26) to partially linearized equation (29) with discontinuous coefficients with boundary conditions (30), (31).

Analytical-numerical solution. We approximate function $t(R,z)$ by variable *z* (Fig. 2) by expression (7) and we shall substitute it in relation (29). As a result, we shall obtain a linear differential equation with partial derivatives relative to the linearizing function (27)

$$\Delta\vartheta=$$
$$=-\sum_{i=1}^{n-1}\left(t_{i+1}-t_i\right)\left[\lambda_0(t_{i+1})-\lambda_1(t_{i+1})\right]S_-(R-r)\delta'_-\left(z-z_i\right) \quad (32)$$

with boundary condition

$$\frac{\partial\vartheta}{\partial z}\Big|_{z=-l}=-q_0S_-(R-r). \quad (33)$$

Applying the integral Hankel transform by coordinate r to equation (32) and boundary conditions (33), we obtain an ordinary differential equation with constant coefficients

$$\frac{d^2\bar{\vartheta}}{dz^2}-\xi^2\bar{\vartheta}=$$
$$=-\frac{R}{\xi}J_1(R\xi)\sum_{i=1}^{n-1}\left(t_{i+1}-t_i\right)\left[\lambda_0(t_{i+1})-\lambda_1(t_{i+1})\right]\delta'_-\left(z-z_i\right) \quad (34)$$

with boundary conditions

$$\frac{d\bar{\vartheta}}{dz}\Big|_{z=l}=0,\ \frac{d\bar{\vartheta}}{dz}\Big|_{z=-l}=-\frac{R}{\xi}q_0J_1(R\xi), \quad (35)$$

where

$$\bar{\vartheta}(\xi,z)=\int_0^\infty r\vartheta(\xi,z)J_0(r\xi)dr$$

is the transformant of function $\vartheta(r,z)$.

Upon solving problem (34), (35) and applying the inverse integral Hankel transform to its solution, we shall receive expression for function ϑ:

$$\vartheta=R\int_0^\infty J_0(r\xi)J_1(R\xi)\times$$
$$\times\left\{\sum_{i=1}^{n-1}\left(t_{i+1}-t_i\right)\left[\lambda_0(t_{i+1})-\lambda_1(t_{i+1})\right]\times\right.$$
$$\times\left[\frac{ch\xi(z+l)}{sh2\xi l}sh\xi(l-z_i)-ch\xi(z-z_i)S_-(z-z_i)\right]+$$
$$\left.+\frac{q_0}{\xi}\frac{ch\xi(z-l)}{sh2\xi l}\right\}d\xi. \quad (36)$$

Substituting the expressions of temperature dependence of coefficient of thermal conductivity of the material of layer and inclusion in relations (27), (36), we obtain a system of nonlinear equations to determine the unknown approximating values of temperature $t_i\,(i=\overline{1,n})$.

The desired thermal field in the indicated structure will be determined using the resulting nonlinear equation with the help of relations (27), (36), after substituting in them specific expressions of the dependence of thermal conductivity coefficient of structural materials on temperature.

A partial example. To solve many practical problems, the following dependence of coefficient of thermal conductivity on temperature [15, 16]:

$$\lambda=\lambda_j^0\left(1-k_jt\right), \quad (37)$$

where λ_j^0,k_j is the reference and temperature coefficient of thermal conductivity of the materials for inclusion (j=0) and layer (j=1).

Taking into account relation (37), from expressions (27), (36) we shall obtain formulas for determining temperature t(r, z) in the region of inclusion

$$t=\frac{1}{k_0}\left(1-\sqrt{1-k_0\left(\frac{2\vartheta}{\lambda_0^0}+\vartheta_1\right)}\right), \quad (38)$$

and in region

$$\Omega_1=\left\{(r,\varphi,z):r>R,0\le\varphi\le2\pi,|z|\le l\right\}$$

of layer (except for inclusion)

$$t=\frac{1}{k_1}\left(1-\sqrt{1-\frac{2k_1\vartheta}{\lambda_1^0}}\right). \quad (39)$$

$$\vartheta_1=\left\{t\left[2-k_0t-\frac{\lambda_1^0}{\lambda_0^0}\left(2-k_1t\right)\right]\right\}\Bigg|_{r=R};$$

$$t\Big|_{r=R}=\frac{1}{k_1}\left(1-\sqrt{1-\frac{2k_1}{\ddot{e}_1^0}\vartheta\Big|_{r=R}}\right).$$

Formulas (38), (39) fully express temperature field in the thermosensitive structure “layer – inclusion”.

4. 4. Thermosensitive isotropic piecewise uniform layer

Object of research and its mathematical model. Let us consider thermosensitive isotropic piecewise uniform layer, relative to the thermal-physical parameters, with a through foreign inclusion of cylindrical shape. Conditions of a perfect thermal contact at the conjugating surfaces

$$S_i=\left\{(r,\varphi,z_i):r>R,\ 0\le\varphi\le 2\pi,\ i=\overline{1,n-1}\right\},$$

$$S_R=\left\{(R,\varphi,z):0\le\varphi\le 2\pi, 0\le z\le z_n\right\}$$

taking into account thermal sensitivity of the structure, we shall write down as

$$t_i=t_{i+1},\ \lambda_i(t)\frac{\partial t_i}{\partial z}=\lambda_{i+1}(t)\frac{\partial t_{i+1}}{\partial z}$$

for

$$z=z_i\ \ (i=\overline{1,n-1});$$

$$t_0=t_i,\ \lambda_0(t)\frac{\partial t_0}{\partial r}=\lambda_i(t)\frac{\partial t_i}{\partial r}$$

$(i=\overline{1,n})$, r=R.

Axisymmetric distribution of temperature $t(r,z)$ by spatial coordinates, taking into account thermal sensitivity, will be obtained by solving a nonlinear equation of thermal conductivity (25) with boundary conditions

$$t\Big|_{r\to\infty}=0,\ \frac{\partial t}{\partial r}\Big|_{r\to\infty}=\frac{\partial t}{\partial z}\Big|_{z=z_n}=0,$$

$$\lambda_0(t)\frac{\partial t}{\partial z}\Big|_{z=0}=-q_0S_-(R-r), \tag{40}$$

where

$$\lambda(r,z,t)=$$
$$=\sum_{i=1}^{n}\{\lambda_i(t)+[\lambda_0(t)-\lambda_i(t)]S_-(R-r)\}N(z,z_{i-1})$$

$\lambda(r,z,t)$ is the coefficient of thermal conductivity of thermosensitive piecewise uniform layer; $\lambda_i(t)$, $\lambda_0(t)$ are the coefficients of thermal conductivity of materials of the *i*-th element of layer and inclusion $z_0=0$;

$$N(z,z_{i-1})=S_+(z-z_{i-1})-S_+(z-z_i).$$

We shall introduce a linearizing function [12]

$$\vartheta(r,z)=\sum_{i=1}^{n}\{N(z,z_{i-1})\int_0^{t(r,z)}\lambda_i(\zeta)d\zeta+S_-(R-r)\times$$
$$\times[N(z,z_{i-1})\int_{t(R,z)}^{t(r,z)}(\lambda_0(\zeta)-\lambda_i(\zeta))d\zeta-S_+(z-z_{i-1})\times$$
$$\times\int_{t(R,z_{i-1})}^{t(r,z_{i-1})}(\lambda_0(\zeta)-\lambda_i(\zeta))d\zeta+$$
$$+S_+(z-z_i)\int_{t(R,z_i)}^{t(r,z_i)}(\lambda_0(\zeta)-\lambda_i(\zeta))d\zeta]-$$
$$-S_+(z-z_{i-1})\int_0^{t(r,z_{i-1})}\lambda_i(\zeta)d\zeta+S_+(z-z_i)\int_0^{t(r,z_i)}\lambda_i(\zeta)d\zeta\}, \tag{41}$$

by differentiating which by variables r and z, we shall obtain

$$\lambda(r,z,t)\frac{\partial t}{\partial r}=\frac{\partial\vartheta}{\partial r}-F_1(r,z),$$

$$\lambda(r,z,t)\frac{\partial t}{\partial z}=\frac{\partial\vartheta}{\partial z}+F_2(r,z), \tag{42}$$

$$F_1(r,z)=S_+(r-R)\sum_{i=1}^{n}\Big\{\ [(\lambda_0(t)-\lambda_i(t))\frac{\partial t}{\partial r}]\Big|_{z=z_{i-1}}S_+(z-z_{i-1})-$$
$$-[(\lambda_0(t)-\lambda_i(t))\frac{\partial t}{\partial r}]\Big|_{z=z_i}S_+(z-z_i)\Big\},$$

With regard to expressions (42), original equation (25) will take the form:

$$\frac{1}{r}\frac{\partial}{\partial r}[r\frac{\partial\vartheta}{\partial r}]+\frac{\partial^2\vartheta}{\partial z^2}-\frac{1}{r}\frac{\partial}{\partial r}[rF_1(r,z)]+\frac{\partial}{\partial z}[F_2(r,z)]=0. \tag{43}$$

Boundary conditions using relation (41) will be written down as:

$$\vartheta\Big|_{r\to\infty}=0,\ \frac{\partial\vartheta}{\partial r}\Big|_{r\to\infty}=\frac{\partial\vartheta}{\partial z}\Big|_{z=z_n}=0,$$

$$\frac{\partial\vartheta}{\partial z}\Big|_{z=0}=-q_0S_-(R-r). \tag{44}$$

Linearizing function (41) allowed us to reduce a non-linear boundary problem (25), (40) to a partially linearized equation with discontinuous coefficients (43) and fully linearized boundary conditions (44).

Analytical-numerical solution. We shall approximate functions $t(R,z)$, $t(r,z_i)$ in the form

$$t(R,z)=t_1^{(iR)}+\sum_{k=1}^{m-1}(t_{k+1}^{(iR)}-t_k^{(iR)})S_-(z-z_k^{(i)*}),$$

$$t(r,z_i)=t_1^{(i)}+\sum_{j=1}^{l-1}(t_{j+1}^{(i)}-t_j^{(i)})S_-(r-r_j), \tag{45}$$

where

$$z_k^{(i)*}\,{}^{\circ}\ ]z_{i-1};z_i[;\ z_1^{(i)*}\le z_2^{(i)*}\le\ldots\le z_{m-1}^{(i)*};$$

$$r_j{}^{\circ}\ ]R;\ r_*[;\ r_1\le r_2\le\ldots\le r_{l-1};$$

l, m is the number of partitions of intervals $]R;r_*[$ and $]z_{i-1};z_i[$, respectively;

$$t_k^{(iR)}(k=\overline{1,m-1}),\ \ t_j^{(i)}(j=\overline{1,l-1})$$

are the unknown approximated values of temperature; r* is the value of radial coordinate, in which temperature t(r, z) is practically equal to zero (to be found from the appropriate linear model).

Substituting expressions (45) in relation (43), we obtain a linear differential equation with partial derivatives relative to the linearizing function $\vartheta(r,z)$:

$$\Delta\vartheta=\sum_{i=1}^{n}[\frac{1}{r}\sum_{j=1}^{l-1}F_i^{(j)}(z)\delta'_-(r-r_j)-S_-(R-r)\sum_{k=1}^{m-1}F_i^{(k)}(z)]. \quad (46)$$

$$F_i^j(z)=r_j\Big\{\Big(t_{j-1}^{(i-1)}-t_j^{(i-1)}\Big)\Big(\lambda_0\Big(t_{j-1}^{(i-1)}\Big)-\lambda_i\Big(t_{j-1}^{(i-1)}\Big)\Big)S_+(z-z_{i-1})-$$
$$-\Big(t_{j-1}^{(i)}-t_j^{(i)}\Big)\Big(\lambda_0\Big(t_{j-1}^{i}\Big)-\lambda_i\Big(t_{j-1}^{i}\Big)\Big)-z_i\big(S_+(z)\big)\Big\}.$$

By applying the integral Hankel transform by coordinate r to equation (46) and boundary conditions (45), we shall obtain an ordinary differential equation with constant coefficients

$$\frac{d^2\bar\vartheta}{dz^2}-\xi^2\bar\vartheta=$$
$$=\sum_{i=1}^{n}[\xi\sum_{j=1}^{l-1}J_1(r_j\xi)F_i^{(j)}(z)-\frac{R}{\xi}J_1(R\xi)\sum_{k=1}^{m-1}F_i^{(k)}(z)] \quad (47)$$

with boundary conditions

$$\left.\frac{d\bar\vartheta}{dz}\right|_{z=0}=-\frac{R}{\xi}q_0J_1(R\xi),\ \left.\frac{d\bar\vartheta}{dz}\right|_{z=z_n}=0. \quad (48)$$

A general solution of equation (47) takes the form:

$$\bar\vartheta=c_1e^{\xi z}+c_2e^{-\xi z}-$$
$$-\frac{1}{\xi}\{\sum_{i=1}^{n}[\sum_{j=1}^{l-1}r_jJ_1(r_j\xi)((1-ch\xi(z-z_{i-1}))S_+(z-z_{i-1})\times$$
$$\times(t_{j+1}^{(i-1)}-t_j^{(i-1)})(\lambda_0(t_{j+1}^{(i-1)})-$$
$$-\lambda_i(t_{j+1}^{(i-1)}))-(1-ch\xi(z-z_i))S_+(z-z_i)(t_{j+1}^{(i)}-t_j^{(i)})\times$$
$$\times(\lambda_0(t_{j+1}^{(i)})-\lambda_i(t_{j+1}^{(i)})))-RJ_1(R\xi)\sum_{k=1}^{m-1}(t_{k+1}^{(iR)}-t_k^{(iR)})(\lambda_0(t_{k+1}^{(iR)})-$$
$$-\lambda_i(t_{k+1}^{(iR)}))ch\xi(z-z_k^{(i)*})S_-(z-z_k^{(i)*})]\}.$$

By using boundary conditions (48), we shall obtain the following solution of problem (47), (48):

$$\bar\vartheta=-\frac{1}{\xi}\left\{\sum_{i=1}^{n}\left[\sum_{j=1}^{l-1}r_jJ_1(r_j\xi)(((1-ch\xi(z-z_{i-1}))S_+(z-z_{i-1})+\right.\right.$$
$$+\frac{ch\xi z}{sh\xi z_n}sh\xi(z_n-z_{i-1}))(t_{j+1}^{(i-1)}-t_j^{(i-1)})(\lambda_0(t_{j+1}^{(i-1)})-\lambda_i(t_{j+1}^{(i-1)}))-$$
$$-((1-ch\xi(z-z_i))S_+(z-z_i)+\frac{ch\xi z}{sh\xi z_n}sh\xi(z_n-z_i))(t_{j+1}^{(i)}-t_j^{(i)})\times$$
$$\times(\lambda_0(t_{j+1}^{(i)})-\lambda_i(t_{j+1}^{(i)})))+RJ_1(R\xi)\sum_{k=1}^{m-1}(t_{k+1}^{(iR)}-$$
$$-t_k^{(iR)})(\lambda_0(t_{k+1}^{(iR)})-\lambda_i(t_{k+1}^{(iR)}))((ch\xi(z-z_k^{(i)*})S_-(z-z_k^{(i)*})-$$
$$-\frac{ch\xi z}{sh\xi z_n}sh\xi(z_n-z_k^{(i)*}))]-\frac{q_0R}{\xi}\frac{ch\xi(z-z_n)}{sh\xi z_n}J_1(R\xi)\}. \quad (49)$$

By applying the inverse integral Hankel transform to relation (49), we shall find expression for function $\vartheta(r,z)$ in the form

$$\vartheta(r,z)=\int_0^\infty\xi J_0(r\xi)\bar\vartheta(\xi,z)\,d\xi. \quad (50)$$

Substituting the expressions of temperature dependence of thermal conductivity coefficient of materials of each element of layer and inclusion into relation (41), (50), we obtain a system of nonlinear equations to determine the unknown approximating values of temperature

$$t_k^{(iR)}(k=\overline{1,m})\text{ and }t_j^{(i)}(j=\overline{1,l}).$$

The desired temperature field in the indicated system will be determined by using the resulting nonlinear equation with the help of relations (41), (50), after substituting specific expressions of dependence of the thermal conductivity coefficient of structural materials on temperature.

A partial example.

Let us consider a dependence of thermal conductivity coefficient on temperature in the form (37), where λ_j^0, k_j are the reference and temperature coefficient of thermal conductivity of the materials for inclusion (j=0) and the *i*-th element of layer (j=i).

With regard to relation (37), from expressions (41), (50) we shall receive formulas for determining temperature t(r, z) for a case of two-element layer (n=2) in the region

$$\Omega_1=\{(r,\varphi,z):r>R,0\le\varphi\le2\pi,0\le z<z_1\}$$

of the 1th element of the layer except for inclusion

$$t=\frac{1-\sqrt{1-2\frac{k_1}{\lambda_1^0}(\vartheta+\vartheta_1)}}{k_1}, \quad (51)$$

in region

$$\Omega_2=\{(r,\varphi,z):r>R,0\le\varphi\le2\pi,z_1\le z\le z_2\}$$

of the 2nd element of the layer except for inclusion

$$t=\frac{1-\sqrt{1-2\frac{k_2}{\lambda_2^0}(\vartheta+\vartheta_2)}}{k_2}, \quad (52)$$

in region

$$\Omega_3=\{(r,\varphi,z):r\le R,0\le\varphi\le2\pi,0\le z<z_1\}$$

of inclusion of the 1th element of the layer

$$t=\frac{1-\sqrt{1-2\frac{k_0}{\lambda_0^0}(\vartheta+\vartheta_3)}}{k_0}, \quad (53)$$

in region

$$\Omega_4=\{(r,\varphi,z):r\le R,0\le\varphi\le2\pi,z_1\le z\le z_2\}$$

of inclusion of the 2nd element of the layer

$$t = \frac{1 - \sqrt{1 - 2\frac{k_0}{\lambda_0^0}(\vartheta + \vartheta_4)}}{k_0}. \quad (54)$$

Here

$$\vartheta_2 = \vartheta_m + \vartheta_1; \ \vartheta_m = \left[(\lambda_2^0 - \lambda_1^0 + \frac{\lambda_1^0 k_1 - \lambda_2^0 k_2}{2} t) t \right]\Bigg|_{z=z_1};$$

$$\vartheta_3 = \vartheta_v^{(1)} - \vartheta_v^{(1)}\Big|_{z=0} + \vartheta_0; \ \vartheta_0 = \lambda_0^0 \left[(1 - \frac{k_0}{2} t) t \right]\Bigg|_{z=0};$$

$$\vartheta_v^{(i)} = \left[(\lambda_0^0 - \lambda_i^0 + \frac{\lambda_i^0 k_i - \lambda_0^0 k_0}{2} t) t \right]\Bigg|_{r=R} \quad i = 1,2;$$

$$\vartheta_4 = \vartheta_v^{(2)} - \vartheta_v^{(1)}\Big|_{z=0} + \vartheta_m\Big|_{r=R} + \vartheta_0;$$

Temperature value t(r,0) will be found by formula (39) and $t(R,z)$, $t(r,z_1)$ by formula (51).

Formulas (51)–(54) fully determine thermal field in the thermosensitive two-element layer with a through inclusion.

5. Application and analysis of results of examining the processes of thermal conductivity

We performed a numerical analysis of dimensionless temperature $t^* = \lambda_0 t / (q_0 R)$ for the following original data: material of layer – ceramics VK94-I, material of inclusion – silver, n=10 is the number of partitions of interval (–l; l); l=R=2mm; q_0=200W. In the temperature range [20 °C; 1230 °C], temperature dependences of thermal conductivity coefficient for the indicated materials are expressed in the form:

$$\lambda_1(t) = 13{,}67 \frac{W}{Km}\left(1 - 0{,}00064 \frac{1}{K} t\right),$$

$$\lambda_0(t) = 422{,}54 \frac{W}{Km}\left(1 - 0{,}00031 \frac{1}{K} t\right), \quad (55)$$

which is a particular case of relation (37).

A spatial dependence was built of dimensionless temperature t^* on the spatial dimensionless radial r*=r/R and axial z*=R/z coordinates for linear variable (Fig. 4) and stable ($\lambda_1 = 13{,}4 W/(Km)$, $\lambda_0 = 419 W/(Km)$) (Fig. 5) coefficient of thermal conductivity of materials in the structure. We shall note that the maximum temperature is achieved in the region of action of the concentrated heat flow.

Obtained results for the chosen materials by the linear dependence of thermal conductivity coefficient on the temperature differ from the results obtained for a stable coefficient of thermal conductivity by 7 %.

Fig. 6, 7 illustrate a change in dimensionless temperature t* for a linearly variable thermal conductivity coefficient of materials in the structure depending on the dimensionless coordinates z* for r*=0 (Fig. 6) and r* for z*=0 (Fig. 7). A behavior of the curves indicates conformity of the mathematical model with the actual physical process, because at surface K_R of the inclusion we observe satisfying conditions for a perfect thermal contact (no temperature jump).

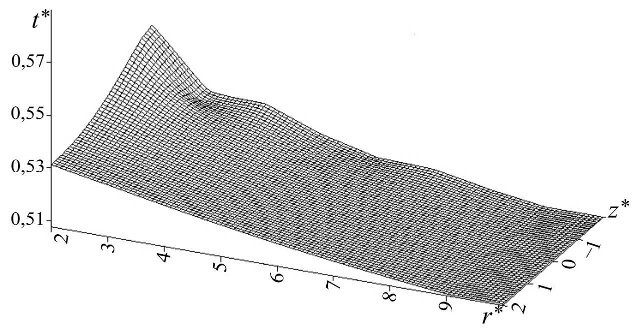

Fig. 4. Dependence of temperature t* on the coordinates r* and z* for linearly variable coefficient of thermal conductivity of materials in the structure

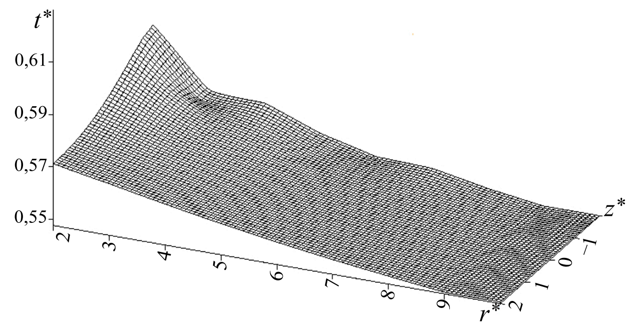

Fig. 5. Dependence of temperature t* on the coordinates r* and z* for a stable coefficient of thermal conductivity of materials in the structure

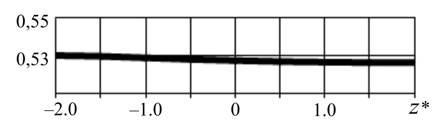

Fig. 6. Dependence of temperature t* on coordinate z* for r*=0

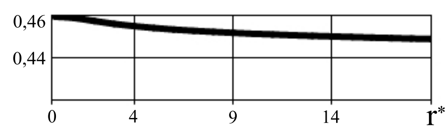

Fig. 7. Dependence of temperature t* on coordinate r* for z*=0

The number of partitions n=10 of interval (–l; l) for the indicated thermal-physical (reference and temperature thermal conductivity coefficients for the materials of layer and inclusion) and geometric (radius and height of the inclusion, thickness of the layer) parameters of the structure allows us to perform calculations with accuracy $\varepsilon = 10^{-6}$.

6. Discussion of results of examining mathematical models of the thermal conductivity process

In the process of development and examination of linear and nonlinear mathematical models of the thermal conductivity process, we obtained analytical-numerical solutions of the corresponding linear boundary problems for the elements of electronic systems, which are geometrically described by the presented piecewise uniform structures.

We introduced linearizing functions, which made it possible to linearize nonlinear boundary problems for the elements of electronic systems, which are geometrically described by the thermosensitive piecewise uniform structures, and presented calculation formulas for determining temperature field in these structures for a linearly variable coefficient of thermal conductivity of their materials.

Using the received analytical-numerical solutions for linear and nonlinear boundary problems for the presented piecewise uniform structures, we devised algorithms and developed, based on them, computing programs that allow us to obtain numerical values of the temperature distribution and to analyze the structures for their thermostability. In future, geometrical structure of the elements of electronic devices will get complicated and, accordingly, the approaches toward studying linear and nonlinear models for the thermal conductivity process will improve while the new ones will be devised.

The methods presented make it possible to solve boundary problems of mathematical physics, which correspond to the linear and nonlinear models of the thermal conductivity process, not only for the elements of electronic systems, but for any objects that have a piecewise uniform structure (layered and with foreign inclusions).

The results obtained were practically implemented in electronic systems, but it is possible to use them in other applied areas. Thus, in modern materials science they develop new materials with improved properties while microelectronics, radio electronics widely apply layered film structures that are exposed to heating in the process of their operation. Overheating may lead to the destruction of individual parts and even entire systems. That is why information on the temperature modes in the media with non-uniform structure is important. In this regard, we report approaches to solving this problem for the examined objects.

7. Conclusions

1. We developed linear and non-linear mathematical models for the thermal conductivity process in complex electronic systems that are geometrically described by a layer and piecewise uniform layer with a through inclusion and obtained analytical-numerical solutions for the corresponding boundary problems.

2. Using the obtained analytical-numerical solutions for boundary problems on thermal conductivity, we devised algorithms and computing programs of their numerical realization for the analysis of temperature modes in the structural elements with piecewise-uniform structure in the devices of modern electronic equipment. Results of analysis of temperature modes allow us to predict operating modes of electronic systems, to identify the unknown parameters and improve temperature resistance, which increases their operating lifecycle.

References

1. Golicyna, E. V. Kvazistacionarnaja trehmernaja zadacha teploprovodnosti vo vrashhajushhemsja sploshnom cilindre iz kompozicionnogo materiala s nelinejnymi granichnymi uslovijami [Text] / E. V. Golicyna, Ju. R. Osipov // Konstrukcii iz kompozicionnyh materialov. – 2007. – Issue 4. – P. 47–58.
2. Kudrjashov, N. A. Priblizhennye reshenija odnomernyh zadach nelinejnoj teploprovodnosti pri zadannom potoke [Text] / N. A. Kudrjashov, V. A. Chmyhov // Zhurnal vychislitel'noj matematiki i matematicheskoj fiziki. – 2007. – Vol. 47, Issue 1. – P. 110–120.
3. Kudinov, V. A. Analiz nelinejnoj teploprovodnosti na osnove opredelenija fronta temperaturnogo vozmushhenija [Text] / V. A. Kudinov, B. V. Averin, E. V. Stefanjuk, S. A. Nazarenko // Teplofizika vysokih temperatur. – 2006. – Vol. 44, Issue 4. – P. 577–585.
4. Popovych, V. S. Nelinijna zadacha teploprovidnosti dlja kuli z teploobminom [Text] / V. S. Popovych, K. S. Ivankiv // Visnyk L'viv un-tu. Ser. Prykl. matematyka ta informatyka. – 2002. – Issue 5. – P. 136–144.
5. Savula, Ja. G. Doslidzhennja variacijnoi' zadachi teploprovidnosti u bagatosharovyh seredovyshhah z tonkymy vkljuchennjamy [Text] / Ja. G. Savula, L. M. Djakonjuk // Visnyk L'viv un-tu. Ser. Prykl. matematyka ta informatyka. – 2000. – Issue 3. – P. 125–130.
6. Bayat, A. Thermo-mechanical analysis of functionally graded thick spheres with linearly time-dependent temperature [Text] / A. Bayat, H. Moosavi, Y. Bayat // Scientia Iranica. – 2015. – Vol. 22, Issue 5. – P. 1801–1812.
7. Mohazzab, A. H. Two-Dimensional Stresses in a Hollow FG Sphere with Heat Source [Text] / A. H. Mohazzab, M. Jabbari // Advanced Materials Research. – 2011. – Vol. 264-265. – P. 700–705. doi: 10.4028/www.scientific.net/amr.264-265.700
8. Podstrigach, Ja. S. Termouprugost' tel neodnorodnoj struktury [Text] / Ja. S. Podstrigach, V. A. Lomakin, Ju. M. Koljano. – Moscow: Nauka, 1984. – 368 p.
9. Koljano, Ju. M. Metody teploprovodnosti i termouprugosti neodnorodnogo tela [Text] / Ju. M. Koljano. – Kyiv: Naukova dumka, 1992. – 280 p.
10. Korn, G. Spravochnik po matematike dlja nauchnyh rabotnikov i inzhenerov [Text] / G. Korn, T. Korn. – Moscow: Nauka, 1977. – 720 p.
11. Gavrysh, V. I. Modeljuvannja temperaturnyh rezhymiv u kuskovo – odnoridnyh strukturah [Text] / V. I. Gavrysh, D. V. Fedasjuk. – Lviv: V-vo Nac. un-tu «L'vivs'ka politehnika», 2012. – 176 p.
12. Gavrysh, V. I. Chislenno – analiticheskoe reshenie nelinejnoj stacionarnoj zadachi teploprovodnosti dlja beskonechnoj termochuvstvitel'noj mnogoslojnoj plastiny [Text] / V. I. Gavrysh // Jelektronnoe modelirovanie. – 2014. – Vol. 36, Issue 3. – P. 59–70.
13. Gavrysh, V. I. Doslidzhennja temperaturnyh rezhymiv u termochutlyvij plastyni z chuzhoridnym naskriznym vkljuchennjam [Text] / V. I. Gavrysh // Fizyko-matematychne modeljuvannja ta informacijni tehnologii'. – 2013. – Issue 18. – P. 43–50.
14. Gavrysh, V. I. Nelinijna krajova zadacha teploprovidnosti dlja sharuvatoi' plastyni z vkljuchennjam [Text] / V. I. Gavrysh // Fizyko-himichna mehanika materialiv. – 2015. – Vol. 51, Issue 3. – P. 32–38.
15. Lomakin, V. A. Teorija uprugosti neodnorodnyh tel [Text] / V. A. Lomakin. – Moscow: Izd- vo Mosk. un- ta, 1976. – 376 p.
16. Berman, R. Teploprovodnost' tverdyh tel [Text] / R. Berman. – Moscow: Mir, 1979. – 288 p.