УДК 550.834 DOI: https://doi.org/10.24028/gj.v47i4.322834

Calculation of Gaussian curvature of the slowness surface in monoclinic media

Yu.V. Roganov*[1], *A. Stovas*[2], *V.Yu. Roganov*[3], *2025

[1]Tesseral Technologies Inc., Kyiv, Ukraine
[2]Norwegian University of Sciences and Technology, Trondheim, Norway
[3]V.M. Glushkov Institute of Cybernetic of National Academy of Sciences of Ukraine, Kyiv, Ukraine
Received 13 February 2025

In this paper, we obtain formulas that determine the type of the slowness surface in the vicinity of a given point (regular or singular) on the vertical axis in a monoclinic medium with a horizontal plane of symmetry. The study uses a method based on a cylindrical coordinate system with the origin at a selected point. In this coordinate system, at a fixed azimuth, the polynomial defining the equation of the slowness surface is expanded into a Taylor series with respect to the distance from the given point. Then, vertical projections of the slowness for different types of waves are found in the form of Taylor series with respect to the distance from the given point. The leading terms of these series determine the shape of the slowness surface in the vicinity of the given point. Gaussians, mean, and principal curvatures are also presented as Taylor series. In this paper, we investigate the leading terms of the Taylor series of Gaussians, mean, and principal curvatures, for regular, double, and triple singular points. It is shown that a double singular point is always a point of the tangential type, i.e., at the double singular point, the slowness surface has a horizontal tangent plane. However, at the singular point, the Gaussian curvature does not exist, and in the vicinity of this point, it depends on the azimuth. Cases with a double singular point, when the leading term of the Gaussian curvature is locally independent of the azimuth, are investigated. The cases are also investigated for which in the vicinity of the singular point or at certain azimuths, the slowness surfaces of S1 and S2 waves are located close to each other. The presented results are demonstrated on two examples of monoclinic media. The analysis can be used to identify the amplitude anomalies in the modelled wavefield in anisotropic media with singularity points, as well as in ray tracing, and solving inverse seismic problems for monoclinic media.

Key words: monoclinic media, Gaussian curvature, singular point, slowness surface, phase velocity, Christoffel matrix.

Introduction. The Gaussian curvature of the slowness surface plays an important role in calculating the geometric spread and amplitudes of the recorded waves. It is also used in ray tracing in elastic media [Gajewski, 1993; Cerveny, 2001; Vavryčuk, 2001; Stovas, 2018; Stovas et al., 2022b]. Gaussian curvature is necessary for solving inverse problems of seismics based on comparing the amplitudes of recorded waves with the amplitudes obtained by modeling. The situation is significantly more complicated if the medium is anisotropic and there are rays with singular directions. In the vicinity of these directions, both the amplitudes and directions of the polarization vectors change rapidly [Alshits, Shuvalov, 1984]. Such anomalies lead to large errors in amplitude comparisons and breakdown the

Citation: Roganov, Yu.V., Stovas, A., & Roganov, V.Yu. (2025). Calculation of Gaussian curvature of the slowness surface in monoclinic media. *Geofizychnyi Zhurnal*, *47*(4), 3—14. https://doi.org/10.24028/gj.v47i4.322834.

ray tracing [Gajewski, Psencik, 1990; Cerveny, 2001; Psencik, Dellinger, 2001].

The classical method for calculating the Gaussian and mean curvatures in the vicinity of a given point on the slowness surface is reduced to calculating the phase velocities using the perturbation method of the eigenvalues of the Christoffel matrix. Then, the principal curvatures of the slowness surface are calculated using the obtained phase velocities [Shuvalov, Every, 1996, 1997, 2000; Shuvalov, 1998]. Shuvalov and Every [1996] obtained formulas for the Gaussian and principal curvatures of the slowness surface in the vicinity of a regular point of a monoclinic medium, as well as the curvature in the vicinity of singular points of anisotropic media with symmetry axes of the third and fourth orders. Stovas et al. [2021, 2022a] analyzed the Gaussian curvature and the principal curvatures in the vicinity of the S1S2 wave singularity point in anelliptic orthorhombic model. The slowness surface shape in symmetry planes near rotational axes has been investigated for a number of symmetry classes [Musgrave, 1957].

Stovas et al. [2025] proposed a method for approximating the slowness surface in the vicinity of a given point (regular or singular) in a triclinic medium by using a cylindrical coordinate system with the origin at a selected point. In this coordinate system, at a fixed azimuth angle, the Christoffel polynomial, defining the equation of the slowness surface, is expanded into a Taylor series with respect to the distance from the given point. Then, vertical projections of the slowness for different types of waves are found in the form of Taylor series with respect to the distance. The leading terms of these series determine the shape of the slowness surface in the vicinity of the given point. The Gaussian, mean, and principal curvatures, are also given in the form of Taylor series.

This paper applies the above calculation scheme to monoclinic media with a horizontal plane of symmetry and a selected point of the slowness surface on the vertical axis. The vertical axis for such a medium is a second-order symmetry axis. There are specific features in curvature computation compared to a triclinic medium. In this paper, the analytical formulas are derived for the leading terms of the Taylor series of the curvatures of the slowness surface in the vicinity of regular, double, and triple singular points. It is shown that the slowness surface in the vicinity of a double singular point has a horizontal tangent plane, i.e., this singular point is of the tangential type of degeneracy. However, the Gaussian curvature does not exist at this singular point. In the vicinity of the singular point, it depends on the azimuth angle; at the singular point, it tends to be discontinuous. We find the criterion when the leading term of the Taylor series of Gaussian curvature in the vicinity of a singular point for S1 and S2 waves is locally azimuthally independent. We also define the classes of monoclinic media, which have the leading term of Taylor series for vertical slowness projection equal for S1 and S2 waves at a certain azimuth angle and for all azimuth angles. The equality of the leading terms results in the closeness of the slowness surfaces for S1 and S2 waves in the vicinity of a singularity point.

The results can be used to solve direct and inverse problems of seismic wave propagation in monoclinic media.

An example of such a monoclinic medium in geophysics is a transversely isotropic medium with several systems of vertical fractures. Ray tracing in such a medium in the vicinity of singular directions requires taking into account the Gaussian curvature [Cerveny, 2001; Vavryčuk, 2001].

Theory. The monoclinic medium with a horizontal plane of symmetry has a vertical axis of symmetry of the second order and is described by the stiffness matrix **C**:

$$\mathbf{C}=\begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{12} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{45} & c_{55} & 0 \\ c_{16} & c_{26} & c_{36} & 0 & 0 & c_{66} \end{pmatrix}. \qquad (1)$$

By rotating the coordinate system around the vertical axis of symmetry, we can always choose an angle (azimuth) at which c_{45}=0 [Fedorov, 1968]. By selecting this coordinate system, we can exclude c_{45} from our analysis. In the following text, we use the notations,

$$d_{12} = c_{12} + c_{66}\,,\; d_{13} = c_{13} + c_{55}\,,$$
$$d_{23} = c_{23} + c_{44}\,,\; \Delta_{mn} = c_{mm} - c_{nn}\,. \qquad (2)$$

The Christoffel matrix of a monoclinic medium for the slowness vector $\mathbf{p}=(p_1, p_2, p_3)^T$ is given by

$$\mathbf{K} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}, \qquad (3)$$

where

$$a_{11} = c_{11}p_1^2 + c_{66}p_2^2 + c_{55}p_3^2 + 2c_{16}p_1p_2\,,$$
$$a_{22} = c_{66}p_1^2 + c_{22}p_2^2 + c_{44}p_3^2 + 2c_{26}p_1p_2\,,$$
$$a_{33} = c_{55}p_1^2 + c_{44}p_2^2 + c_{33}p_3^2\,,$$
$$a_{12} = c_{16}p_1^2 + c_{26}p_2^2 + d_{12}p_1p_2\,,$$
$$a_{13} = d_{13}p_1p_3 + c_{36}p_2p_3\,,$$
$$a_{23} = c_{36}p_1p_3 + d_{23}p_2p_3\,. \qquad (4)$$

The Christoffel equation $F(p_1, p_2, p_3)$= =det(**K**–**I**)=0, where **I** is the identity 3×3 matrix, defines the slowness surfaces for P, S1, and S2 waves.

Since the medium has a vertical axis of symmetry of the second order, the identity is valid

$$F(p_1, p_2, p_3) = F(-p_1, -p_2, p_3)\,. \qquad (5)$$

At p_1=p_2=0, the vertical projection p_3 of the slowness vector satisfies the equation,

$$F(0,0,p_3) =$$
$$= \left(c_{44}p_3^2 - 1\right)\left(c_{55}p_3^2 - 1\right)\left(c_{33}p_3^2 - 1\right) = 0\,. \qquad (6)$$

We consider three cases. In **the first case,** all values (c_{44}, c_{55}, c_{33}) are different. Then, on the vertical axis, there are three regular points $p_{3d} = 1/\sqrt{c_{33}}$, $p_{3d} = 1/\sqrt{c_{44}}$, and $p_{3d} = 1/\sqrt{c_{55}}$. In **the second case,** we assume that c_{33}>c_{55}=c_{44}. In this case, the equation (6) has a double root (a double S1S2 wave singular point on the vertical axis) at $p_3 = 1/\sqrt{c_{44}}$. **The third case** occurs when c_{33}=c_{55}=c_{44}. In this case, the equation (6) has a triple root (a triple PS1S2 wave singular point on the vertical axis) at $p_3 = 1/\sqrt{c_{44}}$.

Let us consider a point of the slowness surface on the vertical axis with coordinates p_{1d}=0, p_{2d}=0, $p_{3d} = p_{3d}^{(\alpha\alpha)} = 1/\sqrt{c_{\alpha\alpha}}$, where α=4 or 5 for S1 and S2 waves and α=3 for *P*-wave.

Let us denote p_1=$p_r$cosφ, p_2=$p_r$sinφ, q_3=p_3– –p_{3d}. The Christoffel equation in variables p_r, q_3, φ takes the form,

$$F(p_1, p_2, p_3) = F_\varphi(p_r, q_3) =$$
$$= Q_{10}q_3 + \left(Q_{20}q_3^2 + Q_{02}p_r^2\right) +$$
$$+ \left(Q_{30}q_3^3 + Q_{12}q_3p_r^2\right) + \qquad (7)$$
$$+ \left(Q_{40}q_3^4 + Q_{22}q_3^2p_r^2 + Q_{04}p_r^4\right) +$$
$$+ \left(Q_{50}q_3^5 + Q_{32}q_3^3p_r^2 + Q_{14}q_3p_r^4\right) +$$
$$+ \left(Q_{60}q_3^6 + Q_{42}q_3^4p_r^2 + Q_{24}q_3^2p_r^4 + Q_{06}p_r^6\right) = 0,$$

where $Q_{m,n}$=$Q_{m,n}$(φ) are the forms of the degree *m*+*n* of the variables sinφ, cosφ. Since $F(0, 0, p_{3d})$=0, then $Q_{0,0}$(φ)=0. Since, –p_1=(–p_r) cosφ, –p_2=(–p_r)sinφ, then from the relations (5) and (7) it follows that the index *n* in the form $Q_{m,n}$(φ) is always even.

Equation $F_\varphi(p_r, q_3)$=0 for fixed φ defines an algebraic curve of degree 6. The leading terms of the series $q_3(p_r,\varphi)$ satisfying the equation $F_\varphi(p_r, q_3)$=0 are defined by the values $Q_{m,n}$(φ) located on the left-hand lower part of the convex hull for points with $Q_{m,n}$(φ)≠0. This convex polygonal line is called the Newton diagram [Walker, 1978]. Fig. 1 shows the Newton diagram for a monoclinic medium with regular (*a*), double (*b*), and triple (*c*) singular points. The slope tangents of the segments of the polygonal line determine the degrees of the leading terms of the Puiseux series [Walker, 1978], and equations for the leading terms are composed by using the points $Q_{m,n}$(φ)≠0 located on the segments.

Let us consider **the first case**, with the point *O*(0,0) being **regular**. One branch of the curve (7) passes through this point and $Q_{1,0}$(φ)≠0 (Fig. 1, *a*). The Newton polygonal line consists of one segment ($Q_{1,0}$, $Q_{0,2}$) with slope tangent 2. In this case, the function $q_3(p_r, \varphi)$ can be represented by a Taylor series in the variable p_r^2 with the leading term r_2(φ)

$$\Delta q_3(p_r, \varphi) = \sum_{k>0} r_{2k}(\varphi)\, p_r^{2k}\,. \qquad (8)$$

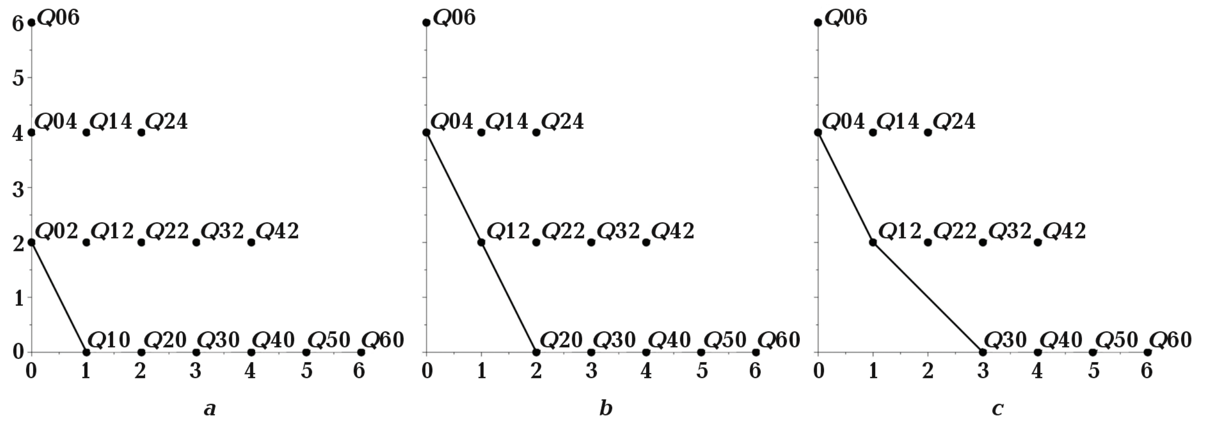


Fig. 1. Non-zero coefficients and the Newton polygonal line for a point $\left(0,0,c_{44}^{-1/2}\right)$ on the slowness surface of a monoclinic medium with a horizontal plane of symmetry: regular point (*a*), double singular point (*b*), triple singular point (*c*).

The $r_2(\varphi)$ satisfies the equation $Q_{1,0}r_2+Q_{0,2}=0$. We substitute the value of q_3 from (8) into the polynomial (7) and group its terms by powers of p_r. Then, consistently equating the coefficients of p_r^{2k} to zero, we find the values of $r_{2k}(\varphi)$:

$$r_2=-\frac{Q_{0,2}}{Q_{1,0}},\quad r_4=-\frac{Q_{0,4}}{Q_{1,0}}+\frac{Q_{0,2}Q_{1,2}}{Q_{1,0}^2}-\frac{Q_{2,0}Q_{0,2}^2}{Q_{1,0}^3},$$

$$r_6=-\frac{Q_6^{(6)}}{Q_{1,0}}+\frac{Q_{0,2}Q_{1,4}+Q_{1,2}Q_{0,4}}{Q_{1,0}^2}-$$

$$-\frac{Q_{0,2}\left(2Q_{2,0}Q_{0,4}+Q_{0,2}Q_{2,2}+Q_{1,2}^2\right)}{Q_{1,0}^3}+$$

$$+\frac{Q_{0,2}^2\left(3Q_{2,0}Q_{1,2}+Q_{0,2}Q_3^{(0)}\right)}{Q_{1,0}^4}-\frac{2Q_{2,0}^2Q_{0,2}^3}{Q_{1,0}^5},\ \ldots \quad (9)$$

If, $p_{3d}=1/\sqrt{c_{44}}$, then

$$Q_{1,0}=\frac{2\Delta_{34}\Delta_{54}}{c_{44}^{3/2}}\neq 0,$$

$$Q_{2,0}=\frac{4c_{44}\left(\Delta_{34}+\Delta_{54}\right)+9\Delta_{34}\Delta_{54}}{c_{44}},$$

$$Q_{0,2}=$$

$$=\frac{\Delta_{54}\left(b_1^{(44)}\cos^2\varphi+2b_{12}^{(44)}\sin\varphi\cos\varphi+b_2^{(44)}\sin^2\varphi\right)}{c_{44}^2},$$

$$r_2=r_2^{(44)}=$$

$$=-\frac{b_1^{(44)}\cos^2\varphi+2b_{12}^{(44)}\sin\varphi\cos\varphi+b_2^{(44)}\sin^2\varphi}{2\Delta_{34}\sqrt{c_{44}}},$$

$$b_1^{(44)}=\Delta_{34}c_{66}-c_{36}^2,$$

$$b_{12}^{(44)}=\Delta_{34}c_{26}-c_{36}d_{23},\ b_2^{(44)}=\Delta_{34}c_{22}-d_{23}^2. \quad (10)$$

If, $p_{3d}=1/\sqrt{c_{55}}$ then

$$Q_{1,0}=\frac{2\Delta_{35}\Delta_{45}}{c_{55}^{3/2}}\neq 0,$$

$$Q_{2,0}=\frac{4c_{55}\left(\Delta_{35}+\Delta_{45}\right)+9\Delta_{35}\Delta_{45}}{c_{55}},$$

$$Q_{0,2}=$$

$$=\frac{\Delta_{45}\left(b_1^{(55)}\cos^2\varphi+2b_{12}^{(55)}\sin\varphi\cos\varphi+b_2^{(55)}\sin^2\varphi\right)}{c_{55}^2},$$

$$r_2=r_2^{(55)}=$$

$$=-\frac{b_1^{(55)}\cos^2\varphi+2b_{12}^{(55)}\sin\varphi\cos\varphi+b_2^{(55)}\sin^2\varphi}{2\Delta_{35}\sqrt{c_{55}}},$$

$$b_1^{(55)}=\Delta_{35}c_{11}-d_{13}^2,$$

$$b_{12}^{(55)}=\Delta_{35}c_{16}-c_{36}d_{13},$$

$$b_2^{(55)}=\Delta_{35}c_{66}-c_{36}^2. \quad (11)$$

If, $p_{3d}=1/\sqrt{c_{33}}$ then

$$Q_{1,0}=\frac{2\Delta_{34}\Delta_{35}}{c_{33}^{3/2}}\neq 0,$$

$$Q_{2,0}=\frac{-4c_{33}\left(\Delta_{34}+\Delta_{35}\right)+9\Delta_{34}\Delta_{35}}{c_{55}},$$

$$Q_{0,2}=\frac{b_1^{(33)}\cos^2\varphi+2b_{12}^{(33)}\sin\varphi\cos\varphi+b_2^{(33)}\sin^2\varphi}{c_{33}^2},$$

$$r_2 = r_2^{(33)} =$$
$$= -\frac{b_1^{(33)}\cos^2\varphi + 2b_{12}^{(33)}\sin\varphi\cos\varphi + b_2^{(33)}\sin^2\varphi}{2\Delta_{34}\Delta_{35}\sqrt{c_{33}}},$$
$$b_1^{(33)} = \Delta_{34}\Delta_{35}c_{55} + \Delta_{34}d_{13}^2 + \Delta_{35}c_{36}^2,$$
$$b_{12}^{(33)} = c_{36}\left(\Delta_{34}d_{13} + \Delta_{35}d_{23}\right),$$
$$b_2^{(33)} = \Delta_{34}\Delta_{35}c_{44} + \Delta_{34}c_{36}^2 + \Delta_{35}d_{23}^2.$$

The leading terms of the Taylor series of the Gaussian, mean, and two principal curvatures of the slowness surface at a regular point p_{1d}=p_{2d}=0, $p_{3d}^{(\alpha)} = \frac{1}{\sqrt{c_{\alpha\alpha}}}$ in the vertical axis are calculated using the equations [Stovas et al., 2025]:

$$G_0 = 2r_2\left(2r_2 + \frac{d^2r_2}{d\varphi^2}\right) - \left(\frac{dr_2}{d\varphi}\right)^2 =$$
$$= \frac{p_{3d}^{(\alpha)\,2}\left(b_1^{(\alpha)}b_2^{(\alpha)} - b_{12}^{(\alpha)\,2}\right)}{\Delta_{3\alpha}^2},$$
$$H_0 = 2r_2 + \frac{1}{2}\frac{d^2r_2}{d\varphi^2} = \frac{-p_{3d}^{(\alpha)}\left(b_1^{(\alpha)} + b_2^{(\alpha)}\right)}{2\Delta_{3\alpha}},$$
$$k_{1,2} = H \pm \sqrt{H^2 - G} = 2r_2 +$$
$$+\frac{1}{2}\frac{d^2r_2}{d\varphi^2} \pm \frac{1}{2}\sqrt{4\left(\frac{dr_2}{d\varphi}\right)^2 + \left(\frac{d^2r_2}{d\varphi^2}\right)^2} = \quad (12)$$
$$= -\frac{p_{3d}^{(\alpha)}}{2\Delta_{3\alpha}}\left(b_1^{(\alpha)} + b_2^{(\alpha)} \pm \sqrt{\left(b_1^{(\alpha)} - b_2^{(\alpha)}\right)^2 + 4b_{12}^{(\alpha)\,2}}\right).$$

The curvatures at a regular point are well-defined and do not depend on azimuth angle φ.

In a particular case, when $p_{3d}^{(\alpha)} = \frac{1}{\sqrt{c_{44}}}$, from (12) we obtain

$$G_0 = \frac{b_1^{(44)}b_2^{(44)} - \left(b_{12}^{(44)}\right)^2}{c_{44}\Delta_{34}^2},$$
$$H_0 = -\frac{b_1^{(44)} + b_2^{(44)}}{2\Delta_{34}\sqrt{c_{44}}}. \quad (13)$$

In **the second case,** the value $p_3 = p_{3d}^{(\alpha)}$ is a double root of the equation $F(0,0,p_3)$=0. We assume that c_{33}>c_{55}=c_{44} and $p_{3d}^{(\alpha)} = 1/\sqrt{c_{44}}$. In this case, $Q_{1,0}$=$Q_{0,2}$=0, $Q_{2,0}$=4Δ_{34}≠0,

$$Q_{3,0} = 4\sqrt{c_{44}}\left(3\Delta_{34} + 2c_{44}\right),$$
$$Q_{1,2} = -4\Delta_{34}\left(r_2^{(44)} + r_2^{(55)}\right),$$

and

$$Q_{0,4} = 4\Delta_{34}r_2^{(44)}r_2^{(55)} - \frac{1}{c_{44}\Delta_{34}}\left(b_{12}^{(44)}\sin^2\varphi +\right.$$
$$\left.+b_{12}^{(55)}\cos^2\varphi + x_{12}\sin\varphi\cos\varphi\right)^2,$$
$$x_{12} = d_{12}\Delta_{34} - c_{36}^2 - d_{13}d_{23}.$$

The quadratic form $Q_{20}q_3^2$ is the leading term of the series (polynomial) $F_\varphi(p_r, q_3)$ from (7). Therefore, at the singular point $p_{3d}^{(\alpha)} = 1/\sqrt{c_{44}}$, the slowness surfaces of S1 and S2 waves have a horizontal tangent plane, and this singular point is a point of tangential type of degeneracy (see Fig. 5).

For a double singular point $p_{3d} = p_{3d}^{(\alpha)}$, the values $r_k(\varphi)$ are derived by the same principle as for a single root. In this case, the Newton polygonal line consists of one segment ($Q_{0,4}$, $Q_{1,2}$, $Q_{2,0}$) of length 2 with a slope tangent of 2 (Fig. 1, *b*). Therefore, the value $r_2(\varphi)$ is found from the quadratic equation

$$Q_{2,0}r_2^2 + Q_{1,2}r_2 + Q_{0,4} = 0. \quad (14)$$

The two roots of the equation (14) are given by

$$r_2 = \frac{-Q_{1,2} \pm \sqrt{Q_{1,2}^2 - 4Q_{2,0}Q_{0,4}}}{2Q_{2,0}} = \frac{r_2^{(44)} + r_2^{(55)}}{2} +$$
$$\pm\frac{1}{2}\sqrt{\left(r_2^{(44)} - r_2^{(55)}\right)^2 + \frac{1}{c_{44}\Delta_{34}^2}\times \quad (15)$$
$$\overline{\times\left(b_{12}^{(44)}\sin^2\varphi + b_{12}^{(55)}\cos^2\varphi + x_{12}\sin\varphi\cos\varphi\right)^2}.$$

The parameter r_2=$r_2(\varphi)$ from (15), taken with the sign (–) or (+), is, respectively, the leading term of the Taylor series (8) for S1 and S2 waves of the slowness surface at a fixed azimuth φ at the singular point. The remaining coefficients $r_{2k}(\varphi)$, k >1 are uniquely defined by the recurrent equations,

$$r_4 = -\frac{Q_{3,0}r_2^3 + Q_{2,2}r_2^2 + Q_{1,4}r_2 + Q_{0,6}}{2Q_{2,0}r_2 + Q_{1,2}},$$

$$r_6 = -\frac{Q_{2,0}r_4^2 + \left(3Q_{3,0}r_2^2 + 2Q_{2,2}^{(2)}r_2 + Q_{1,4}\right)r_4}{2Q_{2,0}r_2 + Q_{1,2}} + $$
$$+\frac{r_2^2\left(Q_{4,0}r_2^2 + Q_{3,2}r_2 + Q_{2,4}\right)}{2Q_{2,0}r_2 + Q_{1,2}}, \ldots \quad (16)$$

The formulas (16) make sense only under the condition $2Q_{2,0}r_2+Q_{1,2}\neq 0$, which is equivalent to the equation (14) having no multiple roots.

If **for certain** φ, the discriminant of the equation (14) is zero, then the equalities $r_2^{(44)} = r_2^{(55)}$ and $b_{12}^{(44)}\sin^2\varphi + b_{12}^{(55)}\cos^2\varphi + x_{12}\sin\varphi\cos\varphi = 0$ are satisfied. In this case, the equation (15) has a double root $r_2 = r_2^{(44)} = r_2^{(55)}$, and the denominators of the expressions (16) are zero, i.e., $2Q_{2,0}r_2+Q_{1,2}=0$ and $Q_{1,2}^2 = 4Q_{2,0}Q_{0,4}$. The numerators of (16) are also zero. Therefore, r_2 additionally satisfies the equation

$$Q_{3,0}r_2^3 + Q_{2,2}r_2^2 + Q_{1,4}r_2 + Q_{0,6} = 0, \quad (17)$$

and r_4 is defined from the quadratic equation ($Q_{2,0}=4\Delta_{34}\neq 0$),

$$Q_{2,0}r_4^2 + \left(3Q_{3,0}r_2^2 + 2Q_{2,2}^{(2)}r_2 + Q_{1,4}\right)r_4 +$$
$$+r_2^2\left(Q_{4,0}r_2^2 + Q_{3,2}r_2 + Q_{2,4}\right) = 0. \quad (18)$$

If the equation (18) has different roots, then the high-order $r_k(\varphi)$, $k>4$ are uniquely defined by equating to zero coefficients at p_r^k, $k>8$ after substitution of q_3 from (8) into the formula (7). The system of equations $r_2^{(44)} = r_2^{(55)}$ and $b_{12}^{(44)}\sin^2\varphi + b_{12}^{(55)}\cos^2\varphi + x_{12}\sin\varphi\cos\varphi = 0$ has a common root φ if and only if the equality (19) is satisfied,

$$\text{resultant}\left(\left(b_1^{(44)} - b_1^{(55)}\right)t^2 + 2\left(b_{12}^{(44)} - b_{12}^{(55)}\right)t + \right.$$
$$\left. +\left(b_2^{(44)} - b_2^{(55)}\right), b_{12}^{(55)}t^2 + x_{12}\,t + b_{12}^{(44)}, t\right) = 0. \quad (19)$$

For example, the equality (19) will be satisfied if for any φ we find the elastic constants c_{11} and c_{16} from the system of two equations $r_2^{(44)} = r_2^{(55)}$ and $b_{12}^{(44)}\sin^2\varphi + b_{12}^{(55)}\cos^2\varphi + x_{12}\sin\varphi\cos\varphi = 0$.

If the discriminant of the equation (14) is zero **for all** φ, then the following equalities are satisfied,

$$b_{11}^{(44)} = b_{11}^{(55)},\ b_{12}^{(44)} = b_{12}^{(55)} = 0,$$
$$b_{22}^{(44)} = b_{22}^{(55)},\ x_{12} = 0, \quad (20)$$

and it follows that

$$\Delta_{34}c_{26} = d_{23}c_{36},\ \Delta_{34}c_{16} = d_{13}c_{36},$$
$$\Delta_{34}d_{12} = c_{36}^2 + d_{13}d_{23},$$
$$\Delta_{34}\Delta_{16} = d_{13}^2 - c_{36}^2,\ \Delta_{34}\Delta_{26} = d_{23}^2 - c_{36}^2. \quad (21)$$

In this case, r_2 it does not depend on the azimuth and is given by

$$r_2 = \frac{c_{36}^2 - \Delta_{34}c_{66}}{2\Delta_{34}\sqrt{c_{44}}}. \quad (22)$$

The coefficient r_4 takes two values, with one value being azimuthally independent,

$$r_{4,1} = -\frac{\left(b_1^{(44)}\right)^2}{8\Delta_{34}^2\sqrt{c_{44}}},\ r_{4,2} = -\frac{\left(b_1^{(44)}\right)^2}{8\Delta_{34}^2\sqrt{c_{44}}} -$$
$$-\frac{\left(c_{44}\,\Delta_{34} - b_1^{(44)}\right)}{2\Delta_{34}^3\sqrt{c_{44}}}\times \quad (23)$$
$$\times\frac{\left[\left(d_{13}\cos\varphi + c_{36}\sin\varphi\right)^2 + \left(d_{23}\sin\varphi + c_{36}\cos\varphi\right)^2\right]}{2\Delta_{34}^3\sqrt{c_{44}}}.$$

For a double singular point of a monoclinic medium with $p_{3d}^{(\alpha)} = 1/\sqrt{c_{44}}$, the leading terms of the series of Gaussian, mean, and two principal curvatures of the slowness surface are computed using the formulas [Stovas et al., 2025]:

$$G_0 = 2r_2\left(2r_2 + \frac{d^2r_2}{d\varphi^2}\right) - \left(\frac{dr_2}{d\varphi}\right)^2,$$
$$H_0 = 2r_2 + \frac{1}{2}\frac{d^2r_2}{d\varphi^2},$$
$$k_{1,2} = 2r_2 + \frac{1}{2}\frac{d^2r_2}{d\varphi^2} \pm \sqrt{\left(\frac{dr_2}{d\varphi}\right)^2 + \frac{1}{4}\left(\frac{d^2r_2}{d\varphi^2}\right)^2}, \quad (24)$$

In these formulas, the value r_2 is taken from (15).

For a singular point, the values G_0 and H_0 are azimuthally dependent. This dependence arises because the curvatures cannot be correctly determined at the singular point. However, the values G_0 and H_0 correctly describe the curvatures in the vicinity of the singular point for each φ at $p_r \to 0$, $p_r \neq 0$. For a monoclinic medium, the graph of the function $p_3 = p_3(p_1, p_2)$ always has a horizontal tangent plane at the singular point, since $q_3 = r_2(\varphi)\, p_r^2 + o\left(p_r^2\right)$.

Differential equations (24) for constant values of G_0 and H_0 have solutions $r_2(\varphi) = c_1\cos(2\varphi) + c_2\sin(2\varphi) + c_3$, where c_1, c_2, c_3 are constant. In this case, the leading terms of the curvatures at the singular point are found by the formulas $H_0 = c_3$, $G_0 = 4\left(c_3^2 - c_1^2 - c_2^2\right)$, $k_{1,2} = 2\left(c_3 \pm \sqrt{c_1^2 + c_2^2}\right)$. The values of G_0 and H_0 do not depend on the azimuth φ only if the radical expression in (15) is a perfect square. This condition is reduced to a system of two equations for the stiffness coefficients of a monoclinic medium (Appendix). One of the solutions of this system, providing the independence of the curvatures from the azimuth angle, is given by the formulas,

$$c_{11} = c_{66} + \frac{2b_{12}^{(55)}\left(b_{12}^{(55)} - b_{12}^{(44)}\right)}{\Delta_{34} x_{12}} + \frac{d_{13}^2 - c_{36}^2}{\Delta_{34}},$$
$$c_{22} = c_{66} + \frac{2b_{12}^{(44)}\left(b_{12}^{(44)} - b_{12}^{(55)}\right)}{\Delta_{34} x_{12}} + \frac{d_{23}^2 - c_{36}^2}{\Delta_{34}}. \quad (25)$$

For a monoclinic medium with the equations (25) are valid, the curvatures of S1 and S2 waves at the singular point are not correctly defined. However, they are constant on the azimuth intervals φ at $p_r \to 0$ and $p_r \neq 0$.

In **the third case,** $p_3 = p_{3d}^{(44)}$ is a triple root of the equation $F(0,0,p_3)=0$. In this case, $c_{33} = c_{55} = c_{44}$, $p_{3d}^{(\alpha)} = 1/\sqrt{c_{44}}$, $Q_{1,0} = Q_{2,0} = Q_{0,2} = 0$, $Q_{3,0} = 8c_{44}^{3/2} \neq 0$, and

$$Q_{1,2} = -\frac{2\left[\left(d_{13}\cos\varphi + c_{36}\sin\varphi\right)^2 + \left(d_{23}\sin\varphi + c_{36}\cos\varphi\right)^2\right]}{\sqrt{c_{44}}}.$$

For a triple singular point $p_3 = p_{3d}^{(\alpha)}$, the Newton polygonal line consists of two segments ($Q_{0,4}$, $Q_{1,2}$) and ($Q_{1,2}$, $Q_{3,0}$) of length 1 and 2 with slope tangent of 2 and 1, respectively (Fig. 1, *c*). Therefore, three branches of the curve $F_\varphi(p_r, q_3)=0$ pass through the origin of the coordinate system, and the leading terms of the Taylor series of the function $q_3(p_r, \varphi)$ satisfy one of the equations,

$$Q_{3,0} r_1^2 + Q_{1,2} = 0, \quad (26)$$

$$Q_{1,2} r_2 + Q_{0,4} = 0. \quad (27)$$

The coefficients $r_1(\varphi)$ are the roots of the equation (26) correspond to $P(-)$ and $S2(+)$ waves,

$$r_1 = \pm\sqrt{-\frac{Q_{1,2}}{Q_{3,0}}} = \quad (28)$$
$$= \pm\frac{\sqrt{\left(d_{13}\cos\varphi + c_{36}\sin\varphi\right)^2 + \left(d_{23}\sin\varphi + c_{36}\cos\varphi\right)^2}}{2c_{44}}.$$

The higher order coefficients $r_k(\varphi)$, $k>1$ are computed by consistently equating to zero the coefficients at p_r^{k+2} in $F_\varphi(p_r, q_3)$,

$$r_2 = -\frac{Q_{0,4} r_1^4 + Q_{2,2} r_1^2 + Q_{4,0}}{3Q_{3,0} r_1^2 + Q_{1,2}} = \frac{Q_{0,4}}{2Q_{1,2}} - \frac{Q_{2,2}}{2Q_{3,0}} + \frac{Q_{4,0} Q_{1,2}}{2Q_{3,0}^2},$$

$$r_3 = \frac{r_1}{8}\left(\frac{3Q_{0,4}^2 Q_{3,0}}{Q_{1,2}^3} - \frac{2Q_{0,4} Q_{2,2}}{Q_{1,2}^2} + \frac{4Q_{1,4}}{Q_{1,2}} - \frac{2Q_{0,4} Q_{4,0} + 4Q_{1,2} Q_{3,2} + Q_{2,2}^2}{Q_{1,2} Q_{3,0}} + \frac{4Q_{1,2} Q_{5,0} + 6Q_{2,2} Q_{4,0} + Q_{2,2}^2}{Q_{3,0}^2} - \frac{5Q_{4,0}^2 Q_{1,2}}{Q_{3,0}^3}\right). \quad (29)$$

If the leading term of the Taylor series satisfies the equation (27), which corresponds to S1 wave, we consistently find:

$$r_2 = -\frac{Q_{0,4}}{Q_{1,2}},$$

$$r_4 = -\frac{Q_{0,6}}{Q_{1,2}} + \frac{Q_{1,4}Q_{0,4}}{Q_{1,2}^2} - \frac{Q_{2,2}Q_{0,4}^2}{Q_{1,2}^3} + \frac{Q_{3,0}Q_{0,4}^3}{Q_{1,2}^4}. \quad (30)$$

The leading terms in the Taylor series for Gaussian and mean curvatures of *P* and S2 wave slowness surfaces in the vicinity of the triple PS1S2 wave singularity point can be computed using equations (A14) from Stovas et al. [2025] by substituting value of $r_1(\varphi)$ from equation (28) and $r_2(\varphi)$, $r_3(\varphi)$ from equation (29). To compute the leading terms in the Taylor series for Gaussian and mean curvatures of S1 wave slowness surface in the vicinity of the triple PS1S2 wave singularity point, we have to use equations (24) and substitute value of $r_2(\varphi)$ from (30). Expression for $Q_{0,4}$ is the quartic form in variables sinφ, cosφ. It results in algebraically complex equations for leading terms of curvatures, which are not shown in this paper.

In Fig. 1, the dots show the location of the non-zero coefficients $Q_{m,n}$ of the Christoffel polynomial (7) in the plane with coordinates (m, n) for regular (a), double (b) and triple (c) singular points. Polygonal line shows the convex hull of points with $Q_{m,n}(\varphi)\neq 0$ (the Newton diagrams).

Numerical examples. In the numerical example and corresponding figures, the reduced stiffness coefficients are given in km^2/s^2, and the velocities are given in km/s. To illustrate the above statements, we consider two anisotropic monoclinic media, M1 and M2, with the reduced (density-normalized) stiffness coefficients from Table. For model M1, the leading term of the Gaussian curvature of the slowness surface S1 and S2 waves at the singular point on the vertical axis is locally azimuthally independent, while for model M2, it is azimuthally dependent everywhere. The values of the elastic constants c_{11}, c_{22} in model M1 satisfy the formulas (25).

Stiffness coefficients c_{ij} of two monoclinic anisotropy models with a double tangential (S1S2) singularity point on the vertical axis. The remaining nine stiffness coefficients are zero

Models, c_{ij}	c_{11}	c_{12}	c_{13}	c_{16}	c_{22}	c_{23}	c_{26}	c_{33}	c_{36}	c_{44}	c_{55}	c_{66}
M1	6.768	3.6	2.25	0.2	7.033	2.4	0.3	5.9375	0.4	0.2	0.2	2.182
M2	9	3.6	2.25	0.2	9.84	2.4	0.3	5.9375	0.4	0.2	0.2	2.182

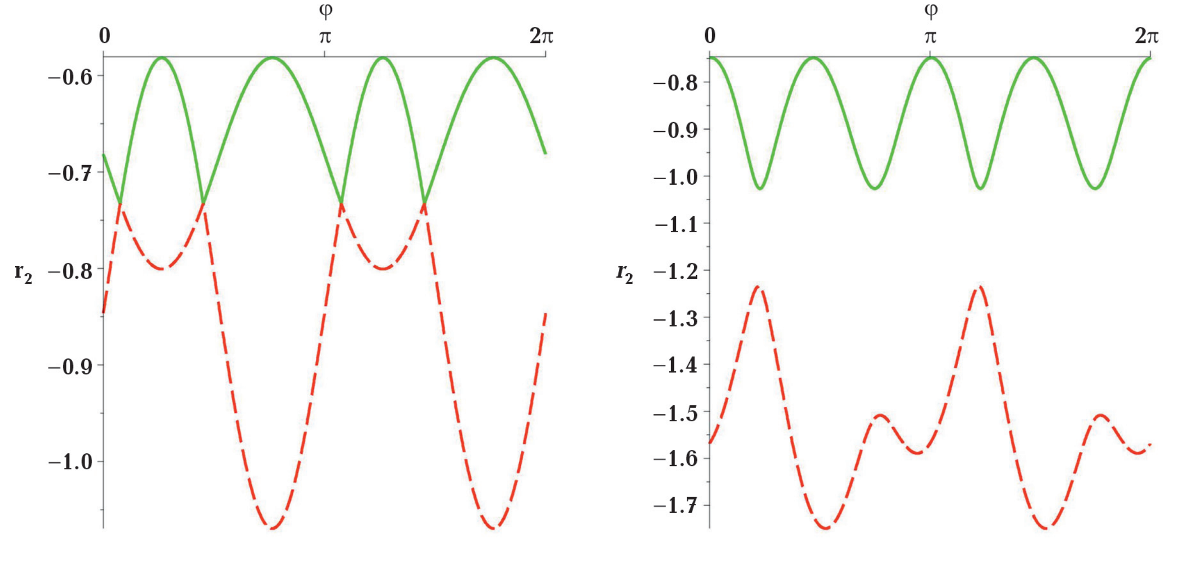
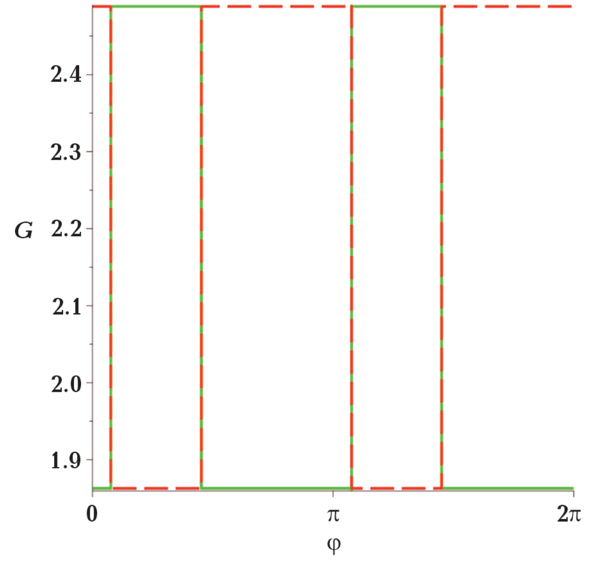
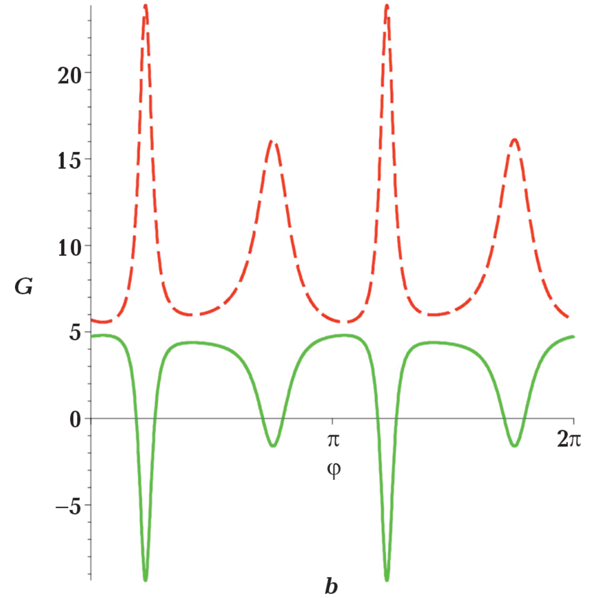


Fig. 2. Leading Taylor series coefficient $r_2(\varphi)$ of the vertical component of the slowness surface S1 (dashed, red) and S2 (solid, green) waves in the vicinity of a double singular point on the vertical axis for the monoclinic medium for models M1 (a), M2 (b).

Fig. 2, *a*, *b* show the azimuthal dependence of the leading coefficient $r_2(\varphi)$ of the Taylor series (8) for monoclinic media M1 and M2, respectively. The graphs of the functions $r_2(\varphi)$ for the S1 wave are shown in green with a solid line and for the S2 wave, they are shown in red with a dashed line. For the M1 model, the graphs $r_2(\varphi)$ are locally shifted sinusoids with argument 2φ. The sinusoids intersect at azimuths for which the values $r_2(\varphi)$ for the S1 and S2 waves coincide. At these azimuths, the vertical projections of the slowness surfaces of the S1 and S2 waves differ from each other only in the fourth-order term in p_r. For the M2 model (Fig. 2, *b*), the graphs $r_2(\varphi)$ are complex functions of φ. In this case, the curve for the S2 wave is above the curve for the S1 wave. The slowness surfaces of the S1 and S2 waves for the M2 model do not intersect.



a



b

Fig. 3. Leading terms of the Gaussian curvature of the slowness surfaces S1 (dashed, red) and S2 (solid, green) waves in the vicinity of a double singular point on the vertical axis for a monoclinic medium for models M1 (*a*), M2 (*b*).

Fig. 3, *a*, *b* show the azimuthal dependence of the leading term $G_0(\varphi)$ of the Gaussian curvature of the slowness surface for monoclinic media M1 and M2. The graphs of the functions $G_0(\varphi)$ for the S1 wave are shown in green and with a solid line, and those for the S2 wave are shown in red and with a dashed line. The leading terms of the Gaussian curvature are constant for the M1 model within the azimuths at which the values $r_2(\varphi)$ for the S1 and S2 waves coincide (Fig. 3, *a*). In this case, the function $r_2(\varphi)$ is given by a single linear trigonometric function of 2φ.The leading terms of the Gaussian curvature for the M2 model (Fig. 3, *b*) depend on the azimuth everywhere and are completely separated from each other: in the vicinity of the singular point, the Gaussian curvature for the S1 wave is greater than the Gaussian curvature for the S2 wave.

Fig. 4, *a*, *b* show horizontal sections of the slowness surfaces of S1 waves (internal, red) and S2 waves (external, green) in the vicinity of a singular point for monoclinic media M1 and M2, respectively. The sections are constructed taking into account only the leading term of the series (8). Fig. 4, *a* shows how the curves of S1 and S2 waves intersect for M1. However, if all terms of the series (8) are taken into account, the curves of S1 and S2 waves will be slightly spaced, and the angle of their intersection will be smoothed. For M2, the curves of S1 and S2 waves are spaced everywhere.

Fig. 5, *a*, *b* show the slowness surfaces S1 (internal, red) and S2 (external, green) of waves in the vicinity of a singular point on the vertical axis for monoclinic media M1 and M2, respectively. The slowness surfaces of S1 and S2 waves for both models have a common

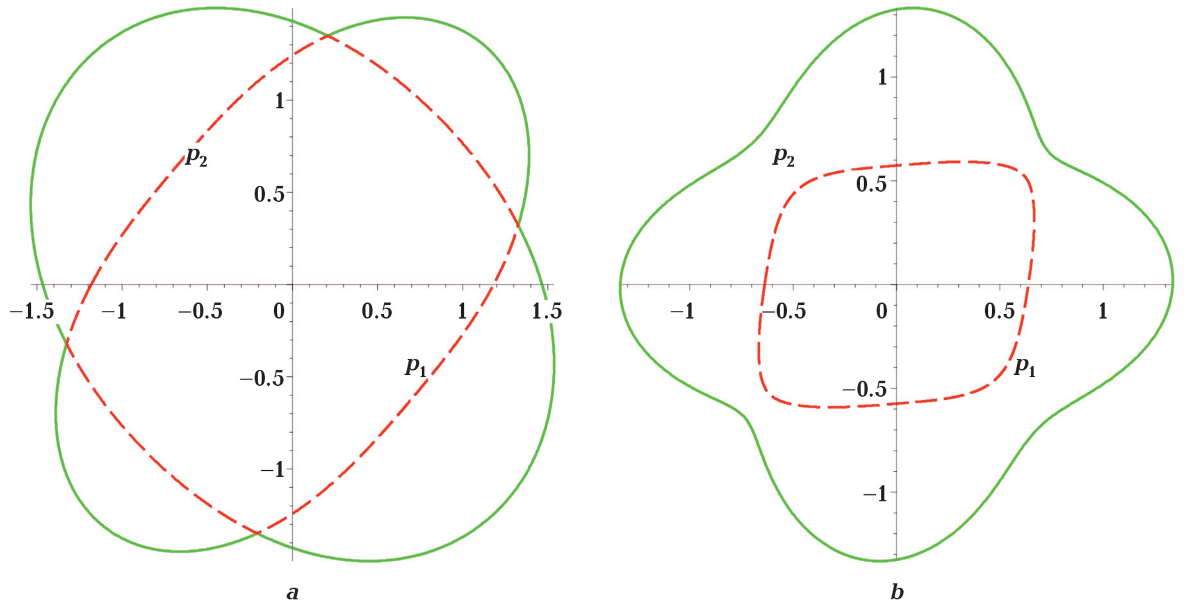


Fig. 4. Horizontal slice of the slowness surfaces S1 (dashed, red) and S2 (solid, green) waves in the vicinity of a double singular point, constructed using the leading term of the Taylor series of function $q_3(p_r,\varphi)=r_2(\varphi)\,p_r^2+o\left(p_r^3\right)$ for a monoclinic medium for models M1 (*a*), M2 (*b*).

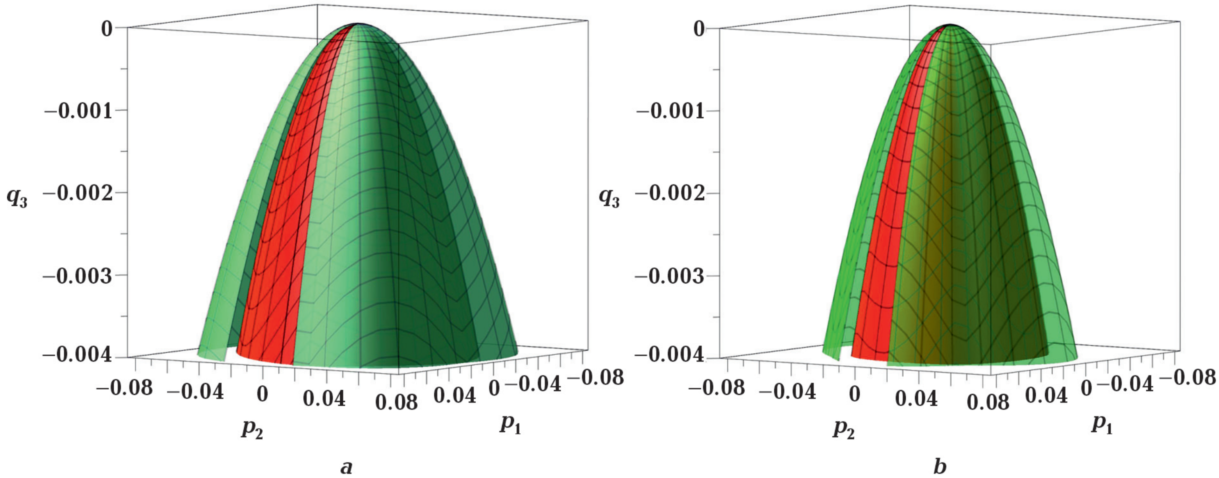


Fig. 5. Slowness surfaces of S1 (internal, red) and S2 (external, green) waves in the vicinity of a double singular point on the vertical axis for a monoclinic medium for models M1 (*a*), M2 (b). The slowness surfaces have a common horizontal tangent plane at the singular point.

horizontal tangent plane at the singular point, which is a tangential-type point.

Conclusions. We define the leading terms of Gaussian, mean, and principal curvatures of the slowness surfaces in the vicinity of regular and singularity points (double and triple) on the vertical axis for monoclinic media with a horizontal plane of symmetry. To compute these terms in curvatures, we use the Christoffel polynomial series and the cylindrical coordinate system associated with the singularity point. In the case of the double singularity point, we always get the tangential class of degeneracy. However, the Gaussian curvature does not exist at the singularity point, and in the vicinity of this point, it depends on the azimuth. The cases of monoclinic media are investigated for which the leading term of the

Gaussian curvature is locally independent of the azimuth. The cases are also investigated for which, in the vicinity of the singularity point or at certain azimuths, the slowness surfaces of S1 and S2 waves are close to each other. The results are demonstrated on two examples of monoclinic media. They can be used to identify the amplitude anomalies in the modelled wave field in monoclinic media with singularity points.

Appendix. Equality $a_4x^4+a_3x^3y+a_2x^2y^2+a_1xy^3+$ $+a_0y^4=(b_2x^2+b_1xy+b_0y^2)^2$ is valid if and only if

$$a_1\sqrt{a_4}=a_3\sqrt{a_0}\,,\ a_2=\frac{a_3^2}{4a_4}+2\sqrt{a_0a_4}\,,$$

$$b_0^2=a_0\,,\ b_2^2=a_4\,,\ b_1=\frac{a_3}{2\sqrt{a_4}}=\frac{a_1}{2\sqrt{a_0}}\,. \quad (31)$$

This statement is proved by substituting x, y=1, 0, –1 into the equality.

In our case, $\left(b_2x^2+b_1x\ y\ +b_0y^2\right)^2=f_1^2+f_2^2$, where are f_1, f_2 — the quadratic forms in variables x, y (see the radical expression in (15)). Therefore, if $b_2x_1^2+b_1x_1y_1+b_0y_1^2=0$ for some x_1, y_1, then this form is the product of two linear forms and is zero also for other values x_2, y_2 (which may coincide with x_1, y_1). Therefore, $f_1(x_1, y_1)=f_1(x_2, y_2)=f_2(x_1, y_1)=f_2(x_2, y_2)=0$ and the three forms $f_1(x, y), f_2(x, y)$, $b_2x^2+b_1xy+b_0y^2$ are proportional to each other. It follows that in (15)

$$c_{11}=c_{66}+\frac{2b_{12}^{(55)}\left(b_{12}^{(55)}-b_{12}^{(44)}\right)}{\Delta_{34}x_{12}}+\frac{c_{36}^2-d_{13}^2}{\Delta_{34}},$$

$$c_{22}=c_{66}+\frac{2b_{12}^{(44)}\left(b_{12}^{(44)}-b_{12}^{(55)}\right)}{\Delta_{34}x_{12}}+\frac{c_{36}^2-d_{23}^2}{\Delta_{34}}\,. \quad (32)$$

On the contrary, if the equalities (32) are satisfied, then the above forms are proportional, and the radical expression in (15) is a perfect square.

References

Alshits, V.I., & Shuvalov, A.L. (1984). Polarization fields of elastic waves near the acoustic axes: *Soviet Physics. Crystallography*, *29*, 373—378.

Cerveny, V. (2001). *Seismic ray theory*. Cambridge Univ. Press. https://doi.org/10.1017/CBO9780511529399.

Fedorov, F.I. (1968). *Theory of Elastic Waves in Crystals*. New York: Plenum Press, 375 p.

Gajewski, D. (1993). Radiation from point sources in general anisotropic media. *Geophysical Journal International*, *113*(2), 299—317. https://doi.org/10.1111/j.1365-246X.1993.tb00889.x.

Gajewski, D., & Psencik, I. (1990).Vertical seismic profile synthetics by dynamic ray tracing in laterally varying layered anisotropic structures. *Journal of Geophysics Research*, *95*(11), 301—315. https://doi.org/10.1029/JB095IB07P11301.

Musgrave, M.J.P. (1957). On whether elastic wave surfaces possess cuspidal edges. *Mathematical Proceedings of the Cambridge Philosophical Society*, *53*(4), 897—906. https://doi.org/10.1017/S0305004100033028.

Stovas, A. (2018). Geometrical spreading in orthorhombic media. *Geophysics*, *83*(1), C61—C73. https://doi.org/10.1190/geo2016-0710.1.

Stovas, A., Roganov, Yu., & Roganov, V. (2022a). Behavior of S waves in vicinity of singularity point in elliptic orthorhombic media. *Geophysics*, *87*(4), C77—C97. https://doi.org/10.1190/geo2021-0522.1.

Stovas, A., Roganov, Yu., & Roganov, V. (2025). Gaussian curvature of the slowness surface in vicinity of singularity point in anisotropic media. *Geophysical Journal International*, *240*(3), 1917—1934. https://doi.org/10.1093/gji/ggaf011.

Stovas, A., Roganov, Yu., & Roganov, V. (2022b). The S waves geometrical spreading in elliptic orthorhombic media. *Geophysical Prospecting*, *70*(7), 1085—1092. https://doi.org/10.1111/1365-2478.13212.

Stovas, A., Roganov, Yu., & Roganov, V. (2021). Wave characteristics in elliptical orthorhombic medium. *Geophysics*, 86(3), C89—C99. https://doi.org/10.1190/geo2020-0509.1.

Shuvalov, A.L. (1998). Topological features of the polarization fields of plane acoustic waves in anisotropic media. *Proceedings of the Royal Society of London, Ser. A, 454*, 2911—2947. http://dx.doi.org/10.1098/rspa.1998.0286.

Shuvalov, A.L., & Every, A.G. (1996). Curvature of acoustic slowness surface of anisotropic solids near symmetry axes. *Physical Review B, 53*, 14906—14916. https://doi.org/10.1103/PhysRevB.53.14906.

Shuvalov, A.L., & Every, A.G. (1997). Shape of the acoustic slowness surface of anisotropic solids near points of conical degeneracy. *The Journal of the Acoustical Society of America, 101*(4), 2381—2382. https://doi.org/10.1121/1.418251.

Shuvalov, A.L., & Every, A.G. (2000). Transverse curvature of the acoustic slowness surface in crystal symmetry planes and associated phonon focusing cusps. *Journal of Acoustic Society of America, 108*(5), 2107—2113. https://doi.org/10.1121/1.1315292.

Vavryčuk, V. (2001). Ray tracing in anisotropic media with singularities. *Geophysical Journal International, 145*, 265—276. https://doi.org/10.1046/j.0956-540x.2001.01387.x.

Walker, R.J. (1978). *Algebraic Curves*. 2nd Edition, Springer, 432.

Обчислення кривизни Гауса поверхні повільності для моноклінних середовищ

Ю.В. Роганов[1], А. Стовас[2], В.Ю. Роганов[3], 2025

[1]Tesseral Technologies Inc., Київ, Україна
[2]Норвезький університет природних і технічних наук, Тронхейм, Норвегія
[3]Інститут кібернетики імені В.М. Глушкова НАН України, Київ, Україна

Отримано формули, що визначають тип поверхні повільності в околі заданої точки (регулярної чи сингулярної) на вертикальній осі в моноклінному середовищі з горизонтальною площиною симетрії. У дослідженні використано метод, заснований на використанні циліндричної системи координат з початком координат у вибраній точці. У цій системі координат при фіксованому азимуті поліном, що визначає рівняння поверхні повільності, розкладається в ряд Тейлора за відстанню від даної точки. Потім знаходяться вертикальні проєкції повільності для різних типів хвиль у вигляді рядів Тейлора за відстанню від даної точки. Початкові члени цих рядів визначають форму поверхні повільності в околі даної точки. Кривизни Гауса, середня та головна, також представлені рядами Тейлора. У цій статті досліджено початкові члени рядів Тейлора кривизн Гауса, середньої та головної, для регулярних, подвійних і потрійних особливих точок. Показано, що подвійна особлива точка завжди є точкою дотичного типу, тобто в подвійній особливій точці поверхня повільності має горизонтальну дотичну площину. Однак в особливій точці кривизни Гауса не існує, а в околі цієї точки вона залежить від азимута. Досліджено випадки з подвійною особливою точкою, коли початковий член кривизни Гауса локально не залежить від азимута, а також випадки, коли в околі особливої точки або на певних азимутах поверхні повільності *S*1 і *S*2 хвиль розташовані близько одна до одної. Представлені результати продемонстровано на двох прикладах моноклінних середовищ. Аналіз, виконаний у цій статті, може бути використаний для ідентифікації аномалій амплітуди, викликаних моделюванням хвильового поля в анізотропному середовищі з особливими точками, а також при трасуванні променів і розв'язанні зворотних сейсмічних задач для моноклінних середовищ.

Ключові слова: моноклінне середовище, кривизна Гауса, особлива точка, поверхня повільності, фазова швидкість, матриця Крістоффеля.