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Anisotropic media with singular slowness surfaces

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It is proved that if an anisotropic medium has an open set of singular directions, then this medium has two slowness surfaces that completely coincide. The coinciding slowness surfaces form one double singular slowness surface. The corresponding anisotropic medium is an elliptical orthorhombic (ORT) medium with equal stiffness coefficients c_{44}=c_{55}=c_{66} rotated to an arbitrary coordinate system. Based on the representation of the Christoffel matrix as a uniaxial tensor and considering that the elements of the Christoffel matrix are quadratic forms in the components of the slowness vector, a system of homogeneous polynomial equations was derived. Then, the identical equalities between homogeneous polynomials are replaced by the equalities between their coefficients. As a result, a new system of equations is obtained, the solution of which is the values of the reduced (density normalized) stiffness coefficients in a medium with a singular surface. Conditions for the positive definite of the obtained stiffness matrix are studied. For the defined medium, the Christoffel equations and equations of group velocity surfaces are derived. The orthogonal rotation matrix that transforms the medium with a singular surface into an elliptic ORT medium in the canonical coordinate system is determined. In the canonical coordinate system, the slowness surfaces S_1 and S_2 waves coincide and are given by a sphere with a radius $c_{44}^{-1/2}$. The slowness surface of qP waves in the canonical coordinate system is an ellipsoid with semi-axes $c_{11}^{-1/2}$, $c_{22}^{-1/2}$, $c_{33}^{-1/2}$. The polarization vectors of S_1 and S_2 waves can be arbitrarily selected in the plane orthogonal to the polarization vector of the qP wave. However, the qP wave polarization vector can be significantly different from the wave vector. This feature should be taken into account in the joint processing and modelling of S and qP waves. The results are illustrated in one example of an elliptical ORT medium.

Key words: singular point, singular surface, phase velocity, Christoffel matrix, elliptical orthorhombic medium.

Introduction. Singular points (directions) in an anisotropic medium are the directions along which the phase velocities of plane waves of different types coincide. Usually, these points are located discretely on the slowness surface, and their number does not exceed 32 [Darinskii, 1994; Vavrychuk, 2005; Roganov et al., 2019]. There are many works devoted to the description of properties and classification of discretely located singular points. An overview of these works can be found, for example, in the article [Stovas et al., 2023b].

According to the differential topology, there are three types of slowness surfaces in the vicinity of a singular point: conical, wedge, and tangential [Alshits, Lothe, 1979; Alshits et al., 1985; Shuvalov, Every, 1997; Shuvalov, 1998; Stovas et al., 2022]. Conical singular points are located discretely and stable with respect to perturbations in the stiffness coefficients. In the

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vicinity of these points, the directions of the polarization vectors of the corresponding waves rapidly change [Alshits, Shuvalov, 1984]. Wedge and tangential singular points are unstable with respect to perturbations in the stiffness coefficients. They can be located discretely and belong to singular lines or singular surfaces. In articles [Roganov et al., 2022; Stovas et al., 2023b], the properties of orthorhombic media with singular lines are studied. The singular lines are also present in pathological media [Musgrave, 1985; Boulanger, Hayes, 1998]. Pathological media are orthorhombic media with some of the values $c_{12}+c_{66}$, $c_{13}+c_{55}$ and $c_{23}+c_{44}$ equal to zero, where c_{ij} are the stiffness coefficients [Roganov et al., 2023; Stovas et al., 2023a].

At the singular point $\mathbf{p}_S=(p_{1S}, p_{2S}, p_{3S})$, the Christoffel equation $F(p_1, p_2, p_3)=0$ has multiple roots in all variables p_1, p_2, p_3. Hence, $F(p_{1S}, p_{2S}, p_{3S})=0$ and $F^{/}_{p_k}\left(p_{1S}, p_{2S}, p_{3S}\right)=0$, $k=1\ldots3$. Therefore, the decomposition of function $F(p_1, p_2, p_3)=0$ in the Taylor series in the vicinity of a point $\mathbf{p}_S$ begins with the quadratic term $1/2\Delta\mathbf{p}^T\mathbf{K}\Delta\mathbf{p}$, where $K_{ij}=F^{//}_{p_i p_j}\left(p_{1S}, p_{2S}, p_{3S}\right)$ is the matrix of second-order derivatives, $\mathbf{p}=(p_1, p_2, p_3)$ and $\Delta\mathbf{p}=\mathbf{p}-\mathbf{p}_S$. If the symmetric matrix $\mathbf{K}$ has a rank one or two, then the singular point is, respectively, a point of tangential or wedge type. If the matrix $\mathbf{K}$ is non-degenerate and has both positive and negative eigenvalues, the singular point is a point of conical type. If $\mathbf{K}=\mathbf{0}$, the singular point is a triple singular point.

In this paper, we prove that if an anisotropic medium has an open set of singular directions, then two slowness surfaces coincide in all directions, and the medium is defined as a rotated elliptical orthorhombic medium with equal stiffness coefficients $c_{44}=c_{55}=c_{66}$. The converse statement was proved in [Stovas et al., 2021].

Theory. Let us consider an anisotropic triclinic medium with a set of singular directions $\mathbf{n}=(n_1, n_2, n_3)$ containing an open set U on the unit sphere $n_1^2+n_2^2+n_3^2=1$. Then, for each direction $(n_1, n_2, n_3)\in U$, the following relations $R_k=0$, $k=1,\ldots,7$ are valid according to [Alshits, Lothe, 1979; Roganov et al., 2019]. The relations $R_k(n_1, n_2, n_3)=0$, $k=1,\ldots,7$ are the criterion for the existence of singularity point in direction $\mathbf{n}=(n_1, n_2, n_3)$. The expressions $R_k(n_1, n_2, n_3)$ are given by homogeneous polynomials of the sixth degree in the variables n_1, n_2, n_3. From homogeneity of R_k, it follows that $R_k(\lambda n_1, \lambda n_2, \lambda n_3)=R_k(n_1, n_2, n_3)=0$ for any λ and $(n_1, n_2, n_3)\in U$. Set of points $(\lambda n_1, \lambda n_2, \lambda n_3)$ at $\lambda>0$ and $(n_1, n_2, n_3)\in U$ is an open set $\bar{U}$ in three-dimensional space: $(\lambda p_1, \lambda p_2, \lambda p_3)\in\bar{U}\subset\mathbf{R}^3$. Because the $R_k=0$ on open set in $\mathbf{R}^3$, all coefficients of these polynomials R_k are equal to zero. Therefore, $R_k(n_1, n_2, n_3)=0$ for any direction $\mathbf{n}=(n_1, n_2, n_3)$, i.e., all directions on the unit sphere are singular. In other words, for this medium, two slowness surfaces identically coincide and form one singular surface.

The elastic properties of this medium are determined by specifying a positive definite Christoffel matrix $\mathbf{A}$, written in terms of the slowness vector components $\mathbf{p}=(p_1, p_2, p_3)$ and reduced (density normalized) stiffness coefficients c_{ij}, $i, j=1\ldots6$ as:

$$\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}, \tag{1}$$

where

$$\begin{aligned}
a_{11}&=c_{11}p_1^2+c_{66}p_2^2+c_{55}p_3^2+2c_{16}p_1p_2+2c_{15}p_1p_3+2c_{56}p_2p_3,\\
a_{22}&=c_{66}p_1^2+c_{22}p_2^2+c_{44}p_3^2+2c_{26}p_1p_2+2c_{46}p_1p_3+2c_{24}p_2p_3,\\
a_{33}&=c_{55}p_1^2+c_{44}p_2^2+c_{33}p_3^2+2c_{45}p_1p_2+2c_{35}p_1p_3+2c_{34}p_2p_3,\\
a_{12}&=c_{16}p_1^2+c_{26}p_2^2+c_{45}p_3^2+d_{12}p_1p_2+d_{14}p_1p_3+d_{25}p_2p_3,\\
a_{13}&=c_{15}p_1^2+c_{46}p_2^2+c_{35}p_3^2+d_{14}p_1p_2+d_{13}p_1p_3+d_{36}p_2p_3,\\
a_{23}&=c_{56}p_1^2+c_{24}p_2^2+c_{34}p_3^2+d_{25}p_1p_2+d_{36}p_1p_3+d_{23}p_2p_3,
\end{aligned} \tag{2}$$

and $d_{12}=c_{12}+c_{66}$, $d_{13}=c_{13}+c_{55}$, $d_{14}=c_{14}+c_{56}$, $d_{23}=c_{23}+c_{44}$, $d_{25}=c_{25}+c_{46}$, $d_{36}=c_{36}+c_{45}$.

If the components p_1, p_2, p_3 of the slowness vector belong to the singular surface, then at this point, the matrix $\mathbf{A}$ has a double eigenvalue 1. The matrix $\mathbf{A}$, at singular points, can be represented as [Fedorov, 1968]:

$$\mathbf{A} = q\mathbf{f}\mathbf{f}^T + \mathbf{I}\,, \tag{3}$$

where $\mathbf{I}$ is identity matrix, q is a number, $\mathbf{f}=(f_1,f_2,f_3)^T$ is a vector, and $1+q\mathbf{f}^T\mathbf{f}$ is the third eigenvalue of the matrix $\mathbf{A}$. After renormalizing the vector $\mathbf{f}$ by multiplying it by $\sqrt{|q|}$, we can assume that $q=\pm 1$.

The case with $q=1$ ***and*** $a_{12}\neq 0$, $a_{13}\neq 0$, $a_{23}\neq 0$***.***

From representation (3), it follows that the following equations

$$a_{12} = f_1 f_2, \quad a_{13} = f_1 f_3, \quad a_{23} = f_2 f_3, \tag{4}$$

$$a_{11} - f_1^2 = a_{22} - f_2^2 = a_{33} - f_3^2\,, \tag{5}$$

are satisfied on the singular surface. From these equations, we can obtain,

$$f_1^2 = \frac{a_{12}a_{13}}{a_{23}}\,,\ f_2^2 = \frac{a_{12}a_{23}}{a_{13}}\,,\ f_3^2 = \frac{a_{13}a_{23}}{a_{12}} \tag{6}$$

and

$$\begin{aligned} a_{13}a_{23}\left(a_{22}-a_{11}\right) &= a_{12}\left(a_{23}^2 - a_{13}^2\right), \\ a_{12}a_{23}\left(a_{33}-a_{11}\right) &= a_{13}\left(a_{23}^2 - a_{12}^2\right), \\ a_{12}a_{13}\left(a_{33}-a_{22}\right) &= a_{23}\left(a_{13}^2 - a_{12}^2\right). \end{aligned} \tag{7}$$

Relations (7) are homogeneous in variables p_1, p_2, p_3, and, therefore, valid for any point (p_1, p_2, p_3) in space $\mathbf{R}^3$. Consequently, the polynomials from the left and right sides of equations (7) coincide as elements of the ring $\mathbf{R}[p_1,p_2,p_3]$.Therefore, in equations (7), we can enjoy properties of divisibility. It follows that a_{23} divides both $a_{12}a_{13}^2$ and $a_{12}^2 a_{13}$. Therefore, a_{23} can also divides the greatest common divisor $\mathrm{GCD}\left(a_{12}^2 a_{13}, a_{12}a_{13}^2\right) = a_{12}a_{13}$. From equations (6), we get that f_1^2 is a homogeneous second-degree polynomial. Similar statements are valid for the f_2^2 and f_3^2.

On the other hand, all polynomials a_{12}, a_{13}, a_{23} are homogeneous polynomials of the second degree. Therefore, on the right-hand sides of the equalities (6), there are only two cases of divisibility of polynomials: either f_1, f_2, f_3 from (4) are linear forms in variables p_1, p_2, p_3, or all polynomials a_{12}, a_{13}, a_{23} are proportional. In the first case, polynomials a_{12}, a_{13}, a_{23} from equations (2) can be decomposed into the linear forms f_1, f_2, f_3 from formulas (4).

The case with components f_1, f_2, f_3 ***of vector*** $\mathbf{f}$ ***are linear forms in variables*** p_1, p_2, p_3 ***and*** $q=1$***.***

In this case, the relations (4)—(7) are homogeneous and satisfied for every p_1, p_2, p_3. Therefore, the Christoffel matrix $\mathbf{A}$ and the equation $F=0$ can be represented as

$$\mathbf{A} = \mathbf{f}\mathbf{f}^T + \left(a_{11} - f_1^2\right)\mathbf{I}\,, \tag{8}$$

$$F = \left(a_{11} + f_2^2 + f_3^2 - 1\right)\left(a_{11} - f_1^2 - 1\right)^2 = 0\,. \tag{9}$$

Christoffel matrix (8) has a single eigenvalue $\lambda_1 = a_{11} + f_2^2 + f_3^2$ with eigenvector $\mathbf{f} = \left(f_1, f_2, f_3\right)^T$ and double eigenvalue $\lambda_{1,2} = a_{11} - f_1^2$ with eigenvectors, perpendicular to the vector $\mathbf{f}$.

To obtain the stiffness coefficients for this medium, we use the formulas for the elements of the Christoffel matrix (2) and introduce linear forms:

$$\begin{aligned} f_1 &= x_1 p_1 + x_2 p_2 + x_3 p_3\,, \\ f_2 &= y_1 p_1 + y_2 p_2 + y_3 p_3\,, \\ f_3 &= z_1 p_1 + z_2 p_2 + z_3 p_3\,, \end{aligned} \tag{10}$$

where x_i, y_i, z_i, i=1,2,3 are some numbers.

By equating the coefficients of polynomials at p_1^2, p_2^2, p_3^2 in equations (4), we obtain the values of nine stiffness coefficients:

$$\begin{aligned} c_{16} &= x_1 y_1, \quad c_{26} = x_2 y_2, \quad c_{45} = x_3 y_3, \\ c_{15} &= x_1 z_1, \quad c_{46} = x_2 z_2, \quad c_{35} = x_3 z_3, \\ c_{56} &= y_1 z_1, \quad c_{24} = y_2 z_2, \quad c_{34} = y_3 z_3. \end{aligned} \tag{11}$$

By equating the coefficients of polynomials at $p_1 p_2$, $p_1 p_3$, $p_2 p_3$ in equations (5) and substituting variables according to the formulas (2), (10) and (11), we obtain a system of equations

$$\begin{aligned} (x_2 - y_1)(x_1 + y_2) &= 0, \\ (x_3 - z_1)(x_1 + z_3) &= 0, \\ (y_3 - z_2)(y_2 + z_3) &= 0. \end{aligned} \tag{12}$$

This system has a solution

$$x_2 = y_1,\, x_3 = z_1,\, y_3 = z_2\,. \tag{13}$$

The relations (13) are necessary and sufficient to determine all the stiffness coefficients for which the polynomials in (4) and (5) are identical.

Substituting variables from equation (13) into equations (4), (5) and equating the coefficients at monoms $p_i p_j$, i,j=1…3, $i \neq j$, we get all the stiffness coefficients, expressed through seven numerical parameters c_{11}, x_1, y_1, y_2, z_1, z_2, z_3:

$$\begin{aligned} & c_{11} = c_{11}, c_{12} = -c_{11} + x_1^2 + x_1 y_2, c_{13} = -c_{11} + x_1^2 + x_1 z_3, c_{14} = x_1 z_2, c_{15} = x_1 z_1, c_{16} = x_1 y_1, \\ & c_{22} = c_{11} - x_1^2 + y_2^2, c_{23} = -c_{11} + x_1^2 + y_2 z_3, c_{24} = y_2 z_2, c_{25} = y_2 z_1, c_{26} = y_1 y_2, \\ & c_{33} = c_{11} - x_1^2 + z_3^2, c_{34} = z_2 z_3, c_{35} = z_1 z_3, c_{36} = y_1 z_3, \\ & c_{44} = c_{11} - x_1^2 + z_2^2, c_{45} = z_1 z_2, c_{46} = y_1 z_2, \\ & c_{55} = c_{11} - x_1^2 + z_1^2, c_{56} = y_1 z_1, \\ & c_{66} = c_{11} - x_1^2 + y_1^2. \end{aligned} \tag{14}$$

The stiffness matrix for this medium has the form,

$$\mathbf{C} = \begin{pmatrix} c_{11} & x_1^2 + x_1 y_2 - c_{11} & x_1^2 + x_1 z_3 - c_{11} & x_1 z_2 & x_1 z_1 & x_1 y_1 \\ & -x_1^2 + y_2^2 + c_{11} & x_1^2 + y_2 z_3 - c_{11} & y_2 z_2 & y_2 z_1 & y_2 y_1 \\ & & -x_1^2 + z_3^2 + c_{11} & z_3 z_2 & z_3 z_1 & z_3 y_1 \\ & & & -x_1^2 + z_2^2 + c_{11} & z_2 z_1 & z_2 y_1 \\ & & & & -x_1^2 + z_1^2 + c_{11} & z_1 y_1 \\ & & & & & -x_1^2 + y_1^2 + c_{11} \end{pmatrix}. \tag{15}$$

According to Sylvester's criterion, the matrix $\mathbf{C}$ is positive defined if and only if $c_{11}>0$ and all the corner minors $m_k=\det(c_{ij})$, $i,j=1,\ldots,k$, $2\le k\le 6$, are positive. In our case,

$$
\begin{aligned}
m_2 &= \left(c_{11}-x_1^2\right)\left(x_1+y_2\right)^2>0,\\
m_3 &= -4\left(c_{11}-x_1^2\right)\left(c_{11}-x_1^2-x_1y_2-x_1z_3-y_2z_3\right)>0,\\
m_4 &= -4\left(c_{11}-x_1^2\right)^3\left(c_{11}-x_1^2-x_1y_2-x_1z_3-y_2z_3+z_2^2\right)>0,\\
m_5 &= -4\left(c_{11}-x_1^2\right)^4\left(c_{11}-x_1^2-x_1y_2-x_1z_3-y_2z_3+z_1^2+z_2^2\right)>0,\\
m_6 &= -4\left(c_{11}-x_1^2\right)^5\left(c_{11}-x_1^2-x_1y_2-x_1z_3-y_2z_3+y_1^2+z_1^2+z_2^2\right)>0.
\end{aligned}
\tag{16}
$$

The system of inequalities (16) is equivalent to double inequality

$$x_1^2<c_{11}<x_1^2+x_1y_2+x_1z_3+y_2z_3-y_1^2-z_1^2-z_2^2\,. \tag{17}$$

Consequently, the matrix $\mathbf{C}$ is positive defined if and only if the inequalities (17) are valid.

For relation with $q=-1$ and components f_1, f_2, f_3 of vector $\mathbf{f}$ expressed as linear forms in p_1, p_2, p_3, we can calculate of the elasticity matrix $\bar{\mathbf{C}}$ in a similar way. Technically, the matrix $\bar{\mathbf{C}}$ can be obtained from the expression (15) for matrix $\mathbf{C}$ by replacing variables $x_1, y_2, y_3, z_1, z_2, z_3$ with pure imaginary values ix_1, iy_1, iy_2, iz_1, iz_2, iz_3, $\left(i=\sqrt{-1}\right)$. After this replacement, the elements of the matrix $\mathbf{ff}^T$ from (3) change the sign to opposite, i.e., we obtain the case with $q_2=-1$. The elements of the elasticity matrix $\bar{\mathbf{C}}$ remain real. However, at $c_{11}>0$, the angular 2×2 — minor of the matrix $\bar{\mathbf{C}}$ is a negative number $-\left(c_{11}+x_1^2\right)\left(x_1+y_2\right)^2$. Consequently, the medium with the $q_2=-1$ and linear forms f_1, f_2, f_3 in p_1, p_2, p_3 does not exist.

Let us continue the case with $q=1$**.**

Let us introduce variables, $\mathbf{M}=\begin{pmatrix} x_1 & y_1 & z_1\\ y_1 & y_2 & z_2\\ z_1 & z_2 & z_3\end{pmatrix}$, $\mathbf{f}=\begin{pmatrix} f_1\\ f_2\\ f_3\end{pmatrix}$, $\mathbf{p}=\begin{pmatrix} p_1\\ p_2\\ p_3\end{pmatrix}$. Obviously, we have $\mathbf{f}=\mathbf{Mp}$.

The Christoffel equation (9) can be represented as

$$F=PQ^2, \tag{18}$$

where $Q=\left(c_{11}-x_1^2\right)\mathbf{p}^T\mathbf{p}-1$ and $P=\mathbf{p}^T\mathbf{M}^2\mathbf{p}+Q$.

Equations $Q=0$ and $P=0$, respectively, define the sphere of radius $1\big/\sqrt{c_{11}-x_1^2}$ and a rotated ellipsoid. The polarization vector of a single wave is determined by the formula $\mathbf{f}=\mathbf{Mp}$, and the two polarization vectors of the double wave are perpendicular to the vector $\mathbf{f}$.

Matrix $\mathbf{M}^2$ has non-negative eigenvalues, i.e., it is positive semi definite. Hence, $\mathbf{p}^T\mathbf{M}^2\mathbf{p}\ge 0$ for any vector $\mathbf{p}$, and the surface defined by the equation $P=0$ is inside the surface defined by the equation $Q=0$. Therefore, the equation $P=0$ defines the slowness surface of qP waves, and the equation $Q=0$ defines double slowness surface of S_1, S_2 waves. These surfaces can be tangent in two points or along a circle if the rank of the matrix $\mathbf{M}$ is 2 or 1, respectively.

The slowness surfaces $P=0$, $Q=0$ in the region of group velocities $\mathbf{v}=(v_1,v_2,v_3)^T$ respectively correspond ellipsoid $G_P=1$ and sphere $G_Q=1$, where

$$G_P=\mathbf{v}^T\left(\mathbf{M}^2+\left(c_{11}-x_1^2\right)\mathbf{I}\right)^{-1}\mathbf{v},\quad G_Q=\left(c_{11}-x_1^2\right)^{-1}\mathbf{v}^T\mathbf{v}\,. \tag{19}$$

The symmetric matrix $\mathbf{M}$ can be diagonalized by the orthogonal matrix $\mathbf{R}$,

$$\mathbf{M}' = \mathbf{R}\mathbf{M}\mathbf{R}^T = \operatorname{diag}\left(x_1', y_2', z_3'\right). \tag{20}$$

In the coordinate system $\mathbf{p}' = \left(p_1', p_2', p_3'\right)^T = \mathbf{R}\mathbf{p}$, the equations $y_1' = z_1' = z_2' = 0$ are valid, and the Christoffel equation has the form $F' = P'Q'^2 = 0$, where

$$P' = \left(x_1'^2 p_1'^2 + y_2'^2 p_2'^2 + z_3'^2 p_3'^2\right) + Q',$$
$$Q' = \left(c_{11}' - x_1'^2\right)\left(p_1'^2 + p_2'^2 + p_3'^2\right) - 1. \tag{21}$$

In the new (canonical) coordinate system, the relations (14) can be represented as

$$\begin{aligned}
&c_{44}' = c_{55}' = c_{66}',\\
&x_1'^2 = c_{11}' - c_{44}' = \Delta_{14}', \quad y_2'^2 = c_{22}' - c_{44}' = \Delta_{24}', \quad z_3'^2 = c_{33}' - c_{44}' = \Delta_{34}',\\
&c_{12}' + c_{44}' = d_{12}' = x_1' y_2' = \sqrt{\Delta_{14}'\Delta_{24}'}, \quad c_{13}' + c_{44}' = d_{13}' = x_1' z_3' = \sqrt{\Delta_{14}'\Delta_{34}'},\\
&c_{23}' + c_{44}' = d_{23}' = y_2' z_3' = \sqrt{\Delta_{24}'\Delta_{34}'},\\
&c_{14}' = c_{15}' = c_{16}' = c_{24}' = c_{25}' = c_{26}' = c_{34}' = c_{35}' = c_{36}' = c_{45}' = c_{46}' = c_{56}' = 0,
\end{aligned} \tag{22}$$

where $\Delta_{ij}' = c_{ii}' - c_{jj}^l$.

From (22), it follows that the polarization vector of qP waves for this medium is defined as $\mathbf{a}_{qP}' = \left(p_1'\sqrt{\Delta_{14}'}, p_2'\sqrt{\Delta_{24}'}, p_3'\sqrt{\Delta_{34}'}\right)^T$, and it is different in orientation from the slowness vector $\mathbf{p}' = \left(p_1', p_2', p_3'\right)^T$. Since the values $c_{11}', c_{22}', c_{33}'$ are different, the stiffness matrix in the canonical coordinate system is determined by four independent parameters $c_{11}', c_{22}', c_{33}', c_{44}'$ and takes the form,

$$\mathbf{C}' = \begin{pmatrix}
c_{11}' & \sqrt{\Delta_{14}'\Delta_{24}'} - c_{44}' & \sqrt{\Delta_{14}'\Delta_{34}'} - c_{44}' & 0 & 0 & 0\\
 & c_{22}' & \sqrt{\Delta_{24}'\Delta_{34}'} - c_{44}' & 0 & 0 & 0\\
 & & c_{33}' & 0 & 0 & 0\\
 & & & c_{44}' & 0 & 0\\
 & & & & c_{44}' & 0\\
 & & & & & c_{44}'
\end{pmatrix}. \tag{23}$$

The Christoffel equation for this medium is represented as $F' = P'Q'^2 = 0$, where

$$P' = c_{11}' p_1'^2 + c_{22}' p_2'^2 + c_{33}' p_3'^2 - 1, \quad Q' = c_{44}'\left(p_1'^2 + p_2'^2 + p_3'^2\right) - 1. \tag{24}$$

The Bond transform by matrix $\mathbf{R}$ transforms the stiffness matrix (15) into the stiffness matrix (23).

Matrix $\mathbf{C}'$ is positive definite if and only if

$$\begin{aligned}
&c_{11}' > c_{44}', \quad c_{22}' > c_{44}', \quad c_{33}' > c_{44}',\\
&0 < c_{44}' < \sqrt{\Delta_{14}'\Delta_{24}'} + \sqrt{\Delta_{14}'\Delta_{34}'} + \sqrt{\Delta_{24}'\Delta_{34}'},
\end{aligned} \tag{25}$$

and the corresponding medium is an orthorhombic elliptic medium [Stovas et al., 2021] with parameters $c_{44}' = c_{55}' = c_{66}'$. For this medium, the slowness surface for S_1, S_2 waves is a sphere with radius $1/\sqrt{c'_{44}}$, and the slowness surface for qP waves is an ellipsoid with semi-axes $1/\sqrt{c'_{11}}$, $1/\sqrt{c'_{22}}$, $1/\sqrt{c'_{33}}$. The slowness surface for qP waves is located inside the slowness surface for S_1, S_2 waves.

Let us consider the case when q=1 **and elements** a_{12}, a_{13}, a_{23} **of matrices A are proportional.** In this case, for some non-zero values t, h, the system of equations has a form,

$$a_{13}=a_{12}t,\quad a_{23}=a_{12}h,\quad a_{22}=a_{11}+a_{12}\left(ht^{-1}-th^{-1}\right),\quad a_{33}=a_{11}+a_{12}\left(th-th^{-1}\right). \tag{26}$$

Equalities (26) are valid for any values of p_1, p_2, p_3 since they are equalities between the homogeneous polynomials of the second degree.

Using the relations (2) and equating the coefficients of the polynomials in (26), we obtain a system of equations. The solution of this system is all stiffness coefficients c_{mn}, expressed through c_{11}, c_{12}, c_{16} and numbers t, h:

$$c_{15}=tc_{16},\quad c_{56}=hc_{16},\quad c_{55}=c_{11}+\frac{t\left(h^2-1\right)}{h}c_{16},\quad c_{66}=c_{11}+\frac{h^2-1}{th}c_{16}, \tag{27}$$

$$c_{14}=t\left(c_{11}+c_{12}\right)-\frac{t^2}{h}c_{16},\quad c_{13}=\left(t^2-1\right)c_{11}+tc_{12}-\frac{t\left(t^2-1\right)}{h}c_{16},$$
$$c_{26}=\frac{h^2-t^2}{2th}\left(c_{11}+c_{12}\right)-\frac{t^4+h^4}{t^2h^2}c_{16}, \tag{28}$$

$$c_{25}=\frac{t^2+h^2}{2h}\left(c_{11}+c_{12}\right)+\frac{\left(t^2-h^2\right)^2-2t^4}{2th^2}c_{16},\quad c_{45}=\frac{t\left(h^2-1\right)}{2h}\left(c_{11}+c_{12}\right)+\frac{t^2+h^2-t^2h^2+h^4}{2h^2}c_{16}, \tag{29}$$

$$c_{36}=\frac{t\left(h^2+1\right)}{2h}\left(c_{11}+c_{12}\right)-\frac{t^2+h^2+t^2h^2-h^4}{2h^2}c_{16},\quad c_{46}=tc_{26},\quad c_{24}=hc_{26},\quad c_{35}=tc_{45},\quad c_{34}=hc_{45}, \tag{30}$$

$$c_{22}=c_{66}+\frac{h^2-t^2}{th}c_{26},\quad c_{33}=c_{55}+\frac{t\left(h^2-1\right)}{h}c_{45},\quad c_{44}=c_{66}+\frac{t\left(h^2-1\right)}{h}c_{26},\quad c_{23}=c_{12}+\frac{h^2-1}{h}c_{25}. \tag{31}$$

However, an anisotropic medium with stiffness coefficients (27)—(31) does not exist since the stiffness matrix **C** for this medium is not positive definite. Indeed, according to the Sylvester's criterion, for a positive definite matrix **C**, the determinant must be positive, but in our case,

$$\det\mathbf{C}=-\frac{\left(h(2h^2+t^2)c_{11}+c_{12}ht^2-t(3h^2+t^2)c_{16}\right)^2}{16h^{10}t^4}\times$$
$$\times\frac{\left(ht(t^2-1)c_{11}+ht(t^2+1)c_{12}-(h^2t^2+t^4+h^2-t^2)c_{16}\right)^4}{16h^{10}t^4}\le 0\,. \tag{32}$$

Let us consider the cases where some of the off-diagonal elements a_{12}, a_{13}, a_{23} ***of matrix A are zero.*** From representation (3), it follows that there maybe two cases: when two elements from a_{12}, a_{13}, a_{23} are zero or when all three of these elements are zero.

If two elements are zero, we assume for definiteness that $a_{12}\neq 0$, $a_{13}=0$, $a_{23}=0$. Then,

$$\mathbf{A}=\begin{pmatrix} a_{11}-a_{33} & a_{12} & 0\\ a_{12} & a_{22}-a_{33} & 0\\ 0 & 0 & 0\end{pmatrix}+a_{33}\mathbf{I}, \tag{33}$$

where **I** is an identity matrix. Then, for some number $f \neq 0$, we obtain the equations,

$$a_{11} - a_{33} = f^{-1} a_{12}, \quad a_{22} - a_{33} = f a_{12}. \tag{34}$$

After substituting in (34) the values a_{ij} from (2) and equating the coefficients of monomials in variables p_1, p_2, p_3, we obtain a system of linear equations for determining the stiffness coefficients c_{mn}. With free parameters a_{45}, a_{16}, a_{66}, f, this system of equations has a solution,

$$\begin{aligned}
&c_{11} = c_{66} - c_{16} f^{-1}\left(f^2 - 1\right), \quad c_{12} = 2f\left(c_{16} - c_{45}\right) - c_{66}, \quad c_{13} = c_{16} f - c_{66}, \quad c_{13} = 0, \quad c_{15} = 0,\\
&c_{22} = c_{66} + c_{16} f\left(f^2 - 1\right) - c_{45} f^{-1}\left(f^2 - 1\right)^2, \quad c_{23} = -c_{66} + c_{16} f - c_{45} f^{-1}\left(f^2 - 1\right),\\
&c_{24} = 0, \quad c_{25} = 0,\\
&c_{26} = c_{16} f^2 + c_{16} f - c_{45}\left(f^2 - 1\right), \quad c_{33} = c_{66} - c_{16} f - c_{45} f^{-1}, \quad c_{34} = 0, \quad c_{35} = 0, \quad c_{36} = -c_{45},\\
&c_{44} = c_{66} - c_{16} f - c_{45} f^{-1}\left(f^2 - 1\right), \quad c_{46} = 0, \quad c_{55} = -c_{16} f + c_{66}, \quad c_{56} = 0.
\end{aligned} \tag{35}$$

However, the determinant of **C** with elements from (35) is not positive,

$$\det \mathbf{C} = -4 f^{-2}\left(c_{16} f^2 - c_{66} f + c_{45}\right)^2 \left(c_{16} f - c_{45} f - c_{66}\right)^4 \leq 0. \tag{36}$$

Therefore, the matrix **C** is not positive definite, and the case defined above is not physically realizable.

Let us consider the last case, when $a_{12}=0$, $a_{13}=0$, $a_{23}=0$**.** For such a matrix **A**, in order to have a double slowness surface, some of its diagonal elements must coincide. For definiteness, we will assume that $a_{11}=a_{22}$. After substituting in equation $a_{11}=a_{22}$ the values a_{ij} from (2) and equating the coefficients at different degrees p_1, p_2, p_3, we obtain a system of equations for determining c_{mn}. This system has a solution,

$$c_{11}=c_{22}=c_{66}=-c_{12}, \; c_{44}=c_{55}=-c_{13}=-c_{23}, \tag{37}$$

where the others $c_{mn}=0$. However, all angular determinants of the corresponding stiffness matrix **C** are not positive. Therefore, the matrix **C** for this medium is not positive definite, and this case is not physically realizable.

Therefore, we have proved that if an anisotropic medium contains an open set of singular directions, then this medium has a double singular surface, and the anisotropic medium is a rotated elliptical orthorhombic medium with the equal stiffness coefficients $c_{44}=c_{55}=c_{66}$ [Stovas et al., 2021]. The stiffness matrix of this medium is determined by the expression (15) and is positive definite if the inequalities (17) are valid.

Numerical example. In the numerical example and corresponding figures, the reduced stiffness coefficients will be given in km^2/s^2, and the velocities are given in km/s.

To illustrate the above statements, we consider an anisotropic medium with the reduced stiffness coefficient matrix satisfied the relation (15),

$$c_{11} = 8.208, \; x_1 = 2.492, \quad y_1 = 0.158, \quad y_2 = 2.566, \quad z_1 = -0.360, \quad z_2 = 0.153, \quad z_3 = 2.373. \tag{38}$$

The reduced (density normalized) stiffness coefficient matrix of this medium has the form,

$$\mathbf{C} = \begin{pmatrix}
8.208 & 4.392 & 3.912 & 0.382 & -0.897 & 0.395\\
 & 8.582 & 4.088 & 0.394 & -0.924 & 0.407\\
 & & 7.631 & 0.364 & -0.855 & 0.376\\
 & & & 2.024 & -0.055 & 0.024\\
 & & & & 2.130 & -0.057\\
 & & & & & 2.025
\end{pmatrix}. \tag{39}$$

For this stiffness matrix **C** value c_{11}>0 and angular minors (16) are also positive:

$$m_2 = 51.150, \quad m_3 = 262.280, \quad m_4 = 523.807, \quad m_5 = 1039.315, \quad m_6 = 2075.415\,. \tag{40}$$

Therefore, the matrix **C** is positive definite and satisfies the criterion of positive definite (17). In our example, the matrix

$$\mathbf{M} = \begin{pmatrix} 2.492 & 0.158 & -0.360 \\ 0.158 & 2.566 & 0.153 \\ -0.360 & 0.153 & 2.373 \end{pmatrix}. \tag{41}$$

Multipliers P, Q from expression (18), which define the slowness surfaces for qP and S_1, S_2 waves, respectively, are given by the formulas:

$$\begin{aligned} &P = 8.363p_1^2 + 8.631p_2^2 + 7.784p_3^2 + 1.492p_1p_2 - 3.455p_1p_3 + 1.401p_2p_3 - 1, \\ &Q = 2\left(p_1^2 + p_2^2 + p_3^2\right) - 1. \end{aligned} \tag{42}$$

Maximum angle between vectors $\mathbf{a}_{qP}$ and $\mathbf{p}$ achieved at p_1=–0.021, p_2=–0.247, p_3=0.662 is equal to 9.82°. Polarization vectors of S_1, S_2 waves are in a plane perpendicular to the vector $\mathbf{a}_{qP}$. They are not perpendicular to the slowness vector $\mathbf{p}$.

The slowness surfaces of qP and S_1, S_2 waves (42) correspond to the group velocities surfaces:

$$\begin{aligned} &G_P = 0.127v_1^2 + 0.118v_2^2 + 0.136v_3^2 - 0.027v_1v_2 + 0.059v_1v_3 - 0.027v_2v_3 - 1, \\ &G_Q = 0.5\left(v_1^2 + v_2^2 + v_3^2\right) - 1. \end{aligned} \tag{43}$$

The matrix **M** defined by expression (41) is reduced to a diagonal form by the orthogonal similarity transformation $\mathbf{M}^{/} = \mathbf{RMR}^T$, where

$$\mathbf{R} = \begin{pmatrix} 0.127 & 0.927 & 0.353 \\ -0.780 & -0.127 & 0.612 \\ 0.612 & -0.354 & 0.707 \end{pmatrix}, \quad \mathbf{M}^{/} = \begin{pmatrix} 2.646 & 0 & 0 \\ 0 & 2.8 & 0 \\ 0 & 0 & 1.984 \end{pmatrix}. \tag{44}$$

The Bond transform using an orthogonal matrix **R** transforms the stiffness matrix **C** into a canonical coordinate system. In this system, the coordinate planes coincide with the symmetry planes of the elliptical orthorhombic medium, and the stiffness matrix has the form,

$$\mathbf{C}^{/} = \begin{pmatrix} 9.0 & 5.408 & 3.25 & 0 & 0 & 0 \\ 5.408 & 9.84 & 3.556 & 0 & 0 & 0 \\ 3.25 & 3.556 & 5.9375 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}. \tag{45}$$

The free parameters in (38) were initially chosen so that in the matrix (45) the values of diagonal elements $c_{11}^{/} = 9.0$, $c_{22}^{/} = 9.84$, $c_{33}^{/} = 5.9375$, $c_{44}^{/} = 2.0$ coincide with the values of the corresponding elements of the stiffness matrix of a standard orthorhombic medium [Schoenberg, Helbig, 1997]. The remaining elements of this matrix satisfy the expression (23), and hence, the matrix (45) defines an orthorhombic elliptic medium [Stovas et al., 2021] with coinciding slowness surfaces of S_1 and S_2 waves.

Fig. 1, *a* shows the slowness surface of a qP wave, which is a rotated ellipsoid, and the slowness surfaces of S_1 and S_2 waves that coincide with each other and are a sphere. The rotation is done using the matrix **R** from (44). Fig. 1, *b* shows the images of these slowness surfaces in the group velocity region, which are also a rotated ellipsoid for qP waves and are a sphere for S_1 and S_2 waves.

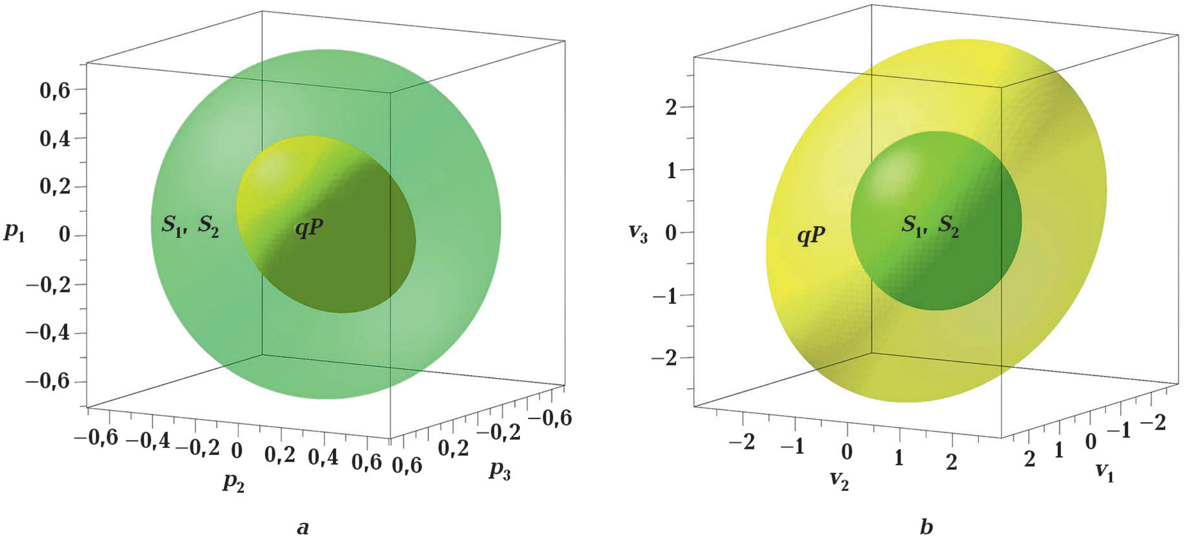

Fig. 1. Slowness surfaces (*a*) and group velocity surfaces (*b*) of qP and S_1, S_2 waves for a medium with the stiffness matrix (39). The slowness surfaces of S_1 and S_2 waves coincide and are a double sphere. The slowness surface of qP wave is a rotated ellipsoid. Similarly, the group velocity surfaces of S_1 and S_2 waves coincide and are a double sphere, and the group velocity surface of qP wave is a rotated ellipsoid.

Conclusions. The paper proves that there is only one class of anisotropic media containing an open set of singular directions — these are elliptic orthorhombic media with stiffness coefficients c_{44}=c_{55}=c_{66} in different coordinate systems. We found equations for the stiffness coefficients of these media in an arbitrary coordinate system and studied the conditions for the positive definite stiffness matrix. We derive equations for slowness and group velocities surfaces for these media. It is shown that in the canonical coordinate system, the slowness surfaces of S_1 and S_2 waves of these media coincide with each other and are a sphere with a radius $c_{44}^{-1/2}$. The slowness surface of qP wave in the canonical coordinate system is an ellipsoid with semi-axes $c_{11}^{-1/2}, c_{22}^{-1/2}, c_{33}^{-1/2}$.

This paper is purely theoretical and devoted to the definition of anisotropic models with isotropic S_1 and S_2 waves (with the same velocity) and anisotropic (elliptic) qP wave. The polarization vectors for S_1 and S_2 waves can be arbitrarily selected in the plane orthogonal to the polarization vector of qP wave. However, the qP wave polarization vector can be significantly different from the wave vector. This feature should be taken into account in the joint processing and modelling of S and qP waves.

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Анізотропні середовища з сингулярними поверхнями повільності

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У статті доведено, що в анізотропному середовищі з відкритою множиною сингулярних напрямків існують дві поверхні повільності, які повністю збігаються. Відповідне анізотропне середовище є еліптичним орторомбічним (ОРТ) середовищем з однаковими коефіцієнтами пружності c_{44}=c_{55}=c_{66}.

На підставі зображення матриці Крістофеля у вигляді одновісного тензора та врахування, що елементами матриці Крістофеля є квадратичні форми від компонент вектора повільності, складено систему однорідних поліноміальних рівнянь, справедливих для всіх векторів повільності. Тотожну рівність поліномів у системі рівнянь замінено на рівність їх коефіцієнтів. У результаті отримано нову систему рівнянь, коренями якої є значення зведених коефіцієнтів пружності. Досліджено умови позитивного визначення отриманої матриці пружності. Для знайденого анізотропного середовища виведено рівняння Крістофеля та рівняння поверхонь групових швидкостей. Визначено ортогональну матрицю повороту отриманого еліптичного ОРТ-середовища в канонічну систему координат. Показано, що в канонічній системі координат поверхні повільності S_1- та S_2-хвиль збігаються між собою та є сферою з радіусом $c_{44}^{-1/2}$. Поверхня повільності *qP*-хвилі для цього середовища в канонічній системі координат є еліпсоїдом з півосями $c_{11}^{-1/2}$, $c_{22}^{-1/2}$, $c_{33}^{-1/2}$. Вектори поляризації S_1- і S_2-хвиль можна довільно вибирати в площині, ортогональній вектору поляризації *qP*-хвилі. Проте вектор поляризації *qP*-хвилі може істотно відрізнятися від хвильового вектора. Цю особливість слід враховувати в разі спільної обробки та моделювання *S*- і *qP*-хвиль. Результати статті продемонстровані на одному прикладі еліптичного ОРТ-середовища.

Ключові слова: сингулярна точка, сингулярна поверхня, фазова швидкість, матриця Крістофеля, еліптичне орторомбічне середовище.