Розв'язана задача відшукання стаціонарних розподілів ймовірностей станів для марковських систем в умовах невизначеності. Передбачається, що параметри аналізованих систем задані нечітко. У задачі аналізу напівмарковських системи оцінка компонентів стаціонарного розподілу ймовірностей станів системи отримана шляхом мінімізації комплексного критерію. Критерій враховує міру відхилення шуканого розподілу від модального, а також рівень компактності функції приналежності нечіткого результату рішення

Ключові слова: марковська і напівмарковська системи, комплексний критерій, відхилення рішення від модального, міра компактності рішення

Решена задача отыскания стационарных распределений вероятностей состояний для марковских систем в условиях неопределенности. Предполагается, что параметры анализируемых систем заданы нечетко. В задаче анализа полумарковской системы оценка компонентов стационарного распределения вероятностей состояний системы получена путем минимизации комплексного критерия. Критерий учитывает меру отклонения искомого распределения от модального, а также уровень компактности функции принадлежности нечеткого результата решения

Ключевые слова: марковская и полумарковская системы, комплексный критерий, отклонение решения от модального, мера компактности решения

UDC 519.2
DOI: 10.15587/1729-4061.2017.97144

FINDING THE PROBABILITY DISTRIBUTION OF STATES IN THE FUZZY MARKOV SYSTEMS

L. Raskin
Doctor of Technical Sciences, Professor, Head of Department*
O. Sira
Doctor of Technical Sciences, Professor*
E-mail: chime@bk.ru
T. Katkova
PhD, Assistant Professor
Department of Information Systems and Technologies
Berdyansk University of Management and Business
Svobody str., 117 a, Berdyansk, Ukraine, 71118
E-mail: 777-kit@ukr.net
*Department of Computer Monitoring and Logistics
National Technical University «Kharkiv Polytechnic Institute»
Kyrpychova str., 2, Kharkiv, Ukraine, 61002

1. Introduction

Traditional problems on describing the behavior and estimation of effectiveness of multiparametric systems are solved on the assumption that basic parameters of the system in the process of its functioning do not change [1–4]. For example, in the analysis of service systems, assumptions are used on that the intensity of incoming flow of requests, as well as the number of system's channels and their productivity, are set and fixed. In this case, under conditions of correctly described incoming flow of requests, it is possible to obtain a closed description of the mathematical model, which defines the process the system functions. In particular, especially simple correlations occur if we consider that the incoming flow of requests is of Poisson kind and service duration is distributed exponentially. At the same time, when solving many practical problems, it is necessary to consider the circumstance that parameters of the analyzed system are not necessarily constants but they may vary stochastically. The models that emerge in this case are beyond the framework of classical theory and thus require studying. It is clear that it is hardly expedient to pose the appropriate problems of systems analysis in the most general statement in view of its insufficient meaningfulness. Given this, we shall confine ourselves to examining the Markov systems.

2. Literature review and problem statement

A traditional procedure for the analysis of Markov systems consists of the following. Infinitesimal matrix $\Lambda=\left(\lambda_{ij}\right)$ of transition intensities is assigned. A vector-function is introduced:

$$P(t)=(p_1(t),p_2(t),...,p_n(t)),$$

whose components define the laws of variation over time of probabilities of the system being on a set of possible states. Then behavior of the system, as is known, is described by the Kolmogorov system of differential equations [1, 2]:

$$\frac{dp_k(t)}{dt}=\sum_{j\in E_k^+}\lambda_{jk}p_j(t)-p_k(t)\sum_{j\in E_k^-}\lambda_{kj},\quad k=1,2,...,n, \tag{1}$$

where λ_{jk} is the intensity of transition from the j-th state into the k-th one; $p_j(t)$ is the probability that in moment t, the system will be in state j; E_k^+ is the set of states, from which a transition is possible into state k in one step; E_k^- is the set of states, into which a transition is possible from state k in one step.

A system of linear differential equations (1) at the assigned initial conditions (for example, $P(0)=(1\quad 0\quad ...\quad 0)$ is solved by known methods [3]. If, in this case, there is interest not in a transient process, but in a stationary distribution of probabilities of states of the system, then

$$\frac{dp_k(t)}{dt}=0,\ k=1,2,\ldots,n,$$

and system (1) reduces to the system of linear algebraic equations [4, 5]:

$$\sum_{j\in E_k^+}\lambda_{jk}p_j-p_k\sum_{j\in E_k^-}\lambda_{kj}=0,\quad k=1,2,\ldots,n. \qquad (2)$$

This system is solved together with the normalization condition:

$$\sum_{j=1}^{n}p_j=1 \qquad (3)$$

and defines the desired distribution of probabilities of states.

The problem, naturally, becomes more complicated if the elements of infinitesimal matrix $\Lambda=(\lambda_{ij})$ of the intensities of transitions cannot be evaluated precisely. We shall consider that in a typical situation with a small sample of initial data, the statistical material available is insufficient for obtaining the adequate theoretically probabilistic description of the processes the system functions. However, these data make it possible to obtain the description in the terms of the theory of fuzzy sets with required quality. Principles of the theory of fuzzy sets are presented in [6–9]. [10–12] examined the methods for systems analysis and decision making under conditions of fuzzy initial data. Let us assume that the elements of matrix of the intensities of transitions are the fuzzy numbers with known membership functions. Assume that in order to assign these fuzzy numbers, we used a Gaussian form of representation, that is:

$$\mu(\lambda_{ij})=\exp\left\{-\frac{\left(\lambda_{ij}-\lambda_{ij}^{(0)}\right)^2}{\sum\sigma_{ij}^2}\right\},\quad i=1,2,\ldots,n,\quad j=1,2,\ldots,n. \qquad (4)$$

In this case, the system of linear algebraic equations (2) will contain the indistinctly described parameters and, given this, traditional methods of its solution are not applicable.

We shall note that difficulties in the analysis of real systems are not limited by the impossibility of precise determining the parameters of Markov system. The problems are of a more general character. Contemporary understanding of the processes of functioning of complex systems testifies to the insufficient adequacy of their Markov descriptions, widely and traditionally employed. The nonexponentiality of the processes of systems being in their possible states they predetermines the expediency of applying in the problems of analysis and synthesis of real systems a more flexible mathematical apparatus – the theory of semi-Markov processes (SMP). Technologies of solving such problems under conditions of SMP are well developed and efficient [13–15].

Assume that the system is assigned by a set of states $E=\{E_1,E_2,\ldots,E_n\}$ and a set of its possible transitions from some states to others. We shall determine SMP in this system by the matrix of conditional distribution functions $F(t)=(F_{ij}(t))$, $i,j\in E$, durations of being in each state before exiting it and by matrix $P=(P_{ij})$ of transition probabilities of the embedded Markov chain (EMC). Then, as is known, the asymptotic behavior of SMP is described by a vector of final probabilities of states of the system whose components are calculated by formula

$$V_i=\frac{\pi_i T_i}{\sum_{i\in E}\pi_i T},\quad i=1,2,\ldots,n, \qquad (5)$$

where π_i is the component of stationary distribution of probabilities of the states of EMC that determines probability of the i-th state, $i\in E$; T_i is the mean duration of SMP being in state i before exiting this state, $i\in E$.

In this case, stationary distribution $\pi=(\pi_1\quad \pi_2\quad \ldots\quad \pi_m)$ of the EMC states is found by using the matrix of transition probabilities P as a result of solving a system of equations:

$$\pi=\pi P, \qquad (6)$$

supplemented by the normalization condition:

$$\sum_{i\in E}\pi_i=1 \qquad (7)$$

and the mean duration of being in E_i before exiting it is determined by relationship:

$$T_i=\sum_{j\in E,j\neq i}P_{ij}\int_0^\infty(1-F_{ij}(t))dt,\quad i\in E. \qquad (8)$$

However, in practice, the situations frequently occur when, for objective reasons, analytical descriptions of basic elements in the Markov and semi-Markov models cannot be obtained precisely. In this case, the least demanding is the representation of these elements of SMP by the means of theory of fuzzy sets.

We shall in this case consider that the analytical descriptions of conditional distribution functions $F_{ij}(t)$ of the duration of being in each of the states before exiting contain a fuzzy parameter θ. Then, determined by the distribution $F_{ij}(t,\theta)$ for a fixed t, the value of the probability of the fact that random duration of being in E_i before passing into E_j will be less than t, becomes a fuzzy number. The membership function of this number is determined by the membership function of fuzzy parameter θ. Analysis of the system by traditional methods in this case is impossible.

Conducted analysis of the known approaches to solving the problems on complex systems analysis allows us to draw the following conclusions. These approaches actually employ the assumption that parameters of the systems are known and determined. This is not the case in real situations, and the level of uncertainty depends considerably on the volume of available statistical material. In this case, in the most frequently occurring situations with a small sample of initial data, the most adequate models are not theoretically-probabilistic, but fuzzy models. This circumstance renders relevance to a problem on developing the mathematical tools to solve the problems of systems theory taking into account the uncertainty in the values of their parameters. Solving this problem is particularly important for the Markov systems whose formal models are maximally parametrized. The range of topics in the publications on this subject is very wide. They examine problems on making fuzzy decisions in the Markov systems [16], the problems of fuzzy control in such systems [17], etc. In all cases, correctness of the result is defined by the level of substantiation of the adopted technology for calculating the stationary probability distributions for the fuzzy Markov systems. This task is of theoretical interest.

Let us state the problem on developing the procedure for systems analysis whose behavior is described by the Markov or semi-Markov process with indistinctly defined parameters.

3. The aim and tasks of the study

The aim of present study is to develop axiomatics and a mathematical toolbox for the theory of fuzzy sets in order to solve problems on the analysis of Markov systems.

To achieve the set aim, the following tasks were formulated:

– development of a procedure for calculating the stationary distribution of probabilities of states of the Markov system with a fuzzy matrix of intensities of transitions;

– calculation of statistical characteristics of the semi-Markov system whose parameters are assigned indistinctly;

– development of a procedure for calculating the stationary distribution of probabilities of states of the semi-Markov system, whose parameters are assigned indistinctly.

4. Analysis of Markov and semi-Markov systems with parameters that are not clearly assigned

Let us examine a problem on finding the final distribution of probabilities of states of the Markov system whose behavior is determined by a fuzzy matrix of the intensities of transitions $\Lambda=\left(\lambda_{ij}\right)$. In connection with this, let us supplement the model of this system, introduced above (2), (3), by relationships (4) that assign the membership functions of fuzzy intensities of transitions. Thus, there appears a system of linear algebraic equations whose parameters are not clearly assigned.

Let us explore the technology of solving such systems in a general form.

Assume that it is necessary to solve system n of linear algebraic equations with n unknowns:

$$\begin{aligned} &a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n-a_{1,n+1}=0,\\ &a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n-a_{2,n+1}=0,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &a_{n1}x_1+a_{n2}x_2+\ldots+a_{nn}x_n-a_{n,n+1}=0. \end{aligned} \tag{9}$$

We shall consider parameters of system (9) to be the Gaussian fuzzy numbers with membership functions. The choice of this form of the membership function is predetermined by the ease of fulfilling the operations over fuzzy numbers of this type [12].

Introduce

$$\mu(a_{ij})=\exp\left\{-\frac{\left(a_{ij}-a_{ij}^{(0)}\right)^2}{2\sigma_{ij}^2}\right\},\ i=1,2,\ldots,n,\ j=1,2,\ldots,n+1. \tag{10}$$

Now we shall point that relationships (9) and (10) assign the fuzzy system of linear algebraic equations.

Introduce a set of fuzzy numbers

$$\begin{aligned} &Z_1=\sum_{j=1}^{n}a_{1j}x_j-a_1,n+1,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &Z_n=\sum_{j=1}^{n}a_{nj}x_j-a_n,n+1. \end{aligned} \tag{11}$$

Let us solve now a generated (9), (10) clear system of linear algebraic equations by using modal values $a_{ij}^{(0)}$ of fuzzy numbers a_{ij}, $i=1,2,\ldots,n$, $j=1,2,\ldots,n+1$:

$$\sum_{j=1}^{n}a_{ij}^{(0)}x_j-a_{i,n+1}^{(0)},\quad i=1,2,\ldots,n. \tag{12}$$

Introduce:

$$A^{(0)}=\left(a_{ij}^{(0)}\right),\quad X=\left\{x_j\right\},$$

$$A_{n+1}^{(0)}=\left(a_{i,n+1}^{(0)}\right),\quad i=1,2,\ldots n,\quad j=1,2,\ldots n.$$

In this case, system of equations (12) in the matrix form is as follows:

$$A^{(0)}X-A_{n+1}^{(0)}=0.$$

Then we shall define a clear solution of problem (9), (10) the set $X=\left\{x_j\right\}$, $j=1,2,\ldots,n$, which minimizes the sum of indicators of compactness of figures, constrained by membership functions $\mu\left(Z_i\right)$ of fuzzy numbers $Z_1,Z_2,\ldots,Z_n$, and being least deviated from $X^{(0)}$. The sense of this criterion is clear. Its application ensures obtaining the set of clear numbers $x_1,x_2,\ldots,x_n$, for which membership functions of numbers $Z_1,Z_2,\ldots,Z_n$, are the least blurred and have modal values that are maximally close to zero. Presented constructive idea of solving the system of linear algebraic equations was for the first time formulated in [18]. A fundamental deficiency in the approach, which realizes this idea in [18], is in the fact that obtained solution will not necessarily satisfy constraints on variables. The deficiency indicated is eliminated here.

We shall write required relationships, which provide for obtaining the solution of problem (9), (10), in the above sense. In accordance with (10), let us determine the membership functions of fuzzy numbers $Z_1,Z_2,\ldots,Z_n$, assigned by (11). In this case:

$$\mu\left(z_i\right)=\mu\left(\sum_{j=1}^{n}a_{ij}x_j-a_{i,n+1}\right)=$$

$$=\exp\left\{-\frac{\left[z_i-\left(\sum_{j=1}^{n}a_{ij}^{(0)}x_j-a_{i,n+1}^{(0)}\right)\right]^2}{2\left(\sum_{j=1}^{n}\sigma_{ij}^2x_j^2+\sigma_{i,n+1}^2\right)}\right\}=\exp\left\{-\frac{(z_i-m_i)^2}{2\sigma_i^2}\right\},$$

$$m_i=\sum_{j=1}^{n}a_{ij}^{(0)}x_j-a_{i,n+1}^{(0)},\quad \sigma_i^2=\sum_{j=1}^{n}\sigma_{ij}^2x_j^2+\sigma_{i,n+1}^2,\quad i=1,2,\ldots,n.$$

If $X^{(0)}=\left\{X_j^{(0)}\right\}$ is the solution of a system of equations (12), parameters of which correspond to the modal values of membership functions (10), then as a measure of deviation of the desired solution of problem X from modal solution $X^{(0)}$, we use:

$$J_1 = \sum_{j=1}^{n}\left(x_j - x_j^{(0)}\right)^2.$$

As an indicator of compactness of the membership function $\mu(Z_i)$, we accept the values of squares of variations σ_i^2 of fuzzy numbers Z_i, $i = 1,2,\ldots,n$. In this case, a generalizing characteristic of the measure of compactness of solution X can be calculated by formula:

$$J_2 = \sum_{i=1}^{n-1}\sum_{j=1}^{n}\sigma_{ij}^2 x_j^2 = \sum_{j=1}^{n}\left(\sum_{i=1}^{n-1}\sigma_{ij}^2\right)x_j^2 = \sum_{j=1}^{n}\sigma_j^2 x_j^2. \quad (13)$$

Relationship (13) took into account that the system of linear algebraic equations (12) contains (n–1) linearly independent equations.

Thus, the problem is reduced to finding the set X that minimizes complex criterion:

$$J(x) = J_1 + J_2 = \sum_{j=1}^{n}\sigma_j^2 x_j^2 + \sum_{j=1}^{n}\left(x_j - x_j^{(0)}\right)^2, \quad (14)$$

on the set of solutions of equation:

$$\sum_{j=1}^{n} x_j = 1, \quad x_j \ge 0, \quad j = 1,2,\ldots,n. \quad (15)$$

We shall obtain a solution of the problem by the method of Lagrange indeterminate coefficients.

A Lagrange function takes the form:

$$\Phi(x) = \sum_{j=1}^{n}\sigma_j^2 x_j^2 + \sum_{j=1}^{n}\left(x_j - x_j^{(0)}\right)^2 - \lambda\left(\sum_{j=1}^{n} x_j - 1\right).$$

Next

$$\frac{dJ(x)}{dx_j} = 2\sigma_j^2 x_j + 2\sum_{j=1}^{n}\left(x_j - x_j^{(0)}\right) - \lambda = 0, \quad j = 1,2,\ldots,n.$$

Hence

$$2x_j\left(\sigma_j^2 + 1\right) = \lambda + 2x_j^{(0)},$$

$$x_j = \frac{\lambda + 2x_j^{(0)}}{2\left(\sigma_j^2 + 1\right)} = \frac{\lambda}{2}\cdot\frac{1}{\sigma_j^2 + 1} + \frac{x_j^{(0)}}{\sigma_j^2 + 1},$$

$$j = 1,2,\ldots,n. \quad (16)$$

By substituting (16) into (15), find $\lambda/2$. We obtain:

$$\sum_{j=1}^{n} x_j = \frac{\lambda}{2}\sum_{j=1}^{n}\frac{1}{\sigma_j^2 + 1} + \sum_{j=1}^{n}\frac{x_j^{(0)}}{\sigma_j^2 + 1} = 1.$$

Then

$$\frac{\lambda}{2} = \frac{1}{\sum_{j=1}^{n}\frac{1}{\sigma_j^2 + 1}}\left(1 - \sum_{j=1}^{n}\frac{x_j^{(0)}}{\sigma_j^2 + 1}\right). \quad (17)$$

By substituting (17) in (16), we shall obtain a relationship for calculating the desired distribution x_j, $j = 1,2,\ldots,n$, of probabilities of states of the system. In this case:

$$x_j = \frac{1}{\sigma_j^2 + 1}\frac{1}{\sum_{j=1}^{n}\frac{1}{\sigma_j^2 + 1}}\left(1 - \sum_{j=1}^{n}\frac{x_j^{(0)}}{\sigma_j^2 + 1}\right) + \frac{x_j^{(0)}}{\sigma_j^2 + 1} =$$

$$\frac{1}{\sigma_j^2 + 1}\left[x_j^{(0)} + \frac{1 - \sum_{j=1}^{n}\frac{x_j^{(0)}}{\sigma_j^2 + 1}}{\sum_{j=1}^{n}\frac{1}{\sigma_j^2 + 1}}\right]. \quad (18)$$

It is easy to see that for two possible extreme situations with uncertainty ($\sigma_j = 0$ or $\sigma_j = \infty$, $j = 1,2,\ldots,n$), formula (18) yields natural results:

$$\lim_{\sigma_j \to 0} x_j = x_j^{(0)}, \quad \lim_{\sigma \to \infty} x_j = \frac{1}{n}, \quad j = 1,2,\ldots,n.$$

Let us pass to the problem on analysis of a fuzzy semi-Markov system. We begin from the uncertainty in the description of functions of distribution of duration of the system being in any specific state before passing to another state.

Assume, for example, that durations of the system being in E_i before passing on to E_j are distributed exponentially, that is:

$$F_{ij}(t) = 1 - e^{-\lambda_{ij}t}, \quad t > 0, \quad (i,j) \in E,$$

in this case, parameters λ_{ij} are the fuzzy numbers with membership functions:

$$\mu_\lambda(\lambda_{ij}) = \begin{cases} 1, \lambda_{ij} \in \left[a_{ij}, b_{ij}\right], \\ 0, \lambda_{ij} \notin \left[a_{ij}, b_{ij}\right]. \end{cases} \quad (19)$$

In this case, the fuzzy value of conditional mean value of duration T_{ij} being in E_i before passing on to E_j is equal to:

$$T_{ij} = \int_0^\infty \left(1 - F_{ij}(t)\right)dt = \int_0^\infty e^{-\lambda_{ij}t}dt = \frac{1}{\lambda_{ij}}. \quad (20)$$

Find the membership function of fuzzy number j= =1, 2,..., n–1 In accordance with the principle of generalization [6], a membership function of the result of executing the operation $z = f(x)$ over fuzzy number x with the membership function $\mu_x(x)$ takes the form $\mu_x\left(f^{-1}(z)\right)$. Then we receive:

$$\mu\left(T_{ij}\right) = \mu_\lambda\left(\frac{1}{T_{ij}}\right) = \begin{cases} 1, \frac{1}{T_{ij}} \in \left[a_{ij}, b_{ij}\right], , \\ 0, \frac{1}{T_{ij}} \notin \left[a_{ij}, b_{ij}\right]. . \end{cases}$$

Hence

$$\mu\left(T_{ij}\right) = \begin{cases} 1, T_{ij} \in \left[\frac{1}{b_{ij}}, \frac{1}{a_{ij}}\right], \\ 0, T_{ij} \notin \left[\frac{1}{b_{ij}}, \frac{1}{a_{ij}}\right]. \end{cases} \quad (21)$$

In fuzzy mathematics, the concept of "mathematical expectation" is lacking, instead of which there is the concept of "expected value". Calculation of the expected value of fuzzy number x with carrier $\left(\pi_1^{(0)}, \pi_2^{(0)}, \ldots, \pi_n^{(0)}\right)$ and membership function $\mu(x)$ is carried out by formula:

$$\mu(x)=\frac{\int_c^d x\mu(x)dx}{\int_c^d \mu(x)dx}. \qquad (22)$$

Let us perform the required actions for calculating the conditional expected value of indistinctly assigned duration T_{ij} of being in E_i under condition of transition to E_j. We obtain:

$$D_{ij}=\alpha_{ij}(1-r_{ij})=\beta_{ij}(1+r_{ij})=\frac{2\alpha_{ij}\beta_{ij}}{\alpha_{ij}+\beta_{ij}}. \qquad (23)$$

$$\int_{\frac{1}{b_{ij}}}^{\frac{1}{a_{ij}}} T_{ij}\mu(T_{ij})dT_{ij}=\frac{1}{2}\left[\left(\frac{1}{a_{ij}}\right)^2-\left(\frac{1}{b_{ij}}\right)^2\right]. \qquad (24)$$

Then, with regard to (22), we receive:

$$M[T_{ij}]=\frac{1}{2}\left(\frac{1}{a_{ij}}+\frac{1}{b_{ij}}\right)=\frac{a_{ij}+b_{ij}}{2a_{ij}b_{ij}}. \qquad (25)$$

Now, using natural analog (7), we compute the unconditional expected value of the duration of being in E_i before passing on into any other state:

$$T_i=\sum_{j\in E, j\neq i} P_{ij}T_{ij}=\sum_{j\in E, j\neq i}\frac{a_{ij}+b_{ij}}{2a_{ij}b_{ij}}P_{ij}. \qquad (26)$$

It is clear that the simplicity of relationship (26) is wholly and entirely predetermined by the simplicity of assigning the membership function with a fuzzy parameter of conditional function of distribution of the duration of being in each of the states of the system. Considerable difficulties in the solution of this problem appear if the distribution functions of durations of being in the states of a semi-Markov system are not exponential. In this case, real situation is as follows: there is some set of observations of the random duration of being in each of the states before exiting. Results of the observations are represented by fuzzy numbers with the assigned membership functions. In this case, a preliminary step is necessary – the restoration of unknown densities of distribution of the random values of the observed magnitudes, for example, by the maximum likelihood method. This scheme leads to a fuzzy problem of mathematical programming with the possible methods of solution examined in [19]. Still more complex is the situation when the descriptions of these functions are constrained by the values of two statistical characteristics – mathematical expectation and dispersion in the corresponding random magnitudes. In this case, it is expedient to solve the problem under the assumption about the worst distribution density, obtained by solving the problem on the continuous linear programming [20, 21]. Finally, this problem becomes most difficult if the indeterminate parameters of distribution functions of the durations of being in the states of the system are described in the terms of inaccurate mathematics [22]. A possible approach to overcoming the problems occurring here is the use of fuzzy models of inaccurate parameters of the problem [23].

Let us return to the task of analysis of a semi-Markov system. Let us examine the procedure to account for possible fuzziness in the description of elements of matrix of transition probabilities $P=\{P_{ij}\}$, assigned by the corresponding membership function $\mu(P_{ij})$. A problem arises when finding the stationary distribution of probabilities of states of the embedded Markov chain due to the need of solving the system of linear algebraic equations (6), (7), whose parameters are not clearly assigned.

Let us write a system of equations (6), (7) in the scalar form:

$$\begin{aligned}
&\pi_1(P_{11}-1)+\pi_2P_{21}+\ldots+\pi_nP_{n1}=0,\\
&\pi_1P_{12}+\pi_2(P_{22}-1)+\ldots+\pi_nP_{n2}=0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\pi_1P_{1,n-1}+\pi_2P_{2,n-1}+\ldots+\pi_{n-1}(P_{n-1,n-1}-1)+\pi_nP_{n,n-1}=0,\\
&\pi_1+\pi_2+\ldots+\pi_{n-1}+\pi_n=1.
\end{aligned} \qquad (27)$$

A general scheme of solving this system of equations corresponds to the procedure, described in the solution of system (9).

Let us solve a system of equations (27) by assigning the values of coefficients P_{ij}, equal to their modal values $P_{ij}^{(0)}$, $i=1,2,..,n$, $j=1,2,..n-1$. Let $\pi^{(0)}=(\pi_1^{(0)},\pi_2^{(0)},\ldots,\pi_n^{(0)})$ be a solution of this system.

Introduce the set of numbers the way we did before:

$$\begin{aligned}
&Z_1=\pi_1(P_{11}-1)+\pi_2P_{21}+\ldots+\pi_nP_{n1},\\
&Z_2=\pi_1P_{12}+\pi_2(P_{22}-1)+\ldots+\pi_nP_{n2},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&Z_{n-1}=\pi_1P_{1,n-1}+\pi_2P_{2,n-1}+\ldots+\pi_nP_{n,n-1}.
\end{aligned} \qquad (28)$$

By applying the rules of fulfilling the operations over fuzzy numbers [6–9], taking into account the assigned membership functions for transition probabilities (P_{ij}), let us find membership functions $\mu(Z_1),\mu(Z_2),..,\mu(Z_{n-1})$ of the fuzzy numbers (28). The modal values of these fuzzy numbers depend on the set $(\pi_1,\pi_2,\ldots,\pi_n)$ and are equal to:

$$\begin{aligned}
&Z_1^{(0)}=\pi_1(P_{11}^{(0)}-1)+\pi_2P_{21}^{(0)}+\ldots+\pi_nP_{n1}^{(0)},\\
&Z_2^{(0)}=\pi_1P_{12}^{(0)}+\pi_2(P_{22}^{(0)}-1)+\ldots+\pi_nP_{n2}^{(0)},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&Z_{n-1}^{(0)}=\pi_1P_{1,n-1}^{(0)}+\pi_2P_{2,n-1}^{(0)}+\ldots+\pi_nP_{n,n-1}^{(0)}.
\end{aligned} \qquad (29)$$

It is obvious that compactness and configuration of the membership functions $\mu(Z_1),\mu(Z_2),..,\mu(Z_{n-1})$ are also defined by the set $(\pi_1,\pi_2,\ldots,\pi_n)$. Now it is possible to pose the problem on finding the clear solution of a fuzzy system of equations (27). As earlier, natural requirements are demanded of this solution. First, the desired set:

$$\pi^*=(\pi_1^*,\pi_2^*,\ldots,\pi_n^*),$$

must minimally differ from set $(\pi_1^{(0)},\pi_2^{(0)},\ldots,\pi_n^{(0)})$, that is, modal values $Z_1^{(0)},Z_2^{(0)},\ldots,Z_{n-1}^{(0)}$ must be maximally close to zero. Second, it is desirable that the uncertainty bodies, assigned by the membership functions

$$\mu(Z_1,\pi^*),\mu(Z_2,\pi^*),..,\mu(Z_{n-1},\pi^*)$$

be maximally compact (the least blurred).

Let fuzzy numbers (P_{ij}) be assigned by membership functions $(L-R)$ – of the type [6, 7], more general than relationships (10), that is:

$$\mu\left(P_{ij}\right)=\begin{cases}L\left(\dfrac{P_{ij}^{(0)}-P_{ij}}{\alpha_{ij}}\right),P_{ij}\le P_{ij}^{(0)},\\ R\left(\dfrac{P_{ij}-P_{ij}^{(0)}}{\beta_{ij}}\right),P_{ij}\rangle P_{ij}^{(0)},\end{cases} \qquad (30)$$

where α_{ij},β_{ij} are the left and right fuzziness coefficients.

If we select L and R functions as Gaussian, then (30) takes the form:

$$\mu\left(P_{ij}\right)=\begin{cases}\exp\left\{-\dfrac{\left(P_{ij}^{(0)}-P_{ij}\right)^2}{2\alpha_{ij}}\right\},P_{ij}\le P_{ij}^{(0)},\\ \exp\left\{-\dfrac{\left(P_{ij}-P_{ij}^{(0)}\right)^2}{2\beta_{ij}}\right\},P_{ij}\rangle P_{ij}^{(0)}.\end{cases} \qquad (31)$$

Let us note that relationship (31) can be recorded in a more compact form:

$$\mu_{ij}\left(P_{ij}\right)=\exp\left\{-\frac{\left(P_{ij}-P_{ij}^{(0)}\right)^2}{2D_{ij}}\left(1+r_{ij}\mathrm{Sign}\left(P_{ij}-P_{ij}^{(0)}\right)\right)\right\}. \qquad (32)$$

Unknown parameters D_{ij} and r_{ij} are easily determined through the assigned values α_{ij} and β_{ij}. Write (32) as follows:

$$\mu\left(P_{ij}\right)=\begin{cases}\exp\left\{-\dfrac{\left(P_{ij}^{(0)}-P_{ij}\right)^2}{2\dfrac{D_{ij}}{1-r_{ij}}}\right\},P_{ij}\le P_{ij}^{(0)},\\ \exp\left\{-\dfrac{\left(P_{ij}-P_{ij}^{(0)}\right)^2}{2\dfrac{D_{ij}}{1+r_{ij}}}\right\},P_{ij}\rangle P_{ij}^{(0)}.\end{cases} \qquad (33)$$

Then $\dfrac{D_{ij}}{1-r_{ij}}=\alpha_{ij}$, $\dfrac{D_{ij}}{1+r_{ij}}=\beta_{ij}$, hence:

$$\frac{1+r_{ij}}{1-r_{ij}}=\frac{\alpha_{ij}}{\beta_{ij}},\quad r_{ij}=\frac{\alpha_{ij}-\beta_{ij}}{\alpha_{ij}+\beta_{ij}},$$

$$D_{ij}=\alpha_{ij}\left(1-r_{ij}\right)=\beta_{ij}\left(1+r_{ij}\right)=\frac{2\alpha_{ij}\beta_{ij}}{\alpha_{ij}+\beta_{ij}}.$$

Now, with regard to (28) and (31), we shall write the membership functions of fuzzy numbers:

$$\mu\left(Z_i\right)=\mu\left(\sum_{j\ne i}\pi_jP_{ij}+\pi_i\left(P_{ii}-1\right)\right)=$$

$$=\begin{cases}\exp\left(-\dfrac{1}{2}\dfrac{\left(Z_i-\overline{Z}_i\right)^2}{\alpha_{ij}}\right),Z_i\le\overline{Z}_i,\\ \exp\left(-\dfrac{1}{2}\dfrac{\left(Z_i-\overline{Z}_i\right)^2}{\beta_{ij}}\right),Z_i>\overline{Z}_i,\end{cases}$$

$$\overline{Z}_i=\sum_{j\ne i}\pi_jP_{ij}^{(0)}+\pi_i\left(P_{ij}^{(0)}-1\right),\quad \alpha_i=\sum_{j=1}^{n}\pi_j^2\alpha_{ij},$$

$$\beta_i=\sum_{j=1}^{n}\pi_j^2\beta_{ij},\quad i=1,2,..,n-1.$$

As an indicator of compactness of the uncertainty body, assigned by the obtained membership function, we may use an area under the appropriate curve, that is:

$$S_i=\int_{-\infty}^{\infty}\mu\left(Z_i\right)dZ_i=\int_{-\infty}^{\overline{Z}_i}\exp\left(-\frac{1}{2}\frac{\left(Z_i-\overline{Z}_i\right)^2}{\alpha_i}\right)dZ_i+$$

$$\int_{\overline{Z}_i}^{\infty}\exp\left(-\frac{1}{2}\frac{\left(Z_i-\overline{Z}_i\right)^2}{\beta_i}\right)dZ_i=\sqrt{\frac{\pi}{2}}\left(\sqrt{\alpha_i}+\sqrt{\beta_i}\right).$$

Then the complex optimality criterion of set

$$\pi^*=\left(\pi_1^*,\pi_2^*,...,\pi_n^*\right)$$

takes the form:

$$J\left(\pi^*\right)=$$

$$=\sqrt{\frac{\pi}{2}}\sum_{i=1}^{n-1}\left(\alpha_i^{\frac{1}{2}}\left(\pi^*\right)+\beta_i^{\frac{1}{2}}\left(\pi^*\right)\right)+\left(\pi^*-\pi^{(0)}\right)\left(\pi^*-\pi^{(0)}\right)^T. \qquad (34)$$

Thus, the problem on finding the stationary distribution of probabilities of the EMC states is reduced to the following problem on mathematical programming: to find the set $\pi^*=\left(\pi_1^*,\pi_2^*,...,\pi_n^*\right)$, which minimizes (34) and satisfies constraints:

$$\sum_{i=1}^{n}\pi_i^*=1,\quad \pi_i^*\ge 0. \qquad (35)$$

Let this set is found. Then by using it, with regard to (5), (8), it is possible to compute the vector of final probabilities of the SMP states. If necessary, complex criterion (34) can be modified through the introduction of weight coefficients, which consider possible differences in the levels of requirements to different components of the criterion.

Finally, we note that as an alternative criterion of compactness of uncertainty bodies for the membership functions $\mu_i\left(Z_i\right)$, $i=1,2,..,n-1$, one may select the summary length of carriers of sets g, corresponding to them, of the level, which is equal to:

$$R=\sum_{i=1}^{n-1}R_i\left(\gamma\right),\quad R_i\left(\gamma\right)=\left\{Z_i:\mu\left(Z_i\right)\ge\gamma\right\}.$$

However, this does not facilitate the task in any way, since, as it is easy to demonstrate, analytical expression for computing

$$R=\sum_{i=1}^{n-1}R_i\left(\gamma\right)$$

repeats the expression for

$$S=\sum_{i=1}^{n-1}S_i$$

with the weight coefficient that depends on g.

We considered the problem on finding the stationary probability distribution for the Markov systems whose parameters are assigned indistinctly. A procedure for obtaining the result is based on a special technology for solving the system of linear algebraic equations with fuzzy parameters. Solution of this problem is achieved in two stages. At the first stage, we obtained a modal solution of the system for modal values of its fuzzy parameters. At the second stage, the complex criterion is minimized, which considers a distance between the desired and modal solutions and the level of compactness in membership function of the desired solution.

5. Conclusions

1. We developed a procedure for calculating the stationary distribution of probabilities of states in the Markov system whose intensities of transitions are not clearly assigned. The procedure is based on the proposed technology for solving the systems of linear algebraic equations with fuzzy coefficients.

2. We described the procedure of calculating the statistical characteristics of stationary distribution of probabilities of states in the semi-Markov system, in which parameters of the laws of distribution of durations of being in states, as well as transition probabilities, are not clearly assigned.

3. A procedure for the calculation of stationary distribution of probabilities of states in a fuzzy semi-Markov system is developed. In this case, we solve an optimization problem on the minimization of complex criterion, which considers deviations in the desired solution from the modal one and the level of compactness of membership function of this solution.

References

1. Bertalanfy, L. Obshchaia teoryia system [Text] / L. Bertalanfy. – Moscow: Prohress, 1969. – 82 p.
2. Mesarovych, M.Obshchaia teoryia system: matematycheskye osnovy [Text] / M. Mesarovych, Ya. Takakhara. – Moscow: MYR, 1978. – 283 p.
3. Volkova, V. N. Teoryia system [Text] / V. N. Volkova. – Moscow: Vysshaia shkola, 2006. – 511 p.
4. Raskin, L. G. Analyz slozhnykh system i elementy teoryy upravlenyia [Text] / L. G. Raskin. – Moscow: Sov. radyo, 1976. – 344 p.
5. Raskin, L. G. Matematycheskye metody yssledovanyia operatsyi y analyza slozhnykh system vooruzhenyia PVO [Text] / L. G. Raskin. – Kharkiv: VYRTA, 1988. – 177 p.
6. Zadeh, L. A. Fuzzy sets [Text] / L. A. Zadeh // Information and Control. – 1965. – Vol. 8, Issue 3. – P. 338–353. doi: 10.1016/s0019-9958(65)90241-x
7. Kofman, A. Vvedenye v teoryiu nechetkykh mnozhestv [Text] / A. Kofman. – Moscow: Radyo y sviaz, 1982. – 486 p.
8. Nehoitse, K. Prymenenye teoryy system k problemam upravlenyia [Text] / К. Негойце. – Moscow: MYR, 1981. – 219 p.
9. Orlovskyi, S. A. Problemy pryniatyia reshenyi pry nechetkoi ynformatsyy [Text] / S. A. Orlovskyi. – Moscow: Nauka, 1981. – 264 p.
10. Zaichenko, Yu. P. Yssledovanye operatsyi. Nechetkaia optymyzatsyia [Text] / Yu. P. Zaichenko. – Kyiv: Vyshcha shkola, 1991. – 191 p.
11. Kaufman, A. Introduction to Fuzzy Arithmetic: Theory and Applications [Text] / A. Kaufman, M. Gupta. – New York, 1985. – 351 p.
12. Raskin, L. G. Nechetkaia matematyka [Text] / L. G. Raskin, O. V. Sira. – Kharkiv: Parus, 2008. – 352 p.
13. Tykhonov, V. Y. Markovskye protsessy [Text] / V. Y. Tykhonov, M. A. Myronov. – Moscow: Sov. Radyo, 1977. – 483 p.
14. Koroliuk, V. S. Polumarkovskye protsessy i ih prylozhenyia [Text] / V. S. Koroliuk, A. F. Turbyn. – Kyiv: Naukova dumka, 1976. – 182 p.
15. Koks, D. R. Teoryia vosstanovlenyia [Text] / D. R. Koks, V. D. Smyt. – Moscow: Sov. radyo, 1967. – 299 p.
16. Fakoor, M. Humanoid robot path planning with fuzzy Markov decision processes [Text] / M. Fakoor, A. Kosari, M. Jafarzadeh // Journal of Applied Research and Technology. – 2016. – Vol. 14, Issue 5. – P. 300–310. doi: 10.1016/j.jart.2016.06.006
17. Cheng, J. Finite-time H fuzzy control of nonlinear Markovian jump delayed systems with partly uncertain transition descriptions [Text] / J. Cheng, J. H. Park, Y. Liu, Z. Liu, L. Tang // Fuzzy Sets and Systems. – 2017. – Vol. 314. – P. 99–115. doi: 10.1016/j.fss.2016.06.007
18. Sira, O. V. A New Approach for Resolving Equations with Fuzzy Parameters [Text] / O. V. Sira, K. H. Al-Shqeerat // European Journal of Scientific Research. – 2009. – Vol. 38, Issue 4. – P. 619–625.
19. Raskin, L. Method of solving fuzzy problems of mathematical programming [Text] / L. Raskin, O. Sira // Eastern-European Journal of Enterprise Technologies. – 2016. – Vol. 5, Issue 4 (83). – P. 23–28. doi: 10.15587/1729-4061.2016.81292
20. Raskin, L. G. Prykladnoe kontynualnoe lyneinoe prohrammyrovanye [Text] / L. G. Raskin, Y. O. Kyrychenko, O. V. Sira. – Kharkiv, 2013. – 293 p.
21. Kostenko, Yu. T. Prohnozyrovanye tekhnycheskoho sostoianyia system upravlenyia [Text] / Yu. T. Kostenko, L. G. Raskin. – Kharkiv: Osnova, 1996. – 303 p.
22. Pawlak, Z. Rough sets [Text] / Z. Pawlak // International Journal of Computer & Information Sciences. – 1982. – Vol. 11, Issue 5. – P. 341–356. doi: 10.1007/bf01001956
23. Raskin, L. Fuzzy models of rough mathematics [Text] / L. Raskin, O. Sira // Eastern-European Journal of Enterprise Technologies. – 2016. – Vol. 6, Issue 4 (84). – P. 53–60. doi: 10.15587/1729-4061.2016.86739