

Розглянуто задачі статички трансверсально-ізоотропних пластин, які знаходяться за дії зосередженої сили. Проведено аналіз фундаментальних розв'язків, отриманих із використанням різних наближених теорій. За допомогою розглянутих теорій тривимірна задача теорії пружності зведена до двовимірної. Розглянуто безмоментний пружний стан і стан вигину для трансверсально-ізоотропних пластин. Проаналізовано фундаментальні розв'язки рівнянь статички трансверсально-ізоотропних пластин, що побудовані на базі узагальненої теорії $\{m,n\}$ -апроксимації для різних наближень

Ключові слова: $\{m,n\}$ -апроксимація, силова дія, рівняння статички, трансверсально-ізоотропна пластина, поліноми Лежандра

Рассмотрены задачи статички трансверсально-ізоотропных пластин, которые находятся под действием сосредоточенной силы. Проведен анализ фундаментальных решений, полученных с использованием различных приближенных теорій. С помощью рассмотренных теорій трехмерная задача теории упругости сведена к двумерной. Рассмотрены безмоментное напряженное состояние и состояние изгиба для трансверсально-ізоотропных пластин. Проанализированы фундаментальные решения уравнений статички трансверсально-ізоотропных пластин, построенные на базе обобщенной теории $\{m,n\}$ -апроксимации для разных приближений

Ключевые слова: $\{m,n\}$ -апроксимация, силовое воздействие, уравнения статички, трансверсально-ізоотропная пластина, полиномы Лежандра

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ANALYSIS OF FUNDAMENTAL SOLUTIONS TO THE EQUATIONS OF STATICS CONSTRUCTED FOR TRANSVERSAL-ISOTROPIC PLATES

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1. Introduction

The thin-walled elements of structures (plates and shells) are the basic bearing elements of many contemporary designs and articles (vessels, aircraft, pipelines, cisterns, etc.). The action of the concentrated or local force influence on a thin-walled element causes dangerous stress concentration, which can lead to the destruction of the structure. That is why examining the stressed state of plates and shells, exposed to considerable force influences, has been, and remains, a theoretically and practically relevant task.

Until recently, the strength calculations of the thin-walled elements of structures have mostly employed the classical theory, based on the hypothesis about undeformable normal elements. In recent years, new composite materials have been utilized in engineering intensively in order to create protective coatings on the friction surfaces, as well as to fabricate different parts of equipment. Such materials typically possess low shear rigidity, which is why the hypotheses of Kirchhoff theory do not hold for them and the classical theory yields significant error in the calculations.

The use of contemporary composite materials leads to the need of constructing the refined theories of plates and shells, which consider phenomena that are connected to transverse

dislocations and compression. However, solving the problems on the theory of elasticity based on the refined theories in a three-dimensional setting is linked with considerable mathematical difficulties. That is why an issue about constructing the refined theories is closely connected to the problem of bringing three-dimensional tasks to the two-dimensional ones.

Thus, examining the SSS of plates under the action of concentrated force influences based on the refined theories is an important and relevant scientific and technical task.

2. Literature review and problem statement

The analytical solution of the axisymmetrical mixed problem for the isotropic half-space with the surface, elastically fixed beyond the circular region of the application of distributed load, is given in article [1]. When solving the given problem, the procedure of transition from the distributed load to the concentrated force was substantiated. They obtained the compact form of precise analytical solution of the problem on the concentrated force, applied to a half-space with elastic surface.

Paper [2] explores the problem on the bend of a rectangular plate with rotationally fixed edges under the action of

concentrated force. The focus is on determining the angular forces and deflections.

The problem on the concentrated force that acts in the inner part of an infinite plate is solved in article [3]. This publication examines an isotropic plate of arbitrary thickness, in which the moduli of elasticity are any assigned functions. Solution of the problem presented is based on the classical solution for the concentrated force in a thin elastic plate.

Paper [4] addresses the displacement of a shallow spherical thin shell under the action of concentrated load. Deflections under the point of the application of concentrated loads on the spherical thin shells are calculated. This work also analyzes and compares the methods, which make it possible to obtain the refining calculations of shell deflections.

The influence of surface stresses on the stressed-strained state of elastic shell in is investigated in article [5]. In this study, the surface stresses are represented in the form of a static load, localized in the ultrathin layers of shell near its surface. The three-dimensional equations of elasticity are analyzed by the asymptotic method with the use of several asymptotic parameters.

Paper [6] presents results of analysis of a flexible plate of different thickness in accordance with the Kirchhoff model, the model of first order shift and the three-dimensional elastic model. Numerical algorithms of the method of difference energy and the method of finite elements are used to solve the problem. Solution of the contact problem and comparative evaluation of the obtained results are given.

An analysis of the publications over recent time allows us to conclude that the refined theory of the {m,n}-approximation has not been commonly applied to study the SSS of plates and shells, which are exposed to the action of concentrated force.

This is linked to the fact that the solutions of problems on the concentrated influence, employing the theory of {m,n}-approximation, is connected with the larger mathematical difficulties than the solution of analogous problems based on classical theory. The mathematical apparatus, which is based on the application of a specialized G-function [7], makes it possible to overcome the indicated difficulties.

Among the publications, which apply a generalized theory of the {m,n}-approximation and the procedure of mapping, built with the help of specialized G-function, those articles should be noted that deal with the construction of fundamental solutions of the equations of statics for the transversal-isotropic plates based on the {1,0}-approximation for the state of bending [8] and based on the {1,2}-approximation for the zero spin stressed state and the state of bending [9, 10]. Worth noting are also the publications, in which authors obtained fundamental solutions of differential equations on thermoelasticity of the {1,0}- and {1,2}-approximation for the transversal-isotropic plates [11, 12].

3. The aim and tasks of the study

The aim of present study is the analysis of fundamental solutions of the equations of statics of the transversal-isotropic plates, built on the basis of a generalized theory of

the {m,n}-approximation for different approximations. This will make it possible to conduct strength calculations of the thin-walled elements of structures that contain dangerous stress concentrators in the form of concentrated and local force impacts.

To achieve the aim, the following tasks were set:

- to analyze the methods for reducing the three-dimensional problems of the theory of elasticity for plates to the two-dimensional ones;

- to compare fundamental solutions, obtained on the basis of the {1,0}- and {1,2}-approximation.

4. Materials and the methods for examining the decrease in order of the three-dimensional problems on the theory of elasticity

4.1. System of equations of the theory of elasticity for transversal-isotropic bodies

The system of equations of the theory of elasticity for the transversal-isotropic bodies includes:

1) the Cauchy ratios [13]

$$e_x = \frac{\partial u_x}{\partial x}; \quad e_y = \frac{\partial u_y}{\partial y}; \quad e_z = \frac{\partial u_z}{\partial z};$$

$$e_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}; \quad e_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}; \quad e_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}; \quad (1)$$

2) Hooke's law for the transversal-isotropic bodies [14]

$$\sigma_x = \bar{E}(e_x + \nu e_y) + \lambda \sigma_z; \quad \sigma_y = \bar{E}(e_y + \nu e_x) + \lambda \sigma_z;$$

$$\sigma_z = \bar{E}'[e_z + \lambda(e_x + e_y)];$$

$$\tau_{xy} = G e_{xy}; \quad \tau_{xz} = G' e_{xz}; \quad \tau_{yz} = G' e_{yz},$$

where

$$\bar{E} = \frac{E}{1-\nu^2}; \quad \bar{E}' = \frac{(1-\nu)E'}{1-\nu-2(\nu')^2 E/E'};$$

$$\lambda = \frac{\nu' E}{1-\nu E'}; \quad G = \frac{E}{2(1+\nu)},$$

where E, E' are the Young moduli for the directions in the planes of isotropy and perpendicular to it; ν, ν', G, G' are the Poisson coefficients and the shear moduli, which correspond to these directions;

3) the equations of equilibrium [14]

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x = 0;$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + F_y = 0;$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + F_z = 0, \quad (2)$$

where $\vec{F} = (F_x, F_y, F_z)$ is the volume force vector.

4. 2. Description of methods for reducing the three-dimensional problems of the theory of elasticity for plates to the two-dimensional ones

Precise solution of the problems of the theory of elasticity for plates in a three-dimensional setting is associated with significant mathematical difficulties, in order to overcome which, the methods of approximation are employed. The method of decreasing the quantity of independent variables is one of them. In this case, a transition from the three-dimensional problems to the two-dimensional ones is accomplished in different ways.

The first group includes the approaches based on the acceptance of various hypotheses. The most popular are the hypotheses of classical Kirchhoff theory for plates. The same group includes an S. P. Timoshenko model, proposed in 1916 for the problems on the dynamics of rods.

Another technique for the reduction of three-dimensional equations to the two-dimensional ones consists in the expansion of the desired functions into exponential or functional series along the normal z coordinate and in the retention of specified sections of these series depending on the required accuracy and character of the problem. This approach is used, in particular, by the refined theory of the {m,n}- approximation [14]. Within the framework of the given theory, when deriving the two-dimensional equations, they employ the method of expansion of the desired and assigned functions of three-dimensional equations (1), (2) into the Fourier series by the Legendre polynomials $P_k = P_k(z/h)$. Moreover, in the theory of {m,n}-approximation, m characterizes a quantity of retained members of the Fourier series for the components of SSS in the xOy plane; n is the number of retained terms of series for the desired and assigned functions in the direction of the Oz axis. In particular, the representations of components of the strain tensor within the {m,n}-approximation take the form [14]

$$e_x = \sum_{k=0}^m e_{xk} P_k (x \rightarrow y); e_{xy} = \sum_{k=0}^m e_{xyk} P_k;$$

$$e_{xz} = \sum_{k=0}^{n+2} e_{xzk} P_k (x \rightarrow y); e_z = \sum_{k=0}^{n+1} e_{zk} P_k,$$

where $P_k = P_k(z/h)$ are the Legendre polynomials.

4. 3. Fundamental relationships and mathematical statement of the problem

We consider a transversal-isotropic plate of thickness 2h in the rectangular Cartesian coordinate system x, y, z. Concentrated force \vec{F}^C , applied at the origin of coordinates (singular point), acts on the plate.

In the case of {1,0}-approximation, the components of displacement vector and stress tensor are represented as:

$$u_x = uP_0 + h\gamma_x P_1; u_y = vP_0 + h\gamma_y P_1; u_z = w_0 P_0; \quad (3)$$

$$\sigma_x = \frac{N_x}{2h} P_0 + \frac{3M_x}{2h^2} P_1 (x \rightarrow y);$$

$$\sigma_z = 0; \tau_{xy} = \frac{S}{2h} P_0 + \frac{3H}{2h^2} P_1;$$

$$\tau_{xz} = \frac{Q_x}{2h} (P_0 - P_2) (x \rightarrow y),$$

where u, v, w₀ are the analogs of displacements of the points of median surface of the plate; γ_x, γ_y are the analogs of the rotation angles of the normal; N_x, N_y, S are the analogs of membrane forces; M_x, M_y, H are the analogs of the bending moment and the torque; Q_x, Q_y are the analogs of transverse forces.

Components of the volume force vector are represented as [14]

$$F_x = \frac{q_x}{2h} P_0 + \frac{3m_x}{2h^2} P_1 (x \rightarrow y); F_z = \frac{q_z}{2h} P_0. \quad (4)$$

The equations of statics in the case of {1,0}-approximation for the transversal-isotropic plates, recorded in the dimensionless coordinate system ($x_1=x/h, x_2=x/h, x_3=x/h$), contain [14]

– equations of Hooke’s law:

$$N_1 = B_0 \left(\frac{\partial u}{\partial x_1} + v \frac{\partial v}{\partial x_2} \right); N_2 = B_0 \left(\frac{\partial v}{\partial x_2} + v \frac{\partial u}{\partial x_1} \right); \quad (5)$$

$$S = \frac{1-v}{2} B_0 \left(\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1} \right); H = \frac{1-v}{2} D_0 \left(\frac{\partial \gamma_1}{\partial x_2} + \frac{\partial \gamma_2}{\partial x_1} \right);$$

$$M_1 = D_0 \left(\frac{\partial \gamma_1}{\partial x_1} + v \frac{\partial \gamma_2}{\partial x_2} \right); M_2 = D_0 \left(\frac{\partial \gamma_2}{\partial x_2} + v \frac{\partial \gamma_1}{\partial x_1} \right);$$

$$Q_1 = \Lambda_0 \left(\gamma_1 + \frac{\partial w_0}{\partial x_1} \right); Q_2 = \Lambda_0 \left(\gamma_2 + \frac{\partial w_0}{\partial x_2} \right),$$

where

$$B_0 = \frac{2}{1-\nu^2}; D_0 = \frac{D}{Eh^2} = \frac{2}{3} \frac{1}{1-\nu^2}; \Lambda_0 = \frac{5}{3E/G'},$$

E/G' is the shear compliance parameter; – equation of equilibrium

$$\frac{\partial N_1}{\partial x_1} + \frac{\partial S}{\partial x_2} + q_1 = 0; \frac{\partial N_2}{\partial x_2} + \frac{\partial S}{\partial x_1} + q_2 = 0; \quad (6)$$

$$\frac{\partial M_1}{\partial x_1} + \frac{\partial H}{\partial x_2} - Q_1 + m_1 = 0; \frac{\partial M_2}{\partial x_2} + \frac{\partial H}{\partial x_1} - Q_2 + m_2 = 0;$$

$$\frac{\partial Q_1}{\partial x_1} + \frac{\partial Q_2}{\partial x_2} + q_3 = 0.$$

Let us note that the system of equations, which describes the zero spin stressed state in the case of {1,0}-approximation (first three equations (5), first two equations (6)), is similar to the system of equations, which describes the flat stressed state of isotropic plates within the framework of classical theory.

Within the framework of the {1,2}-approximation, the representation of components of displacement vector u_x, u_y , of stress tensor $\sigma_x, \sigma_y, \tau_{xy}$ and the components of volume force vector F_x, F_y takes the same form as in the case of {1,0}-approximation (formulas (3), (4)), while the remaining components are determined as [14]:

$$u_z = w_0 P_0 + w_1 P_1 + w_2 P_2;$$

$$\tau_{xz} = \frac{Q_{x0}}{2h}(P_0 - P_4) + \frac{3Q_{x1}}{2h}(P_1 - P_3) + \frac{5Q_{x2}}{2h}(P_2 - P_4) \quad (x \rightarrow y);$$

$$\sigma_z = \frac{R_0}{2h}(P_0 - P_2) + \frac{3R_1}{2h}(P_1 - P_3);$$

$$F_z = \frac{q_{z0}}{2h}P_0 + \frac{3q_{z1}}{2h}P_1 + \frac{5q_{z2}}{2h}P_2.$$

The system of equations for transversal-isotropic plates in the case of {1,2}-approximation, recorded in the dimensionless coordinate system, contains

– equations of Hooke's law:

$$N_1 = B_0 \left(\frac{\partial u}{\partial x_1} + \nu \frac{\partial v}{\partial x_2} \right) + \lambda_0 R_0;$$

$$N_2 = B_0 \left(\frac{\partial v}{\partial x_2} + \nu \frac{\partial u}{\partial x_1} \right) + \lambda_0 R_0; \quad (7)$$

$$S = \frac{1-\nu}{2} B_0 \left(\frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2} \right); \quad H = \frac{1-\nu}{2} D_0 \left(\frac{\partial \gamma_2}{\partial x_1} + \frac{\partial \gamma_1}{\partial x_2} \right);$$

$$M_1 = D_0 \left(\frac{\partial \gamma_1}{\partial x_1} + \nu \frac{\partial \gamma_2}{\partial x_2} \right) + \lambda_0 R_1;$$

$$M_2 = D_0 \left(\frac{\partial \gamma_2}{\partial x_2} + \nu \frac{\partial \gamma_1}{\partial x_1} \right) + \lambda_0 R_1;$$

$$Q_{10} = \Lambda_0 \left[\gamma_1 + \frac{\partial}{\partial x_1} \left(w_0 - \frac{w_2}{14} \right) \right];$$

$$Q_{20} = \Lambda_0 \left[\gamma_2 + \frac{\partial}{\partial x_2} \left(w_0 - \frac{w_2}{14} \right) \right];$$

$$Q_{11} = \Lambda'_0 \frac{\partial w_1}{\partial x_1}; \quad Q_{21} = \Lambda'_0 \frac{\partial w_1}{\partial x_2};$$

$$Q_{12} = -\frac{\Lambda_0}{14} \left[\gamma_1 + \frac{\partial}{\partial x_1} (w_0 - 2w_2) \right];$$

$$Q_{22} = -\frac{\Lambda_0}{14} \left[\gamma_2 + \frac{\partial}{\partial x_2} (w_0 - 2w_2) \right];$$

$$R_0 = \Omega'_0 \left[w_1 + \lambda_0 \left(\frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial x_2} \right) \right];$$

$$R_1 = \Omega_0 \left[w_2 + \frac{\lambda_0}{3} \left(\frac{\partial \gamma_1}{\partial x_1} + \frac{\partial \gamma_2}{\partial x_2} \right) \right],$$

where

$$B_0 = \frac{2}{1-\nu^2}; \quad \lambda_0 = \frac{\nu'}{1-\nu} E^*; \quad \Lambda_0 = \frac{28}{15} \frac{1}{E/G};$$

$$\Omega_0 = \frac{7}{5} \frac{(1-\nu)/E^*}{1-\nu-2(\nu')^2 E^*};$$

$$\Omega'_0 = \frac{25}{21} \Omega_0; \quad D_0 = \frac{2}{3} \frac{1}{1-\nu^2}; \quad \Lambda'_0 = \frac{3}{4} \Lambda_0; \quad E^* = \frac{E}{E'};$$

– equation of equilibrium

$$\frac{\partial N_1}{\partial x_1} + \frac{\partial S}{\partial x_2} + q_1 = 0; \quad \frac{\partial N_2}{\partial x_2} + \frac{\partial S}{\partial x_1} + q_2 = 0; \quad (8)$$

$$\frac{\partial M_1}{\partial x_1} + \frac{\partial H}{\partial x_2} - Q_{10} + m_1 = 0; \quad \frac{\partial M_2}{\partial x_2} + \frac{\partial H}{\partial x_1} - Q_{20} + m_2 = 0;$$

$$\frac{\partial Q_{10}}{\partial x_1} + \frac{\partial Q_{20}}{\partial x_2} + q_3 = 0; \quad \frac{\partial Q_{11}}{\partial x_1} + \frac{\partial Q_{21}}{\partial x_2} - R_0 + q_4 = 0;$$

$$\frac{\partial Q_{12}}{\partial x_1} + \frac{\partial Q_{22}}{\partial x_2} - 3R_1 + q_5 = 0.$$

The moments in relationships (5)–(8) are determined with accuracy to the magnitude Eh^2 , and the generalized efforts – to Eh .

A fundamental solution of system (5), (6) and system (7), (8) has a certain mechanical sense – this is the solution of the problem on the action of concentrated force on a plate. Dirac's delta function [15] is located in the place of load functions in the right sides of resolving equations. Therefore, if the concentrated force \bar{F}^C , acts on the plate, then the volume force vector must be taken in the form

$$\bar{F}(\bar{r}) = \bar{F}^C \delta(\bar{r}),$$

where $\bar{r} = (x, y, z)$ is the radius-vector of the force application point in the initial (dimensional) coordinate system; $\bar{F}^C = (F_x^C, F_y^C, F_z^C)$.

Then we obtain the expressions in the right sides of equations of equilibrium (6), (8) in the dimensionless coordinate system

$$m_i = m_i^* \delta(x_1, x_2); \quad q_i = q_i^* \delta(x_1, x_2), \quad (9)$$

where $\delta(x_1, x_2)$ is the two-dimensional Dirac's delta function.

5. Results of examining the fundamental solutions, built with the use of refined theories

Let us examine the fundamental solutions, built on the basis of theory of the {m,n}-approximation. The fundamental solution of the equations of statics at bending in the case of {1,0}-approximation is obtained in article [8]. The statement of a flat problem within the framework of this approximation coincides with the formulation of analogous problem for isotropic plates based on the Kirchhoff theory. The fundamental solution of the specified problem is built in paper [16]. The fundamental solutions for a flat problem and bending in the case of {1,2}-approximation are received in articles [11, 12].

The expressions of inner force factors, obtained with the use of the generalized theory, take the form:

– for the {1,0}-approximation

$$N_1 = -\frac{1}{2\pi} \left[q_1^* \left\{ \Psi_1(x_1, x_2) + (\nu+2) \Psi_2(x_2, x_1) \right\} + q_2^* \left\{ \nu \Psi_1(x_2, x_1) - \Psi_2(x_1, x_2) \right\} \right]; \quad (10)$$

$$N_2 = -\frac{1}{2\pi} \left[q_1^* \{ v\Psi_1(x_1, x_2) - \Psi_2(x_2, x_1) \} + q_2^* \{ \Psi_1(x_2, x_1) + (v+2)\Psi_2(x_1, x_2) \} \right];$$

$$S = -\frac{1}{2\pi} \left[q_1^* \{ \Psi_1(x_2, x_1) - v\Psi_2(x_1, x_2) \} + q_2^* \{ \Psi_1(x_1, x_2) - v\Psi_2(x_2, x_1) \} \right];$$

$$M_1 = -\frac{1}{2\pi} \left[m_1^* \{ \Psi_1(x_1, x_2) + v\Psi_2(x_2, x_1) + 2\Psi_3(x_2, x_1) \} + m_2^* \{ v\Psi_1(x_2, x_1) + \Psi_2(x_1, x_2) - 2\Psi_3(x_1, x_2) \} - q_3^* \{ \Psi_5(x_1, x_2) + v\Psi_5(x_2, x_1) \} \right];$$

$$M_2 = -\frac{1}{2\pi} \left[m_1^* \{ v\Psi_1(x_1, x_2) + \Psi_2(x_2, x_1) - 2\Psi_3(x_2, x_1) \} + m_2^* \{ \Psi_1(x_2, x_1) + v\Psi_2(x_1, x_2) + 2\Psi_3(x_1, x_2) \} - q_3^* \{ v\Psi_5(x_1, x_2) + \Psi_5(x_2, x_1) \} \right];$$

$$H = -\frac{1}{2\pi} \left[m_1^* \{ (1-v)\Psi_2(x_1, x_2) - \Psi_3(x_1, x_2) + \Psi_4(x_2, x_1) \} + m_2^* \{ (1-v)\Psi_2(x_2, x_1) - \Psi_3(x_2, x_1) + \Psi_4(x_1, x_2) \} - (1-v)q_3^* \Psi_6(x_1, x_2) \right];$$

$$Q_1 = \frac{a^2}{2\pi} \left[m_1^* \Psi_7(x_2, x_1) - m_2^* \Psi_8(x_1, x_2) \right] - \frac{q_3^*}{2\pi} \frac{x_1}{r^2};$$

$$Q_2 = \frac{a^2}{2\pi} \left[m_2^* \Psi_7(x_1, x_2) - m_1^* \Psi_8(x_1, x_2) \right] - \frac{q_3^*}{2\pi} \frac{x_2}{r^2},$$

where

$$\Psi_1(x_1, x_2) = \frac{x_1(x_1^2 + 3x_2^2)}{2r^4};$$

$$\Psi_3(x_1, x_2) = \frac{x_2}{2r^2} G_{0,1}(ar) + \frac{x_2(3x_1^2 - x_2^2)}{2r^4} G_{1,2}(ar);$$

$$\Psi_2(x_1, x_2) = -\frac{x_2(x_1^2 - x_2^2)}{2r^4};$$

$$\Psi_5(x_1, x_2) = -\frac{1}{2} \ln \frac{\gamma r}{2} - \frac{1}{4} \frac{x_1^2 - x_2^2}{r^2};$$

$$\Psi_4(x_1, x_2) = \frac{3x_1}{2r^2} G_{0,1}(ar) + \frac{x_1(x_1^2 - 3x_2^2)}{2r^4} G_{1,2}(ar);$$

$$\Psi_6(x_1, x_2) = -\frac{x_1 x_2}{2r^2};$$

$$\Psi_7(x_1, x_2) = \frac{1}{2} \left\{ G_{0,0}(ar) + \frac{x_1^2 - x_2^2}{r^2} G_{1,1}(ar) \right\};$$

$$\Psi_8(x_1, x_2) = \frac{x_1 x_2}{r^2} G_{1,1}(ar); \quad a^2 = 3(1+v)\Lambda_0; \quad r = \sqrt{x_1^2 + x_2^2};$$

$G_{n,v}(z)$ is the specialized G-function [7];
 – for the {1,2}-approximation

$$N_1 = -\frac{1}{2\pi} \left[q_1^* \frac{B_0}{A} \Phi_1(x_1, x_2) + q_2^* \frac{B_0}{A} \Phi_2(x_1, x_2) + q_1^* a_0^2 \Phi_3(x_1, x_2) + q_2^* a_0^2 \Phi_4(x_1, x_2) + q_3^* \lambda_0 a_0^2 \Phi_5(x_1, x_2) + 2q_1^* \Phi_6(x_2, x_1) - 2q_2^* \Phi_6(x_1, x_2) + q_1^* \frac{B_0 v}{A} \Phi_2(x_2, x_1) + q_2^* \frac{B_0 v}{A} \Phi_1(x_2, x_1) + q_1^* a_0^2 v \Phi_4(x_2, x_1) + q_2^* a_0^2 v \Phi_3(x_2, x_1) + q_3^* \lambda_0 a_0^2 v \Phi_5(x_2, x_1) \right] - \frac{\lambda_0^2 \Omega_0'}{2\pi A} \left[q_1^* \Phi_7(x_1, x_2) + q_2^* \Phi_7(x_2, x_1) - q_3^* \frac{B_0}{\lambda_0 \Lambda_0'} \Phi_9(x_1, x_2) \right]; \quad (11)$$

$$N_2 = -\frac{1}{2\pi} \left[q_1^* \frac{B_0}{A} \Phi_2(x_2, x_1) + q_2^* \frac{B_0}{A} \Phi_1(x_2, x_1) + q_1^* a_0^2 \Phi_4(x_2, x_1) + q_2^* a_0^2 \Phi_3(x_2, x_1) + q_3^* \lambda_0 a_0^2 \Phi_5(x_2, x_1) - 2q_1^* \Phi_6(x_2, x_1) + 2q_2^* \Phi_6(x_1, x_2) + q_1^* \frac{B_0 v}{A} \Phi_1(x_1, x_2) + q_2^* \frac{B_0 v}{A} \Phi_2(x_1, x_2) + q_1^* a_0^2 v \Phi_3(x_1, x_2) + q_2^* a_0^2 v \Phi_4(x_1, x_2) + q_3^* \lambda_0 a_0^2 v \Phi_5(x_1, x_2) \right] - \frac{\lambda_0^2 \Omega_0'}{2\pi A} \left[q_1^* \Phi_7(x_1, x_2) + q_2^* \Phi_7(x_2, x_1) - q_3^* \frac{B_0}{\lambda_0 \Lambda_0'} \Phi_9(x_1, x_2) \right];$$

$$S = -\frac{1-v}{2\pi} \left[q_1^* \frac{B_0}{A} \Phi_2(x_1, x_2) + q_2^* \frac{B_0}{A} \Phi_2(x_2, x_1) + q_1^* a_0^2 \Phi_4(x_1, x_2) + q_2^* a_0^2 \Phi_4(x_2, x_1) + q_3^* \lambda_0 a_0^2 \Phi_8(x_1, x_2) - q_1^* \frac{1}{v-1} \Phi_{10}(x_2, x_1) + q_2^* \frac{1}{v-1} \Phi_6(x_2, x_1) + q_1^* \frac{1}{v-1} \Phi_6(x_1, x_2) - q_2^* \frac{1}{v-1} \Phi_{10}(x_1, x_2) \right];$$

$$M_1 = -\frac{1}{2\pi} \left\{ m_1^* a_1^2 \Phi_{11}(x_1, x_2) + m_2^* a_1^2 \Phi_{12}(x_1, x_2) - q_5^* \frac{\lambda_0 a_1^2}{3} \Phi_{13}(x_1, x_2, a_1) - q_4^* \left[\frac{D_0}{A_1} - \frac{\lambda_0 a_1^2}{42} \right] \Phi_{13}(x_1, x_2, a_1) + m_1^* \frac{D_0}{A_1} \Phi_{14}(x_1, x_2, a_1) + m_2^* \frac{D_0}{A_1} \Phi_{15}(x_1, x_2, a_1) - q_4^* a_1^2 \Phi_{16}(x_1, x_2) + 2m_1^* \Phi_{15}(x_2, x_1, b_1) - 2m_2^* \Phi_{15}(x_1, x_2, b_1) + m_1^* a_1^2 v \Phi_{12}(x_2, x_1) + m_2^* a_1^2 v \Phi_{11}(x_2, x_1) + q_5^* \frac{\lambda_0 a_1^2 v}{3} \Phi_{13}(x_2, x_1, a_1) - q_4^* v \left[\frac{D_0}{A_1} - \frac{\lambda_0 a_1^2}{42} \right] \Phi_{13}(x_2, x_1, a_1) + m_1^* \frac{v D_0}{A_1} \Phi_{15}(x_2, x_1, a_1) + m_2^* \frac{v D_0}{A_1} \Phi_{14}(x_2, x_1, a_1) - q_4^* a_1^2 v \Phi_{16}(x_2, x_1) \right\} + \frac{\lambda_0^2 \Omega_0}{6\pi A_1} \left\{ q_4^* \left[1 + \frac{A_1 a_1^2}{14\lambda_0 \Omega_0} \right] \Phi_{17}(x_1, x_2) - q_5^* \frac{A_1 a_1^2}{\lambda_0 \Omega_0} \Phi_{17}(x_1, x_2) - m_1^* \Phi_{18}(x_1, x_2) - m_2^* \Phi_{18}(x_2, x_1) \right\};$$

$$\begin{aligned}
 M_2 = & -\frac{1}{2\pi} \left\{ m_1^* a_1^2 \Phi_{12}(x_2, x_1) + m_2^* a_1^2 \Phi_{11}(x_2, x_1) + \right. \\
 & + q_5^* \frac{\lambda_0 a_1^2}{3} \Phi_{13}(x_2, x_1, a_1) - q_4^* \left[\frac{D_0}{A_1} - \frac{\lambda_0 a_1^2}{42} \right] \Phi_{13}(x_2, x_1, a_1) + \\
 & + m_1^* \frac{D_0}{A_1} \Phi_{15}(x_2, x_1, a_1) + m_2^* \frac{D_0}{A_1} \Phi_{14}(x_2, x_1, a_1) - \\
 & - q_4^* a_1^2 \Phi_{16}(x_2, x_1) - 2m_1^* \Phi_{15}(x_2, x_1, b_1) + 2m_2^* \Phi_{15}(x_1, x_2, b_1) + \\
 & + m_1^* a_1^2 v \Phi_{11}(x_1, x_2) + m_2^* a_1^2 v \Phi_{12}(x_1, x_2) + \\
 & + q_5^* \frac{\lambda_0 a_1^2 v}{3} \Phi_{13}(x_1, x_2, a_1) - q_4^* v \left[\frac{D_0}{A_1} - \frac{\lambda_0 a_1^2}{42} \right] \Phi_{13}(x_1, x_2, a_1) + \\
 & + m_1^* \frac{v D_0}{A_1} \Phi_{14}(x_1, x_2, a_1) + m_2^* \frac{v D_0}{A_1} \Phi_{15}(x_1, x_2, a_1) - \\
 & - q_4^* a_1^2 v \Phi_{16}(x_1, x_2) \left. \right\} + \frac{\lambda_0^2 \Omega_0}{6\pi A_1} \left\{ q_4^* \left[1 + \frac{A_1 a_1^2}{14\lambda_0 \Omega_0} \right] \Phi_{17}(x_1, x_2) + \right. \\
 & + q_5^* \frac{A_1 a_1^2}{\lambda_0 \Omega_0} \Phi_{17}(x_1, x_2) - m_1^* \Phi_{18}(x_1, x_2) - m_2^* \Phi_{18}(x_2, x_1) \left. \right\};
 \end{aligned}$$

$$\begin{aligned}
 H = & -\frac{1-v}{2\pi} \left\{ m_1^* a_1^2 \Phi_{12}(x_1, x_2) + m_2^* a_1^2 \Phi_{12}(x_2, x_1) + \right. \\
 & + q_5^* \frac{\lambda_0 a_1^2}{3} \Phi_{19}(x_1, x_2, a_1) - q_4^* \left[\frac{D_0}{A_1} - \frac{\lambda_0 a_1^2}{42} \right] \Phi_{19}(x_1, x_2, a_1) + \\
 & + m_1^* \frac{D_0}{A_1} \Phi_{15}(x_1, x_2, a_1) + m_2^* \frac{D_0}{A_1} \Phi_{15}(x_2, x_1, a_1) - \\
 & - q_4^* a_1^2 \Phi_{20}(x_1, x_2) - m_1^* \frac{1}{v-1} \Phi_{14}(x_2, x_1, b_1) - \\
 & - m_2^* \frac{1}{v-1} \Phi_{14}(x_1, x_2, b_1) + \\
 & + m_1^* \frac{1}{v-1} \Phi_{15}(x_1, x_2, b_1) + m_2^* \frac{1}{v-1} \Phi_{15}(x_2, x_1, b_1) \left. \right\};
 \end{aligned}$$

$$\begin{aligned}
 Q_{10} = & \frac{1}{2\pi} \left\{ m_1^* b_1^2 \Phi_{13}(x_2, x_1, b_1) - m_2^* b_1^2 \Phi_{19}(x_1, x_2, b_1) - \right. \\
 & - q_4^* \Phi_{18}(x_1, x_2) - q_4^* a_1^2 \Phi_{21}(x_1, x_2) \left. \right\};
 \end{aligned}$$

$$\begin{aligned}
 Q_{20} = & \frac{1}{2\pi} \left\{ -m_1^* b_1^2 \Phi_{19}(x_1, x_2, b_1) + m_2^* b_1^2 \Phi_{13}(x_1, x_2, b_1) - \right. \\
 & - q_4^* \Phi_{18}(x_2, x_1) - q_4^* a_1^2 \Phi_{21}(x_2, x_1) \left. \right\};
 \end{aligned}$$

$$Q_{11} = \frac{\lambda_0 \Omega_0'}{2\pi A} \left[q_1^* \Phi_5(x_1, x_2) + q_2^* \Phi_8(x_1, x_2) - q_3^* \frac{A}{\lambda_0 \Omega_0'} \Phi_7(x_1, x_2) \right];$$

$$Q_{21} = \frac{\lambda_0 \Omega_0'}{2\pi A} \left[q_1^* \Phi_8(x_1, x_2) + q_2^* \Phi_5(x_2, x_1) - q_3^* \frac{A}{\lambda_0 \Omega_0'} \Phi_7(x_2, x_1) \right];$$

$$\begin{aligned}
 Q_{12} = & \frac{1}{28\pi} \left\{ -m_1^* b_1^2 \Phi_{13}(x_2, x_1, b_1) + m_2^* b_1^2 \Phi_{19}(x_1, x_2, b_1) + \right. \\
 & + m_1^* \frac{14\lambda_0 \Omega_0}{A_1} \Phi_{13}(x_1, x_2, a_1) + m_2^* \frac{14\lambda_0 \Omega_0}{A_1} \Phi_{19}(x_1, x_2, a_1) - \\
 & - 14q_5^* \Phi_{18}(x_1, x_2) + q_4^* \left[a_1^2 + \frac{14\lambda_0 \Omega_0}{A_1} \right] \Phi_{21}(x_1, x_2) \left. \right\};
 \end{aligned}$$

$$\begin{aligned}
 Q_{22} = & \frac{1}{28\pi} \left\{ m_1^* b_1^2 \Phi_{19}(x_1, x_2, b_1) - m_2^* b_1^2 \Phi_{13}(x_1, x_2, b_1) + \right. \\
 & + m_1^* \frac{14\lambda_0 \Omega_0}{A_1} \Phi_{19}(x_1, x_2, a_1) + \\
 & + m_2^* \frac{14\lambda_0 \Omega_0}{A_1} \Phi_{13}(x_2, x_1, a_1) - 14q_5^* \Phi_{18}(x_2, x_1) + \\
 & + q_4^* \left[a_1^2 + \frac{14\lambda_0 \Omega_0}{A_1} \right] \Phi_{21}(x_2, x_1) \left. \right\};
 \end{aligned}$$

$$R_0 = -\frac{\lambda_0^2 \Omega_0'}{2\pi A} \left[q_1^* \Phi_7(x_1, x_2) + q_2^* \Phi_7(x_2, x_1) - q_3^* \frac{B_0}{\lambda_0 \Lambda_0'} \Phi_9(x_1, x_2) \right];$$

$$\begin{aligned}
 R_1 = & \frac{\lambda_0 \Omega_0}{6\pi A_1} \left\{ q_4^* \left[1 + \frac{A_1 a_1^2}{14\lambda_0 \Omega_0} \right] \Phi_{17}(x_1, x_2) + \right. \\
 & + q_5^* \frac{A_1 a_1^2}{\lambda_0 \Omega_0} \Phi_{17}(x_1, x_2) - m_1^* \Phi_{18}(x_1, x_2) - m_2^* \Phi_{18}(x_2, x_1) \left. \right\},
 \end{aligned}$$

where

$$a_0^2 = \frac{B_0 \Omega_0'}{\Lambda \Lambda_0'}; \quad a_1^2 = \frac{196 D_0 \Omega_0}{9 A_1 \Lambda_0}; \quad b_1^2 = \frac{2 \Lambda_0}{D_0 (1-v)};$$

$$\Lambda = B_0 + \lambda_0^2 \Omega_0'; \quad A_1 = D_0 + \frac{1}{3} \lambda_0^2 \Omega_0;$$

$$\Phi_1(x_1, x_2) = \frac{3x_1}{2r^2} G_{0,1}(a_0 r) + \frac{x_1(x_1^2 - 3x_2^2)}{2r^4} G_{1,2}(a_0 r);$$

$$\Phi_2(x_1, x_2) = \frac{x_2}{2r^2} G_{0,1}(a_0 r) + \frac{x_2(3x_1^2 - x_2^2)}{2r^2} G_{1,2}(a_0 r);$$

$$\Phi_3(x_1, x_2) = -\frac{3}{8} x_1 G_{1,0}(a_0 r) - \frac{x_1(x_1^2 - 3x_2^2)}{8r^2} G_{2,1}(a_0 r);$$

$$\Phi_4(x_1, x_2) = -\frac{1}{8} x_2 G_{1,0}(a_0 r) - \frac{x_2(3x_1^2 - x_2^2)}{8r^2} G_{2,1}(a_0 r);$$

$$\Phi_5(x_1, x_2) = \frac{1}{2} \left[G_{0,0}(a_0 r) + \frac{x_1^2 - x_2^2}{r^2} G_{1,1}(a_0 r) \right];$$

$$\Phi_6(x_1, x_2) = -\frac{x_2(x_1^2 - x_2^2)}{2r^4}; \quad \Phi_7(x_1, x_2) = \frac{2x_1}{r^2} G_{0,1}(a_0 r);$$

$$\Phi_8(x_1, x_2) = \frac{x_1 x_2}{r^2} G_{1,1}(a_0 r); \quad \Phi_9(x_1, x_2) = G_{0,0}(a_0 r);$$

$$\Phi_{10}(x_1, x_2) = \frac{x_1(x_1^2 + 3x_2^2)}{2r^4};$$

$$\Phi_{11}(x_1, x_2) = -\frac{3}{8} x_1 G_{1,0}(a_1 r) - \frac{x_1(x_1^2 - 3x_2^2)}{8r^2} G_{2,1}(a_1 r);$$

$$\Phi_{12}(x_1, x_2) = -\frac{1}{8} x_2 G_{1,0}(a_1 r) - \frac{x_2(3x_1^2 - x_2^2)}{8r^2} G_{2,1}(a_1 r);$$

$$\Phi_{13}(x_1, x_2, c) = \frac{1}{2} \left[G_{0,0}(cr) + \frac{x_1^2 - x_2^2}{r^2} G_{1,1}(cr) \right];$$

$$\Phi_{14}(x_1, x_2, c) = \frac{3x_1}{2r^2} G_{0,1}(cr) + \frac{x_1(x_1^2 - 3x_2^2)}{2r^4} G_{1,2}(cr);$$

$$\Phi_{15}(x_1, x_2, c) = \frac{x_2}{2r^2} G_{0,1}(cr) + \frac{x_2(3x_1^2 - x_2^2)}{2r^4} G_{1,2}(cr);$$

$$\Phi_{16}(x_1, x_2) = -\frac{1}{2a_1} \ln \frac{r}{2} - \frac{1}{2a_1^2} G_{0,0}(a_1 r) - \frac{x_1^2 - x_2^2}{8} G_{2,0}(a_1 r);$$

$$\Phi_{17}(x_1, x_2) = G_{0,0}(a_1 r); \quad \Phi_{18}(x_1, x_2) = \frac{2x_1}{r^2} G_{0,1}(a_1 r);$$

$$\Phi_{19}(x_1, x_2, c) = \frac{x_1 x_2}{r^2} G_{1,1}(cr);$$

$$\Phi_{20}(x_1, x_2) = -\frac{x_1 x_2}{4} G_{2,0}(a_1 r);$$

$$\Phi_{21}(x_1, x_2) = -\frac{x_1}{2} G_{1,0}(a_1 r).$$

Graphs of internal force factors (10), (11) (Fig. 1–8) are constructed along the X-axis in the dimensionless Cartesian coordinate system x_1, x_2 , determined with accuracy to the half-thickness of plate h . Fig. 1–3 show the generalized membrane forces for a flat problem. Fig. 4–6 show generalized bending and twisting moments; Fig. 7–8 – generalized transverse forces for the state of bending.

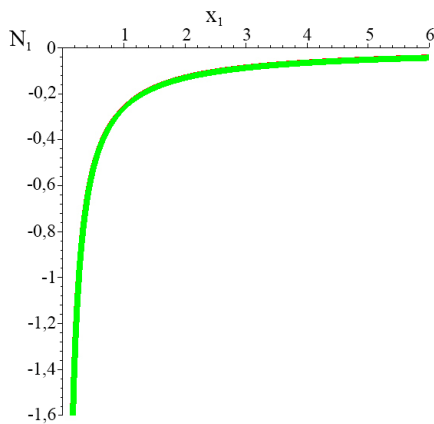


Fig. 1. Generalized membrane effort N_1 ; curve of green color – generalized effort, obtained within the framework of {1,0}-approximation; curve of red color – within the framework of {1,2}-approximation

When conducting the numerical studies, components of the expansion of volume force vector were determined from formulas (9), in which

$$m_1^* = m_2^* = q_1^* = q_2^* = q_3^* = q_4^* = q_5^* = 1.$$

The computations are performed for the following values of elastic constants of the transversal-isotropic material:

$$E^* = 5; \quad \nu = 0,3; \quad \nu' = 0,07; \quad E/G' = 40.$$

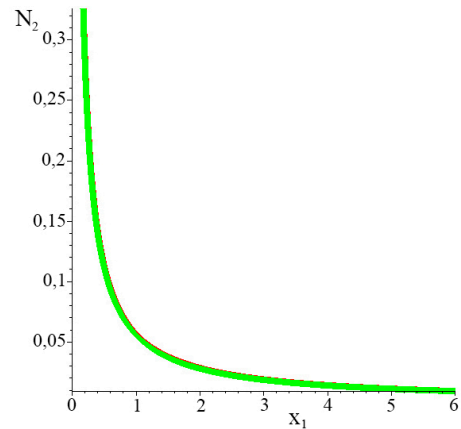


Fig. 2. Generalized membrane effort N_2 ; curve of green color – generalized effort, obtained within the framework of {1,0}-approximation; curve of red color – within the framework of {1,2}-approximation

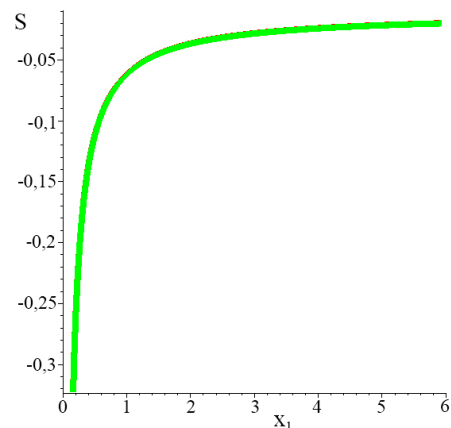


Fig. 3. Generalized membrane effort S ; curve of green color – generalized effort, obtained within the framework of {1,0}-approximation; curve of red color – within the framework of {1,2}-approximation

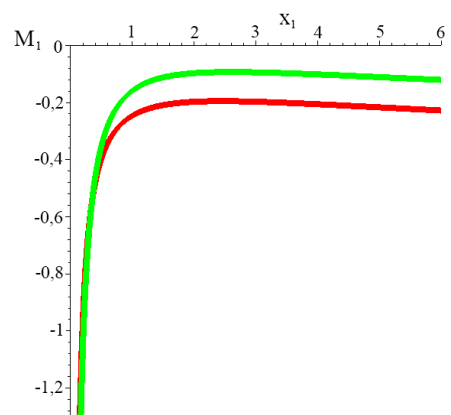


Fig. 4. Generalized bending moment M_1 ; curve of green color – generalized moment, obtained within the framework of {1,0}-approximation; curve of red color – within the framework of {1,2}-approximation

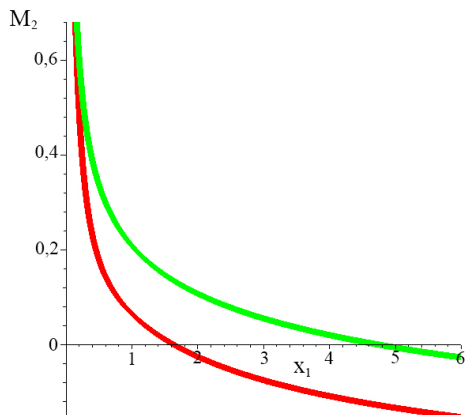


Fig. 5. Generalized bending moment M_2 ; curve of green color – generalized moment, obtained within the framework of $\{1,0\}$ -approximation; curve of red color – within the framework of $\{1,2\}$ -approximation

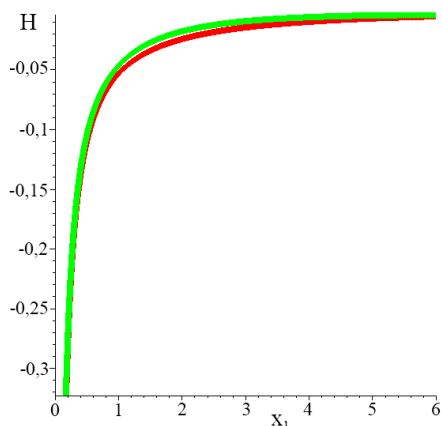


Fig. 6. Generalized torque H ; curve of green color – generalized moment, obtained within the framework of $\{1,0\}$ -approximation; curve of red color – within the framework of $\{1,2\}$ -approximation

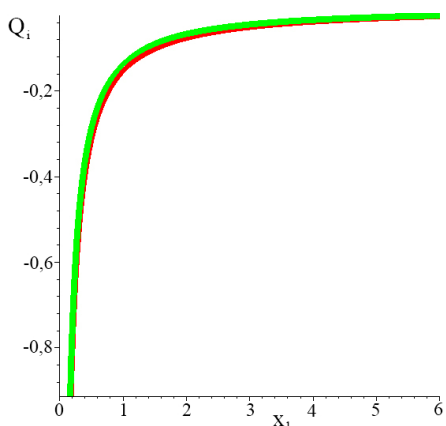


Fig. 7. Generalized transverse forces Q_i ; curve of green color – generalized transverse force, obtained within the framework of $\{1,0\}$ -approximation ($i=1$); curve of red color – within the framework of $\{1,2\}$ -approximation ($i=10$)

We carried out numerical calculations, which make it possible to compare the fundamental solutions (10), (11), built on the basis of generalized theory in the variants of $\{1,0\}$ - and $\{1,2\}$ -approximation, respectively. These studies

allowed us to estimate the refinement, introduced by the use of equations of the $\{1,2\}$ -approximation for transversal-isotropic plates, exposed to the action of concentrated force, instead of applying the analogous equations of $\{1,0\}$ -approximation.

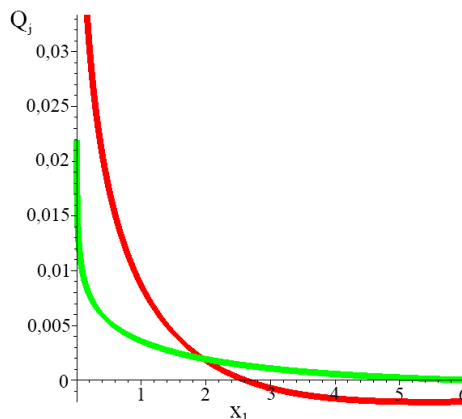


Fig. 8. Generalized transverse forces Q_j ; curve of green color – generalized transverse force, obtained within the framework of $\{1,0\}$ -approximation ($j=2$); curve of red color – within the framework of $\{1,2\}$ -approximation ($j=20$)

6. Discussion of results, obtained with the help of approximated theories

We built fundamental solutions of equations of the theory of elasticity on the basis of refined theory of the $\{m,n\}$ -approximation. The selected theory is the most acceptable for reducing the three-dimensional equations to the two-dimensional ones since it is not based on any hypotheses, but employs the method of I. N. Vekua for the expansion of desired functions into the Fourier series by the Legendre polynomials [14]. This approach makes it possible to examine not only the thin plates, but also the plates of medium and large thickness. In this case, the accuracy of the obtained solutions depends on the number of retained summands in the expansions of the assigned and desired functions. In addition, this theory makes it possible to consider transverse shearing and normal stresses.

The numerical studies conducted have demonstrated that the inner force factors of the zero spin stressed state and the state of bending, obtained with the use of equations of both the $\{1,0\}$ - and $\{1,2\}$ -approximation, have identical character, and their numerical values differ insignificantly. This research confirms that the generalized theory in the variant of $\{1,0\}$ -approximation makes it possible to ensure sufficiently high accuracy of the approximation of a three-dimensional problem of the theory of elasticity to the two-dimensional one.

7. Conclusions

1. We examined the methods of reducing the three-dimensional problems of the theory of elasticity for plates to the two-dimensional ones. An analysis of the approaches to decreasing the quantity of independent variables allowed us to conclude that when calculating the thin-wall elements of structures, made of contemporary composite materials, it is necessary to use the refined theories of plates and shells

for the concentrated force influences. These theories make it possible to evaluate the phenomena, connected to taking account of transverse shifts and compression. Since the classical theory of Kirchhoff-Love does not consider these phenomena, then this theory yields significant error during calculations. The refined theories also make it possible to examine not only the thin plates, but also the plates of medium and large thickness.

2. We analyzed the fundamental solutions of the equations of statics, obtained on the basis of generalized theory in the variants of $\{1,0\}$ - and $\{1,2\}$ -approximation for the purpose of determining the refinement, which is introduced by the retention of a large number of terms in the expansion series of the desired functions. The numerical studies are con-

ducted, which confirmed that the expansion of the desired functions into series by the Legendre polynomials from the thickness coordinate and the retention of particular sections of these series makes it possible to obtain a solution of the problem with required accuracy. The order of the resolving system of equations depends on the selection of approximation and at large m and n it can be sufficiently high. The use in practice of such equations is connected with the considerable mathematical complexities. That is why, depending on the specific character of the examined problems, one selects such approximation so that the resolving equations would take the simpler form and, at the same time, would reflect the specific character of mechanical behavior of plate with a sufficient degree of accuracy.

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