УДК 550.834 DOI: https://doi.org/10.24028/gj.v45i2.278334

Properties of singular points in a special case of orthorhombic media

Yu.V. Roganov*[1], *A. Stovas*[2], *V.Yu. Roganov*[3], *2023

[1]Tesseral Technologies Inc., Kyiv, Ukraine
[2]Norwegian University of Science and Technology, Trondheim, Norway
[3]V. M. Glushkov Institute of Cybernetic of the National Academy of Sciences of Ukraine, Kyiv, Ukraine
Received 28 December 2022

The position of singular lines for orthorhombic (ORT) media with fixed diagonal elements of the elasticity matrix c_{ij}, i=1…6 is studied under the condition that c_{11}, c_{22}, $c_{33}>c_{66}>c_{44}>c_{55}$. In this case, the off-diagonal coefficients of the elasticity matrix c_{12}, c_{13}, c_{23} are chosen so that some of the values of $d_{12}=c_{12}+c_{66}$, $d_{13}=c_{13}+c_{55}$, $d_{23}=c_{23}+c_{44}$ are zero. For orthorhombic medium, where the only one of d_{12}, d_{13}, d_{23} is zero, contains only singular points in the planes of symmetry. If two or all three d_{ij} are zero, then the ORT medium contains singular lines and discrete singular points. We call such media pathological. A degenerate ORT medium with positive d_{12}, d_{13}, d_{23} has at most two singular lines, which are the intersection of a quadratic cone with a sphere. The pathological media may have up to 6 singular lines on the surface of the slowness. Singular lines for pathological media are described by more complex equations than conventional degenerate ORT models. The article proposes to using squares x, y, z of the components of the slowness vector in the equations. In a new coordinate system, equations defining singular lines for pathological media become linear or quadratic. Intersecting with the plane $x+y+z=1$, they define the straight lines, ellipses, or hyperbolas. If non-zero values d_{12}, d_{13}, d_{23} increase, the singular lines pass through four fixed points on the plane $x+y+z=1$, which makes it possible to describe the evolution of their change. Conditions are derived under which the singular curves of pathological ORT models are limiting the singular curves for degenerate ORT models with positive values of d_{12}, d_{13}, d_{23}. Formulas are derived for transforming surfaces of slowness and singular lines of pathological media into the region of group velocities. The results are demonstrated with examples of pathological models obtained from the standard model of the ORT medium by changing the elasticity coefficients c_{12}, c_{13}, c_{23} so that some of the values d_{12}, d_{13}, d_{23} are zero.

Key words: singular point, phase velocity, Christoffel matrix, orthorhombic medium.

Introduction. Singular points (axes, directions) in an anisotropic medium are the directions along which the phase velocities of plane waves of different types coincide. The relevance of studying the properties of singular points arises from the fact that their presence leads to a complication of seismic wave fields: in the vicinity of singular points, the directions of polarization vectors quickly change, and the wave amplitudes become anomalous. In this regard, there are problems with the modeling of wave fields and inversion of seismic data. Many works are devoted to studying the properties of singular points for media with different types of anisotropy [Khatkevich, 1962; Fedorov, 1968; Alshits, Lothe, 1979; Crampin, Yedlin, 1981; Alshits, Shuvalov, 1984; Musgrave, 1985; Crampin, 1991; Holm, 1992; Darinskii, 1994; Boulanger, Hayes 1998; Shuvalov, 1998; Norris, 2004;

Citation: *Roganov, Yu.V., Stovas, A., & Roganov, V.Yu.* (2023). Properties of singular points in a special case of orthorhombic media. *Geofizicheskiy Zhurnal*, *45*(2), 94—107. https://doi.org/10.24028/gj.v45i2.278334.

Vavryčuk, 2005; Roganov et al., 2019, 2022; Stovas et al., 2022, 2023]. Orthorhombic (ORT) media are of special interest for geophysics due to the fact that they are associated with a fractured medium and the use of S1- and S2-waves makes it possible to determine the direction of fractures [Zeng, MacBeth, 1993; Roganov, Roganov, 2011].

In [Alshits et al., 1985; Shuvalov, Every, 1997] shows that there are three types of singular points: conical, wedge and tangent. Singular points are usually located discretely, but there are cases when they are located on singular lines. Such media will be called degenerate. A typical case of a degenerate medium is a transversally isotropic medium. In [Stovas et al., 2023; Roganov et al., 2022] studied the structure of singular curves in degenerate ORT media with positive values of off-diagonal elements of the Christoffel matrix $d_{12}=c_{12}+c_{66}$, $d_{13}=c_{13}+c_{55}$, $d_{23}=c_{23}+c_{44}$, where c_{ij} are the density normalized stiffness coefficients. It is shown that for fixed values of the diagonal elements of the elasticity matrix c_{ij}, $i=1…6$ degenerates belong to the family of ORT models with one parameter. Singular curves in these models are given by the intersection of quadratic cones with sheets of the slowness surface. It is shown that the phase velocities along the singular curves are constant — they are the same at all points of the singular line.

ORT media that have two or three values of the d_{12}, d_{13}, d_{23} equal to zero, are degenerate with more complicated singular curves than in the case described above. If one of the d_{12}, d_{13}, d_{23} is equal to zero, the ORT model contains only discrete singular points located in the planes of symmetry [Boulanger, Hayes 1998].

In this paper, we continue the study of singular curves for ORT media with two or three zero values d_{12}, d_{13}, d_{23}. We will call these media pathological. The change of singular curves in pathological ORT models with an increase of non-zero values d_{12}, d_{13}, d_{23} are studies in the article. Conditions are derived under which the singular curves of pathological ORT media are the limiting singular curves for degenerate ORT models with positive values d_{12}, d_{13}, d_{23}. Formulas for the images of the slowness surfaces and singular lines of pathological media in the region of group velocities are derived. The results are demonstrated on examples of pathological ORT models obtained from the standard ORT model [Schoenberg, Helbig, 1997] by changing the elasticity coefficients c_{12}, c_{13}, c_{23} so that some of the values d_{12}, d_{13}, d_{23} were equal to zero.

Theory. An orthorhombic medium is described by the Christoffel matrix

$$\mathbf{A}=\left(a_{ij}\right)=\begin{pmatrix} a_{11} & d_{12}n_1n_2 & d_{13}n_1n_3 \\ d_{12}n_1n_2 & a_{22} & d_{23}n_2n_3 \\ d_{13}n_1n_3 & d_{23}n_2n_3 & a_{33} \end{pmatrix},\quad(1.1)$$

where

$a_{11}=c_{11}n_1^2+c_{66}n_2^2+c_{55}n_3^2$, $a_{22}=c_{66}n_1^2+c_{22}n_2^2+c_{44}n_3^2$, $a_{33}=c_{55}n_1^2+c_{44}n_2^2+c_{33}n_3^2$, $d_{12}=c_{12}+c_{66}$, $d_{13}=c_{13}+c_{55}$, $d_{23}=c_{23}+c_{44}$, c_{ij} — density normalized stiffness coefficients, c_i — components of direction vector, $n_1^2+n_2^2+n_3^2=1$.

Denote $\Delta_{ij}=c_{ii}-c_{jj}$, $i\neq j$. We also assume that

$$\left(c_{11},c_{22},c_{33}\right)>c_{66}>c_{44}>c_{55}>0,$$
$$d_{12}\geq 0,\quad d_{13}\geq 0,\quad d_{23}\geq 0.\quad(1.2)$$

When the inequalities (1.2) are satisfied, there are no singular points in an orthorhombic medium located on the symmetry axes.

The singular point (direction) is the value of the direction vector $\mathbf{n}=\left(n_1,n_2,n_3\right)$ or slowness vector $\mathbf{p}=\left(p_1,p_2,p_3\right)$, at which the phase velocities of S1, S2 or qP waves are equal.

Let us denote

$$q_1=c_{11}p_1^2+c_{66}p_2^2+c_{55}p_3^2-1,$$
$$q_2=c_{66}p_1^2+c_{22}p_2^2+c_{44}p_3^2-1,$$
$$q_3=c_{55}p_1^2+c_{44}p_2^2+c_{33}p_3^2-1.\quad(1.3)$$

If two or three values d_{12}, d_{13}, d_{23} are equal to zero, then the Christoffel matrix is represented by two or three blocks, respectively. We will use the splitting of the slowness surface into sheets, taking into account this block structure of the Christoffel matrix. For example, if $d_{12}=d_{13}=d_{23}=0$, then the sheets 1, 2, 3 are determined by the equations $q_1=0$, $q_2=0$, $q_3=0$.

Location of singular points in planes of symmetry. The position of the singular point in the ORT-medium in the plane of symmetry (m, k) m,k=1, 2, 3, $m<k$ is determined by the equality to zero of the three main 2×2 — minors of matrix $\mathbf{A}-v_f^2\mathbf{I}$, where v_f — phase velocity, $\mathbf{I}$ — unit 3×3 — matrix. The remaining three (off-diagonal) minors of this matrix in the symmetry planes are equal zero identically. If $d_{mk}>0$, the resulting system of equations for the plane of symmetry (m, k) reduces to a single equation [Musgrave, 1981; Roganov et al., 2022] for the variable $x=n_k^2/n_m^2$:

$$x^2+2R_{mk}x+T_{mk}=0\,, \qquad (1.4)$$

where

$$R_{12}=\frac{-d_{12}^2+\Delta_{15}\Delta_{24}+\Delta_{65}\Delta_{64}}{2\Delta_{64}\Delta_{24}}\,,\; T_{12}=\frac{\Delta_{15}\Delta_{65}}{\Delta_{64}\Delta_{24}}\,,$$
$$R_{13}=\frac{d_{13}^2-\Delta_{16}\Delta_{34}-\Delta_{45}\Delta_{65}}{2\Delta_{45}\Delta_{34}}\,,\; T_{13}=\frac{\Delta_{16}\Delta_{65}}{\Delta_{45}\Delta_{34}}\,, \qquad (1.5)$$
$$R_{23}=\frac{-d_{23}^2+\Delta_{26}\Delta_{35}-\Delta_{45}\Delta_{64}}{2\Delta_{45}\Delta_{35}}\,,\; T_{23}=-\frac{\Delta_{26}\Delta_{64}}{\Delta_{45}\Delta_{35}}\,.$$

Therefore, the position of singular points in the plane of symmetry (m, k) at $d_{mk}>0$ determined by the roots of the quadratic equation (1.4), and the number of singular points in each plane of symmetry in the first octant does not exceed 2.

If $d_{mk}=0$, in the plane of symmetry (m, k) Christoffel matrix (1.1) is diagonal and direction vectors of singular points in this plane satisfy the equation $(a_{11}-a_{22})(a_{11}-a_{33})(a_{22}-a_{33})=0$. When $d_{12}=0$ there is one singular point in the plane of symmetry 12 in the first octant:

$$\frac{n_2^2}{n_1^2}=\frac{\Delta_{16}}{\Delta_{26}}\,. \qquad (1.6)$$

If $d_{13}=0$, there are 3 singular points in the plane of symmetry 13 in the first octant:

$$\left(\frac{n_3^2}{n_1^2}\right)^{(1)}=\frac{\Delta_{65}}{\Delta_{34}}\,,\; \left(\frac{n_3^2}{n_1^2}\right)^{(2)}=\frac{\Delta_{16}}{\Delta_{45}}\,,$$
$$\left(\frac{n_3^2}{n_1^2}\right)^{(3)}=\frac{\Delta_{15}}{\Delta_{35}}\,, \qquad (1.7)$$

and if $d_{23}=0$, there are 2 singular points in the plane of symmetry 23 in the first octant:

$$\left(\frac{n_3^2}{n_2^2}\right)^{(1)}=\frac{\Delta_{64}}{\Delta_{35}}\,,\; \left(\frac{n_3^2}{n_2^2}\right)^{(2)}=\frac{\Delta_{24}}{\Delta_{34}}\,. \qquad (1.8)$$

With different combinations of zeros and positive numbers among d_{12}, d_{13}, d_{23}, formulas (1.4)—(1.8) determine all singular points in the planes of symmetry of the ORT model. There are no singular lines inside the planes of symmetry. However, a singular point in a plane of symmetry may belong to a spatial singular line that intersects this plane of symmetry.

Location of singular points in-between the planes of symmetry. We will assume that in the singular point $n_1n_2n_3\neq0$, i.e., this point is in-between the planes of symmetry of the ORT medium. Equality to zero of three off-diagonal matrix minors $\mathbf{A}-v_f^2\mathbf{I}$ leads to the system of equations

$$\begin{aligned}&\left(a_{33}-v_f^2\right)d_{12}=d_{13}d_{23}n_3^2\,,\\ &\left(a_{22}-v_f^2\right)d_{13}=d_{12}d_{23}n_2^2\,,\\ &\left(a_{11}-v_f^2\right)d_{23}=d_{12}d_{13}n_1^2\,.\end{aligned} \qquad (1.9)$$

If $d_{12}>0$, $d_{13}>0$, $d_{23}>0$, from the system of equations (1.9) it follows the vanishing of another three (diagonal) minors of the matrix $\mathbf{A}-v_f^2\mathbf{I}$:

$$\begin{aligned}&\left(a_{11}-v_f^2\right)\left(a_{22}-v_f^2\right)=d_{12}^2n_1^2n_2^2\,,\\ &\left(a_{11}-v_f^2\right)\left(a_{33}-v_f^2\right)=d_{13}^2n_1^2n_3^2\,,\\ &\left(a_{22}-v_f^2\right)\left(a_{33}-v_f^2\right)=d_{23}^2n_2^2n_3^2\,.\end{aligned} \qquad (1.10)$$

Therefore, in this case, all 2×2 minors of matrix $\mathbf{A}-v_f^2\mathbf{I}$ are equal to zero and the roots of the system of equations (1.9) determine the position of singular points or singular lines in-between the planes of symmetry [Roganov et al., 2019].

Consider the ORT model for which some values d_{ij}, $i<j$ are equal to zero.

Let, $d_{12}=d_{13}=d_{23}=0$. In this case, the Christoffel equation is given by

$$\left(v_f^2-a_{11}\right)\left(v_f^2-a_{22}\right)\left(v_f^2-a_{33}\right)=0\,. \qquad (1.11)$$

The equation (1.11) has a multiple root at a point (n_1, n_2, n_3), if at this point is true the equality

$$u_{12}u_{13}u_{23}=0\,, \qquad (1.12)$$

where

$$u_{12} = a_{11} - a_{22} = \Delta_{16} n_1^2 - \Delta_{26} n_2^2 - \Delta_{45} n_3^2 ,$$
$$u_{13} = a_{11} - a_{33} = \Delta_{15} n_1^2 + \Delta_{64} n_2^2 - \Delta_{35} n_3^2 , \quad (1.13)$$
$$u_{23} = a_{22} - a_{33} = \Delta_{65} n_1^2 + \Delta_{24} n_2^2 - \Delta_{34} n_3^2 .$$

Therefore, when $d_{12}=d_{13}=d_{23}=0$ there are three singular lines satisfying, respectively, the equations $u_{12}=0$, $u_{13}=0$, $u_{23}=0$. These three singular lines coincide if the expressions u_{12} and u_{13} proportional. For this it is necessary and sufficient that the relations

$$t_1 = \Delta_{26}\Delta_{35} + \Delta_{45}\Delta_{64} = 0 ,$$
$$t_2 = \Delta_{16}\Delta_{34} - \Delta_{45}\Delta_{65} = 0 , \quad (1.14)$$
$$t_3 = \Delta_{15}\Delta_{24} - \Delta_{64}\Delta_{65} = 0 ,$$

are satisfied. These relations are a consequence of the equality to zero 2×2 — minors of the matrix composed of the coefficients in the expressions u_{12} and u_{13}.

When the inequalities $t_i > 0,\ i = 1,2,3$ are satisfied, three singular lines (1.13) are different and intersect at a point with coordinates $n_1^2 : n_2^2 : n_3^2 = t_1 : t_2 : t_3$. This is the triple singular point at which all three sheets of the slowness surface intersect. Variables t_1, t_2, t_3 are linearly dependent, so that if two of them are equal to zero, then the third variable is also equal to zero.

Let among the values d_{12}, d_{13}, d_{23} only one d_{ij} is equal to zero. Such a case in the article [Boulanger, Hayes, 1998] is called pathological. If a $d_{12}=0$, $d_{13}\neq 0$, $d_{23}\neq 0$, then from the first equation (1.9) follows that $n_3=0$. Likewise, from $d_{13}=0$, $d_{12}\neq 0$, $d_{23}\neq 0$ and the second equation (1.9) follows that $n_2=0$ and finally from $d_{23}=0$, $d_{12}\neq 0$, $d_{13}\neq 0$ and the third equation (1.9) follows that $n_1=0$. So, if only one value d_{ij} is equal to zero, there are no singular points and lines in-beween the planes of symmetry.

Let us consider ORT model with two values $d_{ij}=0$, and the third value $d_{mk}>0$.

Let, $d_{12}>0$, $d_{13}=d_{23}=0$. In this case, the Christoffel equation is given by

$$F = \left(a_{33} - v_f^2\right)\left[\left(a_{11} - v_f^2\right)\times\right.$$
$$\left.\times\left(a_{22} - v_f^2\right) - d_{12}^2 n_1^2 n_2^2\right] = 0 . \quad (1.15)$$

This equation has multiple roots when

$$u_{13}u_{23} - d_{12}^2 n_1^2 n_2^2 = 0 , \quad (1.16)$$

or when the discriminant D of quadratic equation $\left(v_f^2 - a_{11}\right)\left(v_f^2 - a_{22}\right) - d_{12}^2 n_1^2 n_2^2 = 0$ relatively to v_f^2 is equal zero. If $n_1 n_2 \neq 0$ and $d_{12}>0$, then $D = \left(a_{11} - a_{22}\right)^2 + 4 d_{12}^2 n_1^2 n_2^2 > 0$ and all singular points, located in-beween the planes of symmetry or in the plane of symmetry 12, are determined by the roots of the equation (1.16).

Similarly, when $d_{12}=0$, $d_{13}>0$, $d_{23}=0$ or $d_{12}=d_{13}=0$, $d_{23}>0$ singular points in-between the planes of symmetry or in the corresponding plane of symmetry 13, 23, are determined, respectively, by the root of equations

$$u_{12}u_{23} + d_{13}^2 n_1^2 n_3^2 = 0 , \quad (1.17)$$
$$u_{12}u_{13} - d_{23}^2 n_2^2 n_3^2 = 0 . \quad (1.18)$$

The equation (1.16) in symmetry planes 13 and 23 is equivalent to the equation $u_{13}u_{23}=0$. Therefore, the singular curves given by the equation (1.16), intersect symmetry planes 13 and 23 at points independent of the value d_{12}. The coordinates of these intersection points also satisfy the equation (1.12). Similar statements are also valid for the equations (1.17) and (1.18). They define singular lines that intersect the planes of symmetry 12 and 23, and, respectively, 12 and 13, at points that do not depend on the values d_{13} and d_{23}.

Let us define the conditions when the singular curves (1.13) are the limit of singular curves of degenerate ORT media with $d_{12}>0$, $d_{13}>0$, $d_{23}>0$ [Roganov et al., 2022].

For the ORT medium to be degenerate and contain a singular curve at $d_{12}>0$, $d_{13}>0$, $d_{23}>0$ it is necessary and sufficient to satisfy the following relations [Roganov et al., 2022]:

$$f_{12} = \Delta_{35} + \Delta_{64} + \frac{\Delta_{64}\Delta_{65}}{v_f^2 - c_{66}} ,$$
$$f_{13} = \Delta_{24} - \Delta_{65} + \frac{\Delta_{45}\Delta_{65}}{v_f^2 - c_{55}} , \quad (1.19)$$
$$f_{23} = \Delta_{16} + \Delta_{45} - \frac{\Delta_{45}\Delta_{64}}{v_f^2 - c_{44}} ,$$

where

$$d_{12}^2 = f_{13} f_{23} ,\ d_{13}^2 = f_{12} f_{23} ,\ d_{23}^2 = f_{12} f_{13} , \quad (1.20)$$

and v_f is the phase velocity on a singular curve, which must belong to one of the intervals (c_{55}, c_{44}) or (c_{44}, c_{66}).

From formulas (1.20), it follows that the limiting cases

$$d_{12}>0,\ d_{13}=0,\ d_{23}=0,$$
$$d_{12}=0,\ d_{13}>0,\ d_{23}=0,$$
$$d_{12}=0,\ d_{13}=0,\ d_{23}>0$$

are equivalent to conditions

$$f_{12}=0, f_{13}>0, f_{23}>0,$$
$$f_{12}>0, f_{13}=0, f_{23}>0$$

and

$$f_{12}>0, f_{13}>0, f_{23}=0,$$

respectively. A similar statement was obtained in [Boulanger, Hayes 1998].

Consider the cases

$$d_{12}>0,\ d_{13}\to+0,\ d_{23}\to+0,$$
$$d_{12}\to+0,\ d_{13}>0,\ d_{23}\to+0,$$
$$d_{12}\to+0,\ d_{13}\to+0,\ d_{23}>0. \qquad (1.21)$$

Substituting values $f_{12}=0, f_{13}=0, f_{23}=0$ in the first, second and third formulas (1.19), we derive for the corresponding limiting values of v_f^2:

$$v_{f,12}^2=\frac{c_{33}c_{66}-c_{44}c_{55}}{\Delta_{34}+\Delta_{65}},$$
$$v_{f,13}^2=\frac{c_{22}c_{55}-c_{44}c_{66}}{\Delta_{24}-\Delta_{65}},$$
$$v_{f,23}^2=\frac{c_{11}c_{44}-c_{55}c_{66}}{\Delta_{15}-\Delta_{64}}. \qquad (1.22)$$

Then, substituting each v_f^2 from equation (1.22) into the other two formulas (1.19) and using the formulas (1.20), we obtain for the cases (1.21) positive (not tending to zero) values of d_{12}, d_{13}, d_{23}:

$$d_{12}^2=\frac{t_1t_2}{\Delta_{34}\Delta_{35}},\ d_{13}^2=\frac{t_1t_3}{\Delta_{24}\Delta_{26}},$$
$$d_{23}^2=\frac{t_2t_3}{\Delta_{15}\Delta_{16}}. \qquad (1.23)$$

Singular line of a degenerate ORT model at $d_{12}>0$, $d_{13}>0$, $d_{23}>0$ satisfies the equation [Roganov et al., 2022]:

$$\frac{n_1^2}{v_f^2-c_{44}}+\frac{n_2^2}{v_f^2-c_{55}}+\frac{n_3^2}{v_f^2-c_{66}}=0. \qquad (1.24)$$

Substituting the values of the phase velocity v_f^2 from the relations (1.22) in (1.24), we obtain the equations of singular curves for the cases (1.21):

$$s_{12}=\Delta_{65}\Delta_{35}n_1^2+\Delta_{64}\Delta_{34}n_2^2-\Delta_{35}\Delta_{34}n_3^2=0,$$
$$s_{13}=\Delta_{65}\Delta_{26}n_1^2+\Delta_{26}\Delta_{24}n_2^2+\Delta_{24}\Delta_{45}n_3^2=0, \qquad (1.25)$$
$$s_{23}=\Delta_{15}\Delta_{16}n_1^2+\Delta_{16}\Delta_{64}n_2^2-\Delta_{15}\Delta_{45}n_3^2=0.$$

Note that if the inequalities (1.2) are satisfied, the equation $s_{13}=0$ has no real roots. Equations $s_{12}=0$, $s_{23}=0$ define the curves with projections onto the plane 12 being ellipses.

When the relations (1.23) are satisfied, formulas (1.16)—(1.18) describing the singular lines for the cases

$$d_{12}>0,\ d_{13}=0,\ d_{23}=0,$$
$$d_{12}=0,\ d_{13}>0,\ d_{23}=0,$$
$$d_{12}=0,\ d_{13}=0,\ d_{23}>0,$$

respectively, can be given by

$$u_{13}u_{23}-d_{12}^2n_1^2n_2^2=\frac{s_{12}g_{12}}{\Delta_{34}\Delta_{35}}=0,$$
$$-u_{12}u_{23}-d_{13}^2n_1^2n_3^2=\frac{s_{13}g_{13}}{\Delta_{24}\Delta_{26}}=0, \qquad (1.26)$$
$$u_{12}u_{13}-d_{23}^2n_2^2n_3^2=\frac{s_{23}g_{23}}{\Delta_{15}\Delta_{16}}=0,$$

where

$$g_{12}=\Delta_{15}\Delta_{34}n_1^2+\Delta_{24}\Delta_{35}n_2^2-\Delta_{34}\Delta_{35}n_3^2,$$
$$g_{13}=-\Delta_{24}\Delta_{16}n_1^2+\Delta_{26}\Delta_{24}n_2^2-\Delta_{26}\Delta_{34}n_3^2, \qquad (1.27)$$
$$g_{23}=\Delta_{15}\Delta_{16}n_1^2-\Delta_{15}\Delta_{26}n_2^2-\Delta_{16}\Delta_{35}n_3^2.$$

Therefore, every singular line consists of two components, one of which is the limiting singular line $s_{ij}=0$ with $d_{ij}=0$. On the limiting singular lines $s_{12}=0$, $s_{13}=0$, $s_{23}=0$ (if they exist) the phase velocities are constant and determined by the formulas (1.22). On singular lines $g_{12}=0$, $g_{13}=0$, $g_{23}=0$, the phase velocities may be different for different points on the curve.

From the relations (1.20), it follows that the degenerate case $d_{12}=0$, $d_{13}=0$, $d_{23}=0$ is the limit of case $d_{12}\to+0$, $d_{13}\to+0$, $d_{23}\to+0$ if and only if two or all three values of f_{12}, f_{13}, f_{23} are zero. From (1.22), it follows that the values $v_{f,12}^2$, $v_{f,13}^2$, $v_{f,23}^2$, corresponding to $f_{ij}=0$, should be the same. Since,

$$v_{f,12}^2 - v_{f,13}^2 = \frac{t_1\Delta_{65}}{(\Delta_{35}+\Delta_{64})(\Delta_{24}-\Delta_{65})},$$

$$v_{f,12}^2 - v_{f,23}^2 = \frac{t_2\Delta_{64}}{(\Delta_{35}+\Delta_{64})(\Delta_{15}-\Delta_{64})}, \quad (1.28)$$

$$v_{f,23}^2 - v_{f,13}^2 = \frac{t_3\Delta_{45}}{(\Delta_{24}-\Delta_{65})(\Delta_{15}-\Delta_{64})}$$

then the limit $d_{12}\to+0$, $d_{13}\to+0$, $d_{23}\to+0$ exists in two cases: when only one of the values t_1, t_2, t_3 equal to zero, or when $t_1=t_2=t_3=0$.

If $t_1=0$ then $f_{12}=f_{13}=0$ and $f_{23}=\frac{t_2}{\Delta_{34}}=\frac{t_3}{\Delta_{24}}$. In this case $u_{23}=\frac{s_{12}}{\Delta_{35}}=-\frac{s_{13}\Delta_{35}}{\Delta_{64}\Delta_{45}}$ and singular line $u_3=0$ coincides with the limiting singular line for velocity $v_{f,12}^2=v_{f,13}^2$ from (1.22).

Similarly, if $t_2=0$, then $f_{12}=f_{23}=0$ and $f_{13}=\frac{t_1}{\Delta_{35}}=\frac{t_3\Delta_{34}}{\Delta_{35}\Delta_{65}}$. In this case, the singular line $u_{23}=0$ coincides with the limiting singular line for velocity $v_{f,12}^2=v_{f,23}^2$.

Finally, if $t_3=0$, then $f_{13}=f_{23}=0$ and $f_{12}=\frac{t_1}{\Delta_{26}}=-\frac{t_2\Delta_{24}}{\Delta_{26}\Delta_{65}}$. In this case, the singular line $u_{12}=0$ coincides with the limiting singular line for velocity $v_{f,13}^2=v_{f,23}^2$.

If $t_1=t_2=t_3=0$, then $f_{12}=f_{13}=f_{23}=0$. In this case $v_f^2=v_{f,12}^2=v_{f,13}^2=v_{f,23}^2$ and all equations $u_{ij}=0$ define the same singular line (triple), which is the intersection of the surfaces $a_{ii}=v_f^2$, $i=1, 2, 3$ This line coincides with the limiting singular line at all $d_{ij}\to+0$.

Note that for any pathological ORT model and a point on its singular line, there exists a non-degenerate ORT model with a singular point arbitrarily close to the chosen point of the pathological medium. To find parameters d_{ij} for this medium enough to solve system of equations (1.9) for d_{ij} at given singular direction (n_1, n_2, n_3).

Equations (1.12), (1.16)—(1.18), (1.24)—(1.26) are homogeneous with respect to the variables n_1, n_2, n_3. Therefore, they remain valid when these variables are replaced by components p_1, p_2, p_3 of slowness vector.

Location of singular lines in the region of group velocities at $d_{12}=d_{13}=d_{23}=0$. When $d_{12}=d_{13}=d_{23}=0$, there are three singular lines that satisfy the equations $u_{12}=0$, $u_{13}=0$, $u_{23}=0$ (1.12). They are intersections of slowness surfaces satisfying the equations $q_1=0$, $q_2=0$, $q_3=0$ respectively. The slowness surfaces are ellipsoids, and their images in the region of group velocities are also ellipsoids. Let us show this by the example of a surface satisfying the equation $q_1=0$. Since, $\nabla q_1 = 2(c_{11}p_1, c_{66}p_2, c_{55}p_3)$ and $\nabla q_1\cdot\mathbf{p}=2$, then the components of the group velocity vector for the first slowness surface are found by the formula $\mathbf{v}^{(1)}=(v_1,v_2,v_3)=\nabla q_1/(\nabla q_1\cdot\mathbf{p})=(c_{11}p_1, c_{66}p_2, c_{55}p_3)$. Substituting values $p_1=v_1/c_{11}$, $p_2=v_2/c_{66}$, $p_3=v_3/c_{55}$ in the expression $q_1=0$, we obtain the equation for the image of the first surface in the region of group velocities

$$\frac{v_1^2}{c_{11}}+\frac{v_2^2}{c_{66}}+\frac{v_3^2}{c_{55}}=1. \quad (1.29)$$

Substituting these values p_1, p_2, p_3 in the expressions $u_{12}=0$, $u_{13}=0$ for singular lines, which are, respectively, the intersection of slowness surfaces 1 and 2, as well as — 1 and 3, we obtain the equations for the images of these singular lines on the surface (1.29):

$$\frac{\Delta_{16}v_1^2}{c_{11}^2}-\frac{\Delta_{26}v_2^2}{c_{66}^2}-\frac{\Delta_{45}v_3^2}{c_{55}^2}=0,$$

$$\frac{\Delta_{15}v_1^2}{c_{11}^2}+\frac{\Delta_{64}v_2^2}{c_{66}^2}-\frac{\Delta_{35}v_3^2}{c_{55}^2}=0. \quad (1.30)$$

Similarly, the images in the region of group velocities of the second and third slowness surfaces $q_2=0$, $q_3=0$ are given by parametric relations $p_1=v_1/c_{66}$, $p_2=v_2/c_{22}$, $p_3=v_3/c_{44}$ and $p_1=v_1/c_{55}$, $p_2=v_2/c_{44}$, $p_3=v_3/c_{33}$, respectively. Consequently, the images of slowness surfaces 2 and 3 in the region of group velocities satisfy, respectively, the following equations

$$\frac{v_1^2}{c_{66}}+\frac{v_2^2}{c_{22}}+\frac{v_3^2}{c_{44}}=1. \quad (1.31)$$

$$\frac{v_1^2}{c_{55}}+\frac{v_2^2}{c_{44}}+\frac{v_3^2}{c_{33}}=1. \quad (1.32)$$

Equations of image of singular lines $u_{12}=0$, $u_{23}=0$, lying in the region of group velocities on the surface (1.31) are given by the form:

$$\frac{\Delta_{16}v_1^2}{c_{66}^2}-\frac{\Delta_{26}v_2^2}{c_{22}^2}-\frac{\Delta_{45}v_3^2}{c_{44}^2}=0\,,$$

$$\frac{\Delta_{65}v_1^2}{c_{66}^2}+\frac{\Delta_{24}v_2^2}{c_{22}^2}-\frac{\Delta_{34}v_3^2}{c_{44}^2}=0\,. \qquad (1.33)$$

Finally, the equations for the images of singular lines u_{13}=0, u_{23}=0, lying in the region of group velocities on the surface (1.32) look like:

$$\frac{\Delta_{15}v_1^2}{c_{55}^2}+\frac{\Delta_{64}v_2^2}{c_{44}^2}-\frac{\Delta_{35}v_3^2}{c_{33}^2}=0\,,$$

$$\frac{\Delta_{65}v_1^2}{c_{55}^2}+\frac{\Delta_{24}v_2^2}{c_{44}^2}-\frac{\Delta_{34}v_3^2}{c_{33}^2}=0\,. \qquad (1.34)$$

Note that each singular point $\mathbf{p}=(p_1, p_2, p_3)$ from the phase region located on the line of intersection of two slowness surfaces, there correspond two points $\mathbf{v}=(v_1, v_2, v_3)$ located on different group velocities surfaces. For these points, the equality, $p_1v_1+p_2v_2+p_3v_3=1$, define a plane with direction vector $\mathbf{p}$ in space with coordinates v_1, v_2, v_3. This plane is tangent to the images of the corresponding slowness surfaces at the two points under consideration $\mathbf{v}$. The image in the region of group velocities of the point of intersection of the three slowness surfaces is the three points of contact of the common tangent plane with the images of the three slowness surfaces.

Location of singular lines in the region of group velocities at $\mathbf{d}_{12}>0$, $\mathbf{d}_{13}=\mathbf{d}_{23}=0$. Among pathological ORT media with $d_{ij}>0$ let's look at one example where $d_{12}>0$, $d_{13}=d_{23}=0$. For this medium, the slowness surface is described with the equation $\left(q_1q_2-d_{12}^2p_1^2p_2^2\right)q_3=0$ and consists of three sheets. The first two of these sheets satisfy the equation $k_{12}=q_1q_2-d_{12}^2p_1^2p_2^2=0$, and the third sheet satisfies the equation q_3=0. Earlier it was shown that the image in the region of group velocities of the third sheet satisfies the relation (1.32). Using parametric relations $p_1=v_1/c_{55}$, $p_2=v_2/c_{44}$, $p_3=v_3/c_{33}$ and equation (1.16), we obtain a relation for the image of singular lines in the region of group velocities located on the surface (1.32)

$$\left(\frac{\Delta_{15}v_1^2}{c_{55}^2}+\frac{\Delta_{64}v_2^2}{c_{44}^2}-\frac{\Delta_{35}v_3^2}{c_{33}^2}\right)\times$$

$$\times\left(\frac{\Delta_{65}v_1^2}{c_{55}^2}+\frac{\Delta_{24}v_2^2}{c_{44}^2}-\frac{\Delta_{34}v_3^2}{c_{33}^2}\right)-$$

$$-\frac{d_{12}^2v_1^2v_2^2}{c_{44}^2c_{55}^2}=0\,. \qquad (1.35)$$

Image of surface k_{12}=0 in the group velocities region is determined by a high degree equation and is not given in the article. However, one can obtain simple parametric formulas for the image of this surface using the formula

$$v_i=\frac{\partial k_{12}}{\partial p_i}\Bigg/\left(p_1\frac{\partial k_{12}}{\partial p_1}+p_2\frac{\partial k_{12}}{\partial p_2}+p_3\frac{\partial k_{12}}{\partial p_3}\right):$$

$$v_1=p_1\left(c_{11}q_2+c_{66}q_1-d_{12}^2p_2^2\right)/\left(q_1+q_2\right),$$

$$v_2=p_2\left(c_{66}q_2+c_{22}q_1-d_{12}^2p_1^2\right)/\left(q_1+q_2\right),$$

$$v_3=p_3\left(c_{55}q_2+c_{44}q_1\right)/\left(q_1+q_2\right). \qquad (1.36)$$

Similar results are valid for the cases $d_{13}>0$, $d_{12}=d_{23}=0$ and $d_{23}>0$, $d_{12}=d_{13}=0$.

Numerical examples. In the numerical examples and in the figures, the density normalized elasticity coefficients will be represented in the dimension km^2/s^2, and the velocities are in the dimension km/s.

To illustrate the above statements, we consider ORT media, in which the diagonal elements of the elasticity matrix coincide with the elements of the standard orthorhombic medium [Schoenberg, Helbig, 1997]

$$c_{11}=9.0,\ c_{22}=9.84,\ c_{33}=5.9375,$$

$$c_{44}=2.0,\ c_{55}=1.6,\ c_{66}=2.182. \qquad (1.37)$$

These elements satisfy the inequalities (1.2). The remaining elements are given by the equalities $c_{12}=d_{12}-c_{66}$, $c_{13}=d_{13}-c_{55}$, $c_{23}=d_{23}-c_{44}$, with different $d_{12}\geq0$, $d_{13}\geq0$, $d_{23}\geq0$. The Christoffel matrix for such a medium is determined by the formula (1.1).

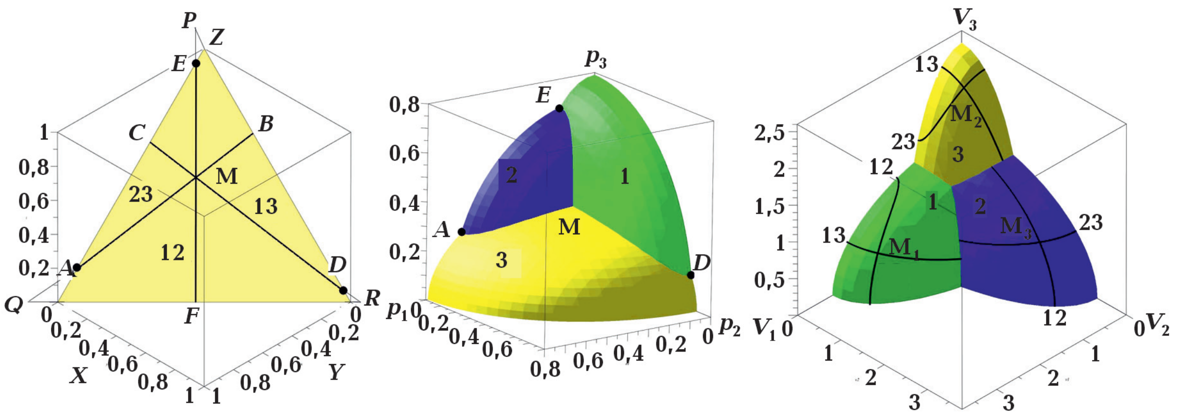
Equations (1.16)—(1.18) are biquadratic with respect to the variables n_1, n_2, n_3. Together with the equation $n_1^2+n_2^2+n_3^2=1$, they determine the position of the singular lines in pathological media with one non-zero value d_{ij}. To understand what the corresponding singular lines look like, we perform the change of variables $x=n_1^2$, $y=n_2^2$, $z=n_3^2$. With new variables, the equations (1.16)—(1.18) define quadratic cones that intersect the plane $x+y+z=1$ along ellipses, straight lines or hyperbolas. The parts of these curves that are in the region $0\leq x, y, z\leq 1$, determine the position of singular curves in the first octant. If all $d_{ij}=0$, then the sheets of the slowness surface (1.12) in coordinates (x, y, z) are planes that intersect along singular straight lines. Position of singular curves in coordinates (x, y, z) on plane $x+y+z=1$ we annotate as a **diagram**.

Fig. 1, *a* shows a diagram for a pathological ORT medium with elasticity coefficients (1.37) and parameters $d_{12}=d_{13}=d_{23}=0$. Fig. 1, *b* shows sheets 1, 2, 3 of the slowness surface of this medium in the first octant, given by the equations $q_1=0$, $q_2=0$, $q_3=0$. Sides *xz*, *yz*, *xy* of the triangle of the diagram in Fig. 1, *a* correspond the planes of symmetry 13, 23, 12 in the coordinates (p_1, p_2, p_3) of the slowness vector in Fig. 1, *b*. The total number of closed singular lines for this medium is six.

The images of the sheets of slowness surface in the region of group velocities are ellipsoids that satisfy the formulas (1.29), (1.31), (1.32). These ellipsoids, together with the images of singular lines, are shown in Fig. 1, *c*. The image of a triple singular point M are three points M_1, M_2, M_3, in which the images of the singular lines intersect. These points are the tangent points of the common tangent plane to the images of the three slowness surfaces 1, 2, 3. Because, at $d_{12}=d_{13}=d_{23}=0$ Christoffel matrix diagonal, its eigenvalues coincide with the diagonal elements, and the eigenvectors are located along the coordinate axes, regardless of the direction vector. The sheets of the slowness surface in Fig. 1, *b* are indicated by numbers 1, 2, 3. The lines of their intersection, in Fig. 1, *a* are indicated by pairs of this numbers. Capital letters indicate the points of intersection of singular lines in the coordinates (x, y, z) with the sides of the triangle or their extensions. From equalities (1.6)—(1.8) the values of the ratios of the coordinates of these points follow:

$$\frac{y_Q}{x_Q}=-\frac{\Delta_{65}}{\Delta_{24}}=-0.074\,,\quad \frac{y_F}{x_F}=\frac{\Delta_{16}}{\Delta_{26}}=0.89\,,$$
$$\frac{y_R}{x_R}=-\frac{\Delta_{15}}{\Delta_{64}}=-40.659\,,$$
$$\frac{z_A}{x_A}=\frac{\Delta_{65}}{\Delta_{34}}=0.148\,,\quad \frac{z_C}{x_C}=\frac{\Delta_{15}}{\Delta_{35}}=1.706\,,$$
$$\frac{z_E}{x_E}=\frac{\Delta_{16}}{\Delta_{45}}=17.045\,, \qquad (1.38)$$
$$\frac{z_D}{y_D}=\frac{\Delta_{64}}{\Delta_{35}}=0.042\,,\quad \frac{z_B}{y_B}=\frac{\Delta_{24}}{\Delta_{34}}=1.991\,,$$
$$\frac{z_P}{y_P}=-\frac{\Delta_{26}}{\Delta_{45}}=-19.145\,.$$

In Fig. 2 are shown sections of the slowness surface by symmetry planes 12, 13, and 23 for the case $d_{12}=d_{13}=d_{23}=0$. Letters A—F denote singular points in the symmetry planes. Fig. 2, *b* shows that all three sheets



a *b* *c*

Fig. 1. ORT media with $d_{12}=d_{13}=d_{23}=0$: diagram (*a*); three sheets of the slowness surface in the first octant (*b*); images in the group velocities region of slowness surfaces and singular lines (*c*).

of the slowness surface intersect in the plane of symmetry 13. Note that for an ORT media with positive values d_{ij} a maximum of two such intersections are possible, since the number of intersections is determined by the number of roots of the quadratic equation (1.4).

In Fig. 3—5 are shown diagrams and singular lines on the corresponding quadratic surfaces q_1=0, q_2=0, q_3=0 for pathological ORT media with a non-zero value of one of the parameters d_{12}, d_{13}, d_{23}.

In Fig. 3 are shown diagrams and slow-

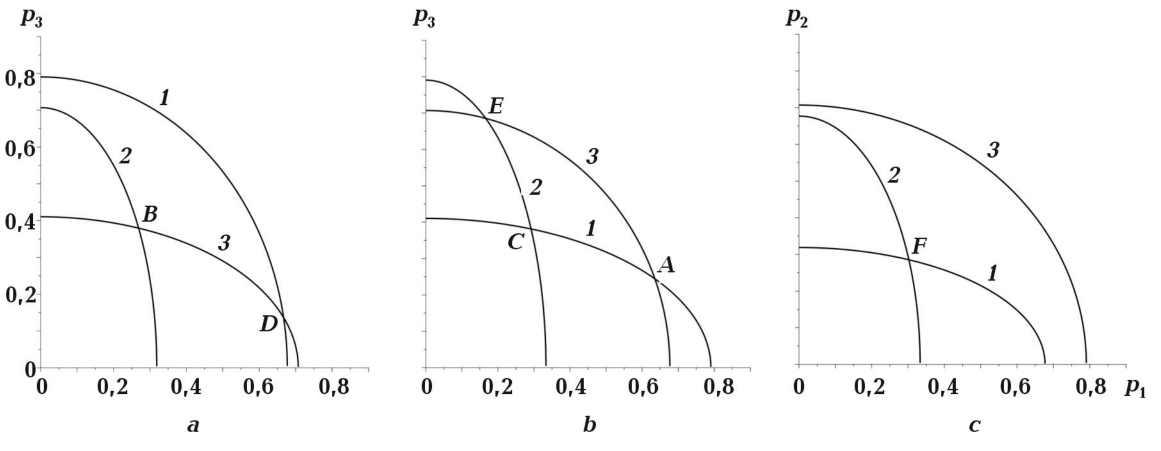


Fig. 2. Sections of the slowness surface at d_{12}=d_{13}=d_{23}=0 in different planes of symmetry: p_1=0 (*a*), p_2=0 (*b*), p_3=0 (*c*).

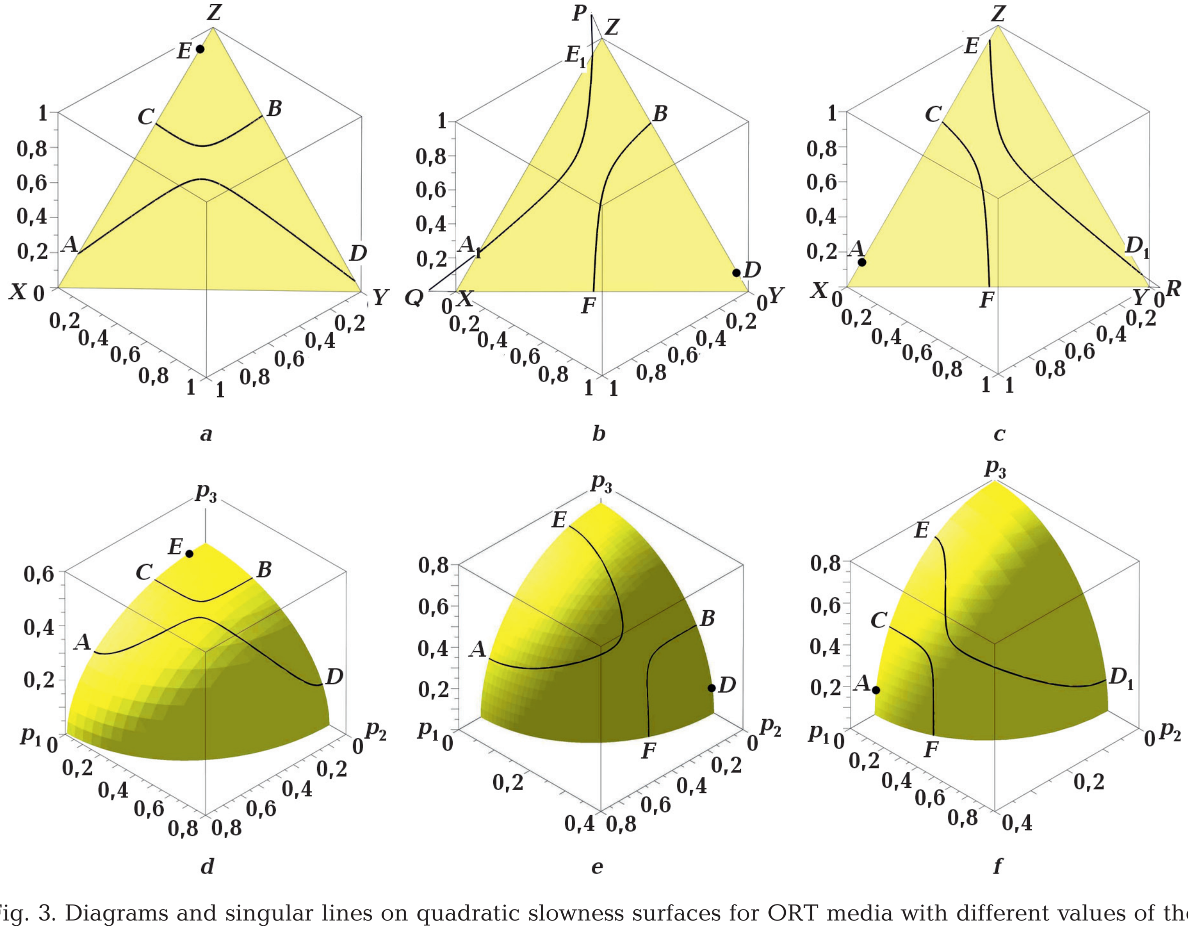


Fig. 3. Diagrams and singular lines on quadratic slowness surfaces for ORT media with different values of the parameters d_{ij}: d_{12}=2, d_{13}=d_{23}=0 (a, d); d_{13}=2, d_{12}=d_{23}=0 (b, e); d_{23}=2, d_{12}=d_{13}=0 (c, f).

ness surfaces for one non-zero d_{ij} with value 2. Comparing Fig. 3, *a*—*c* with Fig. 1, *a*, we see how the positions of singular lines on the diagram changes under perturbation of one parameter d_{ij}. In this case straight line *ij* on the diagram, formed by the intersection of sheets *i* and *j*, disappears and the other two straight lines are deformed into hyperbolas. One end of the line *ij*, belonging to the plane of symmetry (*ij*), disappears, since it is controlled by one of the equations (1.16)—(1.18). The second end of this line is in a plane of symmetry different from (*ij*). This point is preserved and becomes a discrete singular point. Discrete singular points E, D, A in Fig. 3—5 are marked with black circles. If value of one of nonzero parameters d_{12}, d_{13}, d_{23} change, the position of the four points, through which the singular curves pass, are preserved: *A*, *B*, *C*, *D* — for $d_{12}>0$, *Q*, *P*, *F*, *B* — for $d_{13}>0$ and *F*, *C*, *R*, *E* — for $d_{23}>0$. Points *A*, *B*, *C*, *D* do not belong to the side *xy* of the triangle, *Q*, *P*, *F*, *B* do not belong to the side *xz* and *F*, *C*, *R*, *E* do not belong to the side *yz*. When we pass from coordinates (*x*, *y*, *z*) to the coordinates of the slowness vector (p_1, p_2, p_3), we obtain singular lines, shown in Fig. 3, *d*—*f*. Singular lines in these figures are placed on the corresponding quadratic slowness surfaces. Note that complex value of the slowness vector corresponds to the point located on the extension of the side of the triangle. These points are needed to study the dependence of the position of singular curves with change of nonzero d_{ij}.

With an increase of non-zero parameters d_{ij} hyperbolas in diagrams change positions with the fixed points of intersection listed above. When the value of one of the parameters d_{12}=7.202, or d_{13}=5.666, or d_{23}=5.527, and zero values of the rest d_{ij}, the hyperbolas in the diagrams are straight lines (Fig. 4, *a*—*c*). These are the cases of pathological ORT media when their singular curves are the limit of singular curves of degenerate ORT media

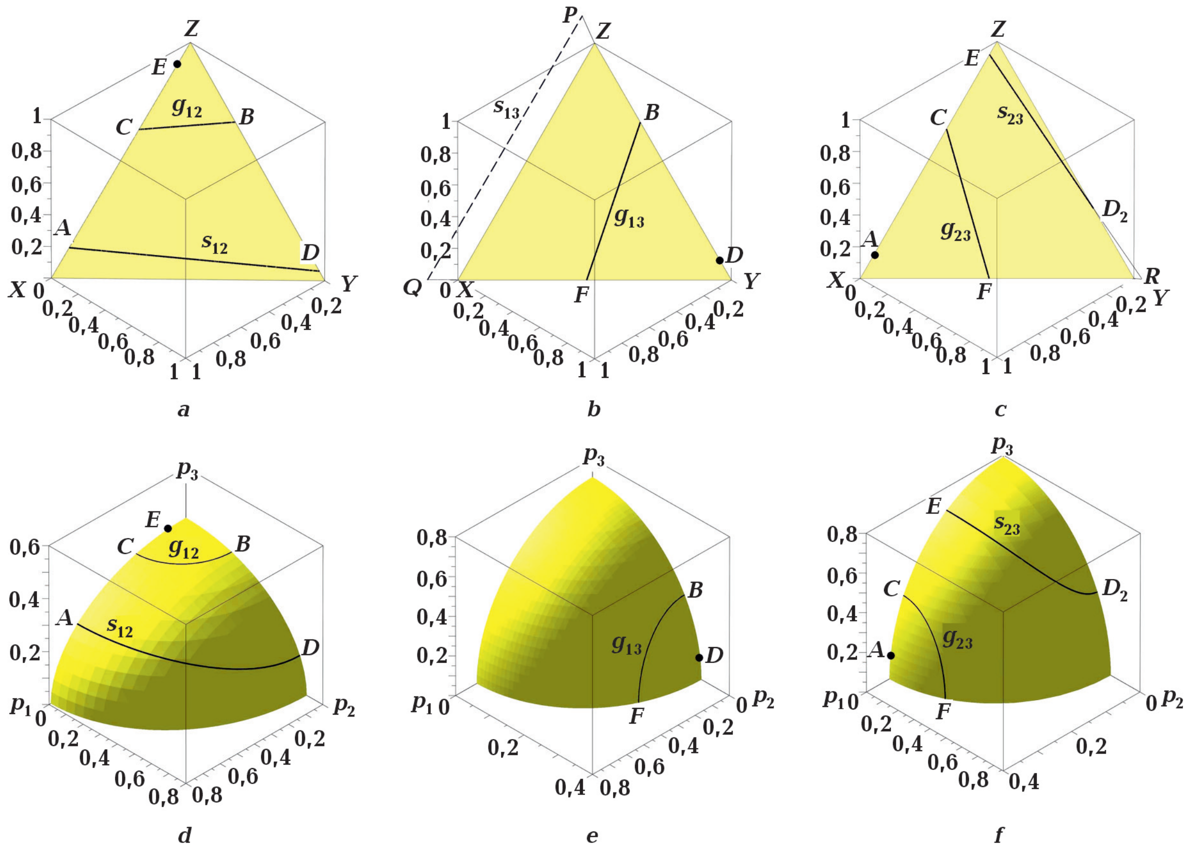


Fig. 4. Diagrams and singular lines on quadratic slowness surfaces for ORT media with different values of the parameters d_{ij}: d_{12}=7.202, d_{13}=d_{23}=0 (*a*, *d*); d_{13}=5.666, d_{12}=d_{23}=0 (*b*, *e*); d_{23}=5.527, d_{12}=d_{23}=0 (*c*, *f*).

with positive values d_{ij}. The equation (1.26) describe two different singular curves defined by equations s_{ij}=0 or g_{ij}=0. The limiting singular curves in Fig. 4 are marked as s_{12}, s_{13}, s_{23}. On these curves, the phase velocities are constant and satisfy the formulas (1.22): v_{f12}=1.469, v_{f13}=1.252, v_{f23}=1.418. On Fig. 4, *b* the singular curve s_{13}=0 located out of triangle.

Further increase of non-zero parameters d_{ij} leads to the deformation of straight singular lines on the diagrams into elliptical ones,

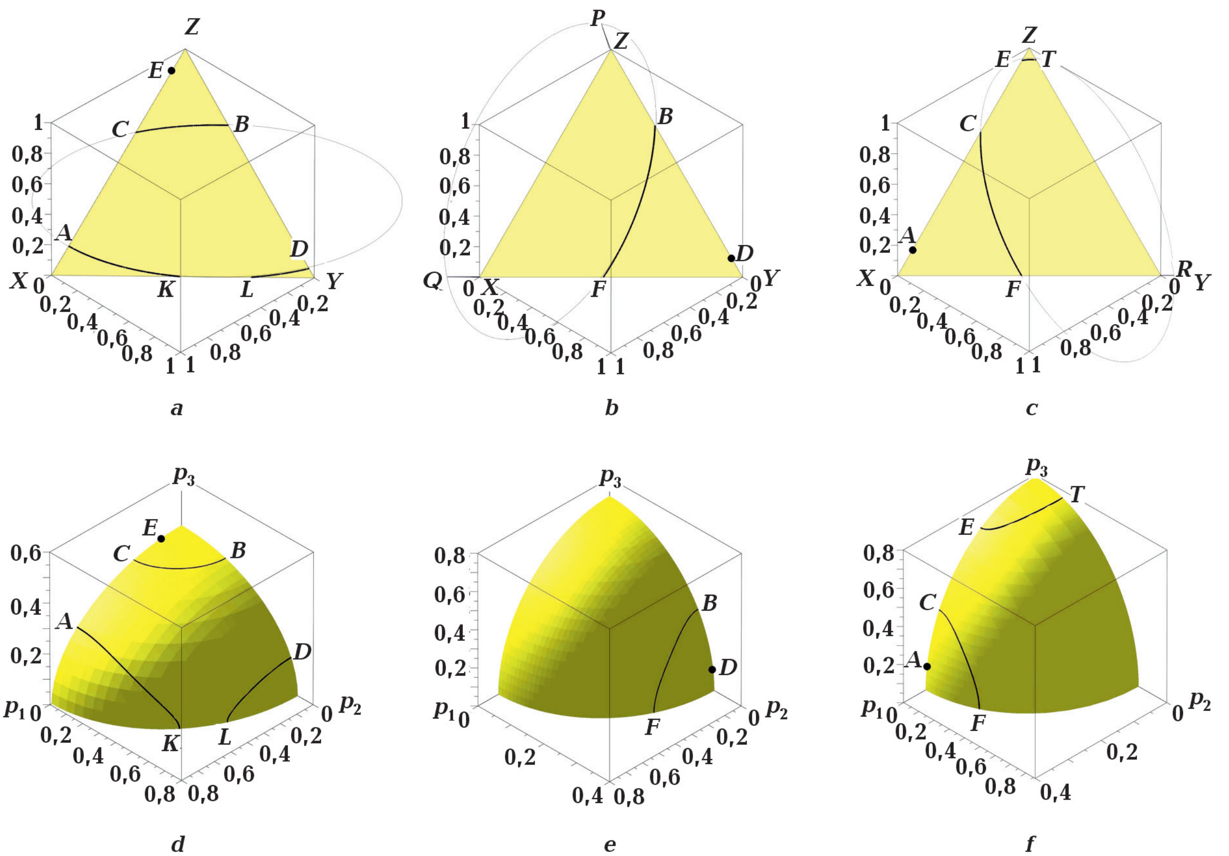


Fig. 5. Diagrams and singular lines on quadratic slowness surfaces for ORT media with different values of the parameters d_{ij}: d_{12}=8, d_{13}=d_{23}=0 (*a*, *d*); d_{13}=8, d_{12}=d_{23}=0 (*b*, *e*); d_{23}=8, d_{12}=d_{13}=0 (*c*, *f*).

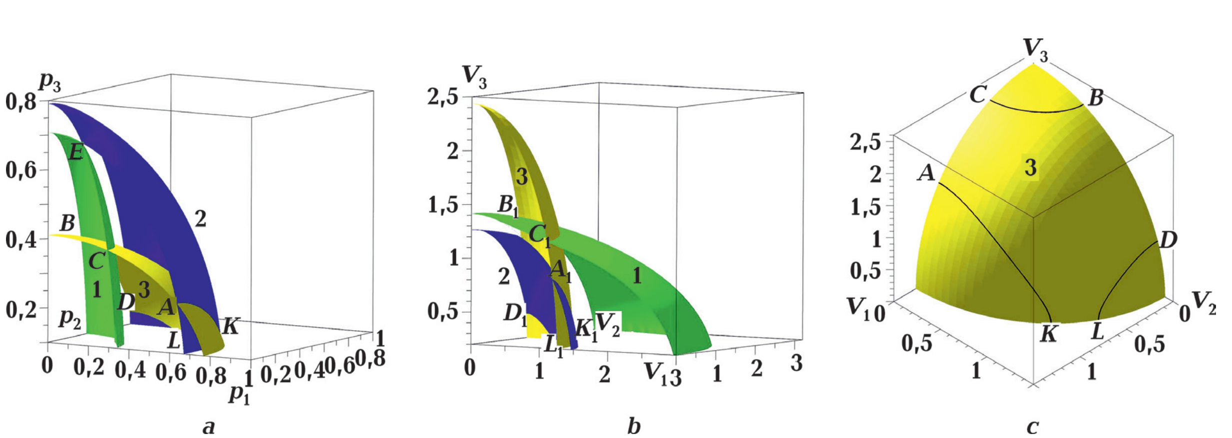


Fig. 6. Slowness surfaces 1, 2, 3 for an ORT medium with d_{12}=8, d_{13}=d_{23}=0 (*a*). Images of these slowness surfaces in the group velocities region (*b*). Image in the group velocities region of the slowness surface 3 and singular lines (*c*).

which pass through the four fixed points listed above. In Fig. 5, *a*—*c* are shown diagrams for nonzero d_{ij}=8. In this case, only sections of the ellipses that are inside of the triangle make sense. The maximum number of such sections is 3, as shown in Fig. 5, *a*. To each section of the ellipse in the region of slowness corresponds a part of a separate singular line, located in the first octant. In Fig. 5, *a*—*f* also shown discrete singular points *E*, *D*, *A*, remaining from the singular lines with perturbation in one of zero d_{ij}.

In Fig. 6, *a* are shown three slowness surfaces for model with d_{12}=8, d_{13}=d_{23}=0. The intersection lines of surfaces 1 and 3 or 2 and 3 defined two singular lines. Also, in Fig. 6, *a* an isolated singular point E is shown. In Fig. 6, *b*, the image of slowness surfaces are shown for the model with d_{12}=8, d_{13}=d_{23}=0 in the region of group velocities. The lines of intersection of the images of slowness surfaces in the region of group velocities do not coincide with the images of the singular lines. Positions of singular lines in region of group velocities are similar to their positions in the region of phase velocities (Fig. 5, *d*). They are shown in Fig. 6, *c* placed on the image in region of group velocities of the slowness surface q_3=0.

Conclusions. The position of singular lines for orthorhombic (ORT) models with fixed diagonal elements of the elasticity matrix c_{ii}, i=1,…,6 is studied under the condition that c_{11}, c_{22}, c_{33}>c_{66}>c_{44}>c_{55}. In this case, the off-diagonal coefficients of the matrix elasticity c_{12}, c_{13}, c_{23} are chosen so that some of the values of d_{12}=c_{12}+c_{66}, d_{13}=c_{13}+c_{55}, d_{23}=c_{23}+c_{44} are equal to zero. An orthorhombic medium, in which only one of the values d_{12}, d_{13}, d_{23} is zero, contains only singular points in the planes of symmetry. If two or all three values d_{ij} are zero, then the ORT model contains singular lines and discrete singular points. We call such media pathological. A degenerate ORT model with positive d_{12}, d_{13}, d_{23} has at most two singular lines, which are the intersection of a quadratic cone with a sphere [Roganov et al., 2022]. The pathological models may have up to 6 singular lines on the surface of the slowness. Singular lines for pathological media are described by more complex equations than the conventional degenerate ORT models. The article proposes to use squares x, y, z of the components of the slowness vector in the equations. In the new coordinate system, equations defining singular lines for pathological media become linear or quadratic. Intersecting with the plane x+y+z =1, they define straight lines, ellipses, or hyperbolas. If non-zero values d_{12}, d_{13}, d_{23} increase, singular lines pass through four fixed points on the plane x+y+z =1, which makes it possible to describe the evolution of their change. Note that some of the coordinates x, y, z these points can be negative or bigger than 1. However, this does not lead to complications, since not the entire curve is interpreted, but only a part of it, where all variables x, y, z are non-negative. Conditions are derived to ensure that the singular curves of pathological ORT media are limiting singular curves for degenerate ORT models with positive values d_{12}, d_{13}, d_{23}. Formulas are derived for transforming slowness surfaces and singular lines of pathological media into the region of group velocities. The results are demonstrated on examples of pathological media obtained from the standard model of the ORT medium by changing the elasticity coefficients c_{12}, c_{13}, c_{23} so that some values d_{12}, d_{13}, d_{23} are equal to zero.

References

Alshits, V.I., & Lothe, J. (1979). Elastic waves in triclinic crystals. Articles I, II, III. *Soviet Physics. Crystallography*, *24*, 387—392, 393—398, 644—648.

Alshits, V.I., & Shuvalov, A.L. (1984). Polarization fields of elastic waves near the acoustic axes: *Soviet Physics. Crystallography*, *29*, 373—378.

Alshits, V.I., Sarychev, A.V., & Shuvalov, A.L. (1985). Classification of degeneracies and analysis of their stability in the theory of elastic waves. *Zhurnal Eksperimentalnoy i Teoreticheskoy Fiziki*, *89*, 922—938 (in Russian).

Boulanger, Ph., & Hayes, M. (1998). Acoustic axes for elastic waves in crystals: Theory and applications. *Proc. Roy. Soc. London, Ser. A, 454*, 2323—2346. https://doi.org/10.1098/rspa.1998.0261.

Crampin, S. (1991). Effects of point singularities on shear-wave propagation in sedimentary basins: *Geophysical Journal International, 107*, 531—543. https://doi.org/10.1111/j.1365-246X.1991.tb01413.x.

Crampin, S., & Yedlin, M. (1981). Shear-wave singularities of wave propagation in anisotropic media. *Journal of Geophysics*, 49, 43—46.

Darinskii, B.M. (1994). Acoustic axes in crystals. *Soviet Physics. Crystallography, 39*, 697—703.

Fedorov, F.I. (1968). *Theory of Elastic Waves in Crystals*. New York: Plenum Press, 375 p.

Khatkevich, A.G. (1962). The acoustic axis in crystals. *Soviet Physics. Crystallography, 7*, 601—604.

Musgrave, M.J.P. (1985). Acoustic axes in orthorhombic media. *Proc. Roy. Soc. London, Ser. A, 401*, 131—143. http://dx.doi.org/10.1098/rspa.1985.0091.

Musgrave, M.J.P. (1981). On an elastodinamic classification of orthorhombic media. *Proc. Roy. Soc. London, Ser. A, 374*, 401—429. https://doi.org/10.1098/rspa.1981.0028.

Norris, A.N. (2004). Acoustic axes in elasticity. *Wave Motion, 40*, 315—328. http://dx.doi.org/10.1016/j.wavemoti.2004.02.005.

Holm, P. (1992). Generic elastic media. *Physica Scripta, 44*, 122—127. http://dx.doi.org/10.1088/0031-8949/1992/T44/019.

Roganov, Yu., & Roganov, V. (2011). Modeling and use of converted wavefields to determine fracture azimuths. *Geofizicheskiy Zhurnal*, 33(2), 64—79. https://doi.org/10.24028/gzh.0203-3100.v33i2.2011.117298 (in Russian).

Roganov, Y.V., Stovas, A., & Roganov, V.Y. (2019). Properties of acoustic axes in triclinic media. *Geofizicheskiy Zhurnal, 41*(3), 3—17. https://doi.org/10.24028/gzh.0203-3100.v41i3.2019.172417 (in Russian).

Roganov, Yu., Stovas, A., & Roganov, V. (2022). Location of singular points in orthorhombic media. *Geofizicheskiy Zhurnal, 44*(3), 3—20. https://doi.org/10.24028/gj.v44i3.261965 (in Ukrainian).

Schoenberg, M., & Helbig, K. (1997). Orthorhombic media: Modeling elastic wave behavior in a vertically fractured earth. *Geophysics, 62*, 1954—1974. https://doi.org/10.1190/1.1444297.

Shuvalov, A.L. (1998). Topological features of the polarization fields of plane acoustic waves in anisotropic media. *Proc. Roy. Soc. London, Ser. A, 454*, 2911—2947. http://dx.doi.org/10.1098/rspa.1998.0286.

Shuvalov, A.L., & Every, A.G. (1997). Shape of the acoustic slowness surface of anisotropic solids near points of conical degeneracy. *The Journal of the Acoustical Society of America, 101*(4), 2381—2382. https://doi.org/10.1121/1.418251.

Stovas, A., Roganov, Yu., & Roganov, V. (2022). Behavior of S waves in vicinity of singularity point in elliptic orthorhombic media. *Geophysics, 87*, C77—C97. https://doi.org/10.1190/geo2021-0522.1.

Stovas, A., Roganov, Yu., & Roganov, V. (2023), On singularity points in elastic orthorhombic media. *Geophysics, 88*, 1—22. https://doi.org/10.1190/GEO2022-0009.1.

Vavryčuk, V. (2005). Acoustic axes in triclinic anisotropy. *The Journal of the Acoustical Society of America, 118*, 647—653. http://dx.doi.org/10.1121/1.1954587.

Zeng, X., & MacBeth, C. (1993). Algebraic processing techniques for estimating shear-wave splitting in near-offset VSP data. *Geophysical Prospecting, 41*, 1033—1066. https://doi.org/10.1111/j.1365-2478.1993.tb00897.x.

Властивості сингулярних точок в особливому випадку орторомбічних середовищ

Ю.В. Роганов[1], А. Стовас[2], В.Ю. Роганов[3], 2023

[1]Tesseral Technologies Inc., Київ, Україна
[2]Норвезький університет природничих і технічних наук, Тронхейм, Норвегія
[3]Інститут кібернетики ім. В. М. Глушкова НАН України, Київ, Україна

Вивчено розташування сингулярних ліній для орторомбічних (ОРТ) середовищ із фіксованими діагональними елементами матриці пружності c_{ij}, i=1…6 за умови, що c_{11}, c_{22}, $c_{33}>c_{66}>c_{44}>c_{55}$. При цьому недіагональні коефіцієнти матриці пружності c_{12}, c_{13}, c_{23} вибрано так, що деякі значення $d_{12}=c_{12}+c_{66}$, $d_{13}=c_{13}+c_{55}$, $d_{23}=c_{23}+c_{44}$ дорівнюють нулю. Орторомбічне середовище, у якого тільки одне із значень d_{12}, d_{13}, d_{23} дорівнює нулю, містить лише сингулярні точки в площинах симетрії. Якщо два чи всі три значення d_{ij} дорівнюють нулю, то це означає, що середовище містить сингулярні лінії і дискретні сингулярні точки. Такі середовища ми називаємо патологічними. Вироджене ОРТ-середовище з позитивними d_{12}, d_{13}, d_{23} має щонайбільше дві сингулярні лінії, які є перетином квадратичного конуса зі сферою. Патологічне середовище може мати до шести сингулярних ліній на поверхні повільності. Сингулярні лінії для патологічних середовищ описують складнішими рівняннями порівняно із звичайними виродженими ОРТ-середовищами. Запропоновано використовувати в рівняннях замість компонентів вектора повільності їхні квадрати x, y, z. У новій системі координат рівняння, що задають сингулярні лінії, стають лінійними чи квадратичними. Перетинаючись із площиною $x+y+z=1$, вони визначають прямі лінії, еліпси чи гіперболи. У разі зростання ненульових значень d_{12}, d_{13}, d_{23}, ці лінії проходять через чотири фіксовані точки на площині $x+y+z=1$, що дає змогу описати еволюцію їх зміни. Виведено умови, за яких сингулярні криві патологічних ОРТ-середовищ є граничними сингулярними кривими для вироджених ОРТ-середовищ з додатними значеннями d_{12}, d_{13}, d_{23}. Виведено формули для перетворення поверхонь повільності та сингулярних ліній патологічних середовищ у область групових швидкостей. Результати продемонстровані на прикладах патологічних середовищ, отриманих з моделі стандартного ОРТ-середовища зміною коефіцієнтів пружності c_{12}, c_{13}, c_{23} так, щоб деякі значення d_{12}, d_{13}, d_{23} дорівнювали нулю.

Ключові слова: сингулярна точка, фазова швидкість, матриця Крістофеля, орторомбічне середовище.