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V.M. Simulik[1], I.Yu. Krivsky[1], I.L. Lamer[1], T.M. Zajac[2]
[1]Institute of Electron Physics, Ukr. Nat. Acad. Sci., 21 Universitetska Str., 88017 Uzhhorod, Ukraine, e-mail: *vsimulik@gmail.com*
[2]Department of Electronic Systems, Uzhhorod National University, 13 Kapitulna Str., 88000, Uzhhorod, Ukraine

ON THE GROUP-THEORETICAL FOUNDATIONS FOR FERMI – BOSE DUALITY OF THE SPINOR FIELD. SOLUTIONS AND CONSERVATION LAWS

The property of Fermi – Bose duality of the Dirac equation with nonzero mass is proved on the basis of generalized Clifford – Dirac algebra and group-theoretical analysis of the spinor field. The Foldy – Wouthuysen representation is used as a starting ground for consideration and the results for standard Dirac formalism are proved to be the simple consequences of such consideration. It is shown that the Dirac equation can describe not only fermionic but also the bosonic states. The proof of our assertion is given on the example of the existence of bosonic symmetries and solutions of the Dirac equation. Furthermore, the bosonic conservation laws are proved to be the consequences of the Dirac equation.

Key words: the spinor field, the Dirac equation, the Foldy – Wouthuysen representation, fermions, bosons.

1. Introduction

The Fermi – Bose (FB) duality of the spinor field has been mentioned at first by L. Foldy [1] The extended consideration has been given in [2]. P. Garbaczewski proved [2] that the Fock space $\mathscr{H}^{\mathrm{F}}\left(\mathbb{H}^{3,\mathrm{M}}\right)$ over the quantum mechanical space $\mathrm{L}_2(\mathrm{R}^3)\otimes\mathbb{C}^{\otimes\mathrm{M}}$ of the particle, which is described by the field ϕ: $\mathrm{M}(1,\mathrm{N})\rightarrow\mathbb{C}^{\otimes\mathrm{N}}$, allows to fulfill the dual, Fermi – Bose (FB), quantization of the field ϕ in $\mathscr{H}^{\mathrm{F}}$. And both the canonical commutation relations (CCR) and anticommutation relations (CAR) were used for the realization of the above mentioned quantization. Moreover, for the both types of quantization the uniqueness of the vacuum in $\mathscr{H}^{\mathrm{F}}$ was proved. The dual FB quantization was illustrated for different examples and in the spaces $\mathrm{M}(1,\mathrm{N})$ of arbitrary dimensions. The massless spinor field was considered in details [2].

In our publications the consideration of the FB duality conception of the field was extended by application of the group-theoretical approach for the problem (FB duality was often called by us as the relationship between the fields of integer and half-integer spins, see e. g. [3, 4] and references therein). As a first step we have considered in details the case of massless Dirac equation. Both Fermi and Bose local representations of the universal covering $\mathscr{P}\supset\mathscr{L}=\mathrm{SL}(2,\mathrm{C})$ of proper ortochronous Poincare group $\mathrm{P}_+^{\uparrow}\supset\mathrm{L}_+^{\uparrow}=\mathrm{SO}(1,3)$, with respect to which the Dirac equation is invariant, were found. The same was realized [3] for the slightly generalized original Maxwell equations, in which the complex valued 4-object $\mathscr{E}(x)=E(x)-iH(x)$ of field strengths is the tensor-scalar (s=1,0) $\mathscr{P}$-covariant. Recently [5] we were able to extend our consideration for the Dirac equation with nonzero mass.

Here we consider the dual (fermionic and bosonic) symmetries [5] and solutions [6] of the Dirac (Foldy – Wouthuysen (FW) [7]) equation with nonzero mass and FB conservation laws for the spinor field. We presented in details the corresponding quantum mechanical stationary complete sets of operators of FB physical quantities. It allows us to demonstrate the statistical aspect of the spinor field FB duality.

The FW representation [7] is used as a starting ground for consideration and the results for standard Dirac formalism are proved to be the simple consequences of such consideration. In the FW representation the additional symmetries have the evident simple form in comparison with the symmetries in the standard Pauli – Dirac (PD) representation (fermionic spin as a symmetry operator is a good example). Therefore, start from the FW representation enabled us to find the new additional conservation laws (CL) even for the standard spinor solutions of the Dirac equation. On the basis of the bosonic solutions [6] of the FW equation not only new but an unexpected bosonic CL for the Dirac field are found. All the results, which are proved in FW representation, are translated very easy in the PD representation by the help of FW transformation [7].

In order to derive the CL on the basis of the Noether theorem one needs to explore the Lagrange approach and the Lagrange function. The Lagrange approach for the spinor field in FW representation has been suggested in [8, 9] in the terms of infinite order derivatives. Our desire to have ordinary type Lagrange approach gives the consideration in [10] and here of the new Lagrange function for the spinor field in FW representation.

For our purposes we use the mathematical formalism of the extended real Clifford – Dirac (ERCD) algebra and proper ERCD algebra [5]. Such generalization of the standard Clifford – Dirac (CD) algebra enabled us [5] to prove the bosonic symmetries of the FW and Dirac equation with nonzero mass.

The property of FB duality of the spinor field was the subject of our 5 reports on the conference, where step by step this duality was demonstrated on the examples of symmetry, solutions and CL. Therefore, this paper covers the material not only from our abstract [11] but also from the abstracts [12–15]. Thus, we present below ***the brief review of our results, which were reported on the conference***.

In Sec. 3 bosonic spin s=(1,0) Poincare symmetry of the Foldy – Wouthuysen and Dirac equations is presented briefly.

Sec. 4 gives the possibility to compare the fermionic and bosonic solutions of the FW (Dirac) equation.

In Sec. 5 the brief consideration of the Lagrange approach for the spinor field in FW representation is presented. On this basis the examples of both fermionic and bosonic conservation laws for the Dirac theory are given.

In Sec. 6 the main conclusions are discussed.

2. Notations and definitions

The system of units $\hbar = c = 1$ and metric $g = (g^{\mu\nu}) = (+---)$, $a^{\mu} = g^{\mu\nu} a_{\nu}$, are taken. Here the Greek indices are changed in the region $0,1,2,3 = \overline{0,3}$, Latin – $\overline{1,3}$, the summation over the twice repeated index is implied.

Our consideration is fulfilled in the rigged Hilbert space $\mathbb{S}^{3,4} \subset \mathbb{H}^{3,4} \subset {}^{\times}\mathbb{S}^{3,4}$, where $\mathbb{H}^{3,4}$ is given by

$$\begin{aligned} &\mathbb{H}^{3,4} = \mathrm{L}_2(\mathrm{R}^3) \otimes \mathbb{C}^{\otimes 4} = \{\phi \equiv (\phi^{\alpha}): \\ &\mathrm{R}^3 \to \mathbb{C}^{\otimes 4};\ \int d^3x \left|\phi(t,\vec{x})\right|^2 < \infty\},\ \alpha = 0,1,2,3, \end{aligned} \tag{1}$$

and symbol «×» in ${}^{\times}\mathbb{S}^{3,4}$ means, that the space of the Schwartz generalized functions ${}^{\times}\mathbb{S}^{3,4}$ is conjugated to the Schwartz test function space $\mathbb{S}^{3,4}$ by the corresponding topology. The application of the rigged Hilbert space allows one to reproduce a detailed consideration of a field theory in mathematically correct form.

For the purposes connected with physics it is useful to consider the corresponding groups and algebras with real parameters (e. q. the parameters $a = (a^{\mu})$, $\omega = (\omega^{\mu\nu})$ of the translations and rotations for the group $\mathrm{P}_{+}^{\uparrow} \supset \mathrm{L}_{+}^{\uparrow} = \mathrm{SO}(1,3)$ are real). Therefore, corresponding generators are anti-Hermitian. The mathematical correctness of such choice of generators is verified in [16, 17].

The ordinary CD algebra is considered here as the algebra of 4×4 Dirac matrices in the standard PD representation with the standard 2×2 Pauli matrices in their most extended explicit forms.

We consider the standard 16-dimensional CD algebra of the γ^{μ} matrices as a real one and add the imaginary unit $i=\sqrt{-1}$ together with the operator C of complex conjugation (the involution operator in the space $\mathbb{C}^{\otimes 4}$) into the set of the CD algebra possible generators. It enabled us to extend the standard CD algebra up to the 64-dimensional extended real CD algebra (ERCD algebra of [5]). Here the subalgebras of the ERCD algebra are considered briefly. The most important are the representations in $\mathbb{C}^{\otimes 4}$ of the 29-dimensional proper ERCD algebra SO(8) spanned on orts

$$\gamma^1,\ \gamma^2,\ \gamma^3,\ \gamma^4\equiv\gamma^0\gamma^1\gamma^2\gamma^3,$$
$$\gamma^5\equiv\gamma^1\gamma^3C,\ \gamma^6\equiv i\gamma^1\gamma^3C,\ \gamma^7\equiv i\gamma^0. \quad (2)$$

The generators (2) satisfy the anticommutation relations [1]

$$\gamma^{\mathrm{A}}\gamma^{\mathrm{B}}+\gamma^{\mathrm{B}}\gamma^{\mathrm{A}}=-2\delta^{\mathrm{AB}};$$
$$\mathrm{A,B}=1,2,...,7\equiv\overline{1,7}, \quad (3)$$

and the generators of proper ERCD algebra (together with unit ort we have 29 independent orts $\alpha^{\tilde{\mathrm{A}}\tilde{\mathrm{B}}}=(\mathrm{I},2s^{\tilde{\mathrm{A}}\tilde{\mathrm{B}}})$)

$$s^{\tilde{\mathrm{A}}\tilde{\mathrm{B}}}=\{s^{\mathrm{AB}}\equiv\frac{1}{4}[\gamma^{\mathrm{A}},\gamma^{\mathrm{B}}],\ s^{\mathrm{A}8}=-s^{8\mathrm{A}}=\frac{1}{2}\gamma^{\mathrm{A}}\},$$
$$\tilde{\mathrm{A}},\tilde{\mathrm{B}}=\overline{1,8}, \quad (4)$$

satisfy the commutation relations of SO(8) algebra

$$[s^{\tilde{\mathrm{A}}\tilde{\mathrm{B}}},s^{\tilde{\mathrm{C}}\tilde{\mathrm{D}}}]=\delta^{\tilde{\mathrm{A}}\tilde{\mathrm{C}}}s^{\tilde{\mathrm{B}}\tilde{\mathrm{D}}}+\delta^{\tilde{\mathrm{C}}\tilde{\mathrm{B}}}s^{\tilde{\mathrm{D}}\tilde{\mathrm{A}}}$$
$$+\delta^{\tilde{\mathrm{B}}\tilde{\mathrm{D}}}s^{\tilde{\mathrm{A}}\tilde{\mathrm{C}}}+\delta^{\tilde{\mathrm{D}}\tilde{\mathrm{A}}}s^{\tilde{\mathrm{C}}\tilde{\mathrm{B}}}. \quad (5)$$

Namely the proper ERCD algebra SO(8), given by the 29 orts (4), is our [5] direct generalization of a standard 16-dimensional CD algebra. It is also the basis for our dual FB consideration of a spinor field, which enabled us to prove the additional bosonic properties of this field. For the physical applications we consider the realizations of the proper ERCD algebra in the field space ${}^{\times}\mathbb{S}(\mathrm{M}(1,3))\otimes\mathbb{C}^{\otimes 4}\equiv{}^{\times}\mathbb{S}^{4,4}$ of the Schwartz generalized functions and in the quantum mechanical Hilbert space $\mathbb{H}^{3,4}$ (1). These realizations are found with the help of transformations $V^{+}\mathrm{SO}(8)V^{-}$, $v\mathrm{SO}(8)v$, where the operators of transformation have the form

$$V^{\pm}\equiv\frac{\pm i\gamma^{\ell}\partial_{\ell}+\widehat{\omega}+m}{\sqrt{2\widehat{\omega}(\widehat{\omega}+m)}},\ v\equiv\begin{vmatrix}\mathrm{I}_2 & 0\\ 0 & C\mathrm{I}_2\end{vmatrix};$$
$$\widehat{\omega}\equiv\sqrt{-\Delta+m^2},\ \mathrm{I}_2\equiv\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix},\ \ell=1,2,3. \quad (6)$$

We put into consideration ERCD algebra (64 orts) and proper ERCD algebra (29 orts) into the FW representation of the spinor field [7] (advantages in comparison with the standard Dirac equation in definitions of coordinate, velocity and spin operators are well known from [7]). In this representation the equation for the spinor field (the FW equation) has the form

$$(\partial_0+i\gamma^0\omega)\phi(x)=0;\quad \omega\equiv\sqrt{-\Delta+m^2},$$
$$x\in\mathrm{M}(1,3),\ \phi\in\mathbb{H}^{3,4}; \quad (7)$$

and is linked with the Dirac equation

$$(\partial_0+iH_{\mathrm{D}})\psi(x)=0,\ H_{\mathrm{D}}\equiv\vec{\alpha}\cdot\vec{p}+\beta m, \quad (8)$$

by the FW transformation $V^{\pm}$:

$$\phi(x)=V^{-}\psi(x),\ \psi(x)=V^{+}\phi(x),$$
$$V^{+}\gamma^0\omega V^{-}=\vec{\alpha}\cdot\vec{p}+\beta m. \quad (9)$$

Below the ERCD algebra and proper ERCD algebra (4) are essentially used in our proofs of bosonic properties of the Dirac and FW equations.

3. Bosonic spin s=(1,0) symmetry of the Foldy – Wouthuysen and Dirac equations

We have found in [5] different bosonic symmetries of the FW equation: (i) irreducible vector (1/2,1/2) and reducible tensor-scalar $(1,0)\oplus(0,0)$ representations of the $SO(1,3)=L_{+}^{\uparrow}$ algebra of the Lorenz group, (ii) the representation of the universal covering $\mathcal{P}\supset\mathcal{L}=SL(2,C)$ of proper ortochronous Poincare group $P_{+}^{\uparrow}\supset L_{+}^{\uparrow}=SO(1,3)$, with respect to which the Dirac (FW) equation is invariant.

Below we demonstrate briefly the example of the construction of the Lie algebra of the bosonic spin (1,0) representation of the Poincare group $\mathcal{P}\supset\mathcal{L}=SL(2,C)$, with respect to which the FW (Dirac) equation is invariant. Namely the representations of this group are important to theoretical physics.

The fundamental assertion is that subalgebra SO(6) of proper ERCD algebra (4), which is determined by the operators

$$\{I,\ \alpha^{\underline{AB}}=2s^{\underline{AB}}\},\ \underline{A},\underline{B}=\overline{1,6}, \qquad (10)$$

$$\{s^{\underline{AB}}\}=\{s^{\underline{AB}}\equiv\frac{1}{4}[\gamma^{\underline{A}},\gamma^{\underline{B}}]\}. \qquad (11)$$

is the algebra of invariance of the Dirac equation in the FW representation (7) (in (11) the six matrices $\{\gamma^{\underline{A}}\}\equiv\{\gamma^1,\gamma^2,\gamma^3,\gamma^4,\gamma^5,\gamma^6\}$ are known from (2)). Algebra SO(6) contains two different realizations of SU(2) algebra for the spin s=1/2 doublet. By taking the sum of the two independent sets of SU(2) generators from (11) one can obtain the SU(2) generators of spin s=(1,0) multiplet, which generate the transformation of invariance of the FW equation (7). Transformation

$$W=\frac{1}{\sqrt{2}}\begin{vmatrix}\sqrt{2}&0&0&0\\0&0&i\sqrt{2}C&0\\0&-C&0&1\\0&-C&0&-1\end{vmatrix},$$

$$W^{-1}=\frac{1}{\sqrt{2}}\begin{vmatrix}\sqrt{2}&0&0&0\\0&0&-C&-C\\0&i\sqrt{2}C&0&0\\0&0&1&-1\end{vmatrix}, \qquad (12)$$

$$WW^{-1}=W^{-1}W=1,$$

translates these generators of spin s=(1,0) multiplet into the bosonic representation

$$\underset{\sim}{s}^1=\frac{1}{\sqrt{2}}\begin{vmatrix}0&0&iC&0\\0&0&-C&0\\-iC&C&0&0\\0&0&0&0\end{vmatrix},$$

$$\underset{\sim}{s}^2=\frac{1}{\sqrt{2}}\begin{vmatrix}0&0&C&0\\0&0&-iC&0\\-C&iC&0&0\\0&0&0&0\end{vmatrix}, \qquad (13)$$

$$\underset{\sim}{s}^3=\begin{vmatrix}-i&0&0&0\\0&i&0&0\\0&0&0&0\\0&0&0&0\end{vmatrix},\ \vec{\underset{\sim}{s}}^2=-1(1+1)\begin{vmatrix}I_3&0\\0&0\end{vmatrix}.$$

The spin operators $\underset{\sim}{s}_{ln}$ (13) of SU(2) algebra commute with the operator $\partial_0+i\gamma^0\omega$ of the FW equation (7). It is important to note that transformation (12) does not change the operator $\partial_0+i\gamma^0\omega$.

On the basis of the spin operators (13) the bosonic spin (1,0) representation of the Poincare group $\mathcal{P}$ is constructed. It is easy to show (after our consideration in [5] and above) that $(p_\mu,\ j^{B}_{\mu\nu})$ generators

$$p_0=-i\gamma_0\omega,\ p_n=\partial_n,\ j^{B}_{ln}=x_l\partial_n-x_n\partial_l+\underset{\sim}{s}_{ln},$$
$$j^{B}_{0k}=x_0\partial_k+i\gamma_0\left\{x_k\omega+\frac{\partial_k}{2\omega}+\frac{(\vec{\underset{\sim}{s}}\times\vec{\partial})_k}{\omega+m}\right\}, \qquad (14)$$

of the group $\mathcal{P}$ commute with the operator of the FW equation (7) and satisfy the commutation relations of the Lie algebra of the group $\mathcal{P}$ in manifestly covariant form. The operators (14) generate in the space $\mathbb{H}^{3,4}$ another than the fermionic $\mathcal{P}^{F}$-generators D-64 – D-67 of [1] unitary $\mathcal{P}$ representation, i. e. the bosonic $\mathcal{P}^{B}$ representation of the group $\mathcal{P}$, with respect to which the FW equation (7) is invariant. For the generators (14) the Casimir operators have the form:

$$p^\mu p_\mu=m^2,\ W^{B}=w^\mu w_\mu=m^2\vec{\underset{\sim}{s}}^2$$
$$=-1(1+1)m^2\begin{vmatrix}I_3&0\\0&0\end{vmatrix}. \qquad (15)$$

Hence, according to the Bargman – Wigner classification, we consider here the spin s=(1,0) representation of the group $\mathcal{P}$.

The corresponding bosonic spin s=(1,0) symmetries of the Dirac equation (8) can be found from the generators (14) with the help of FW operator $V^{\pm}$ (6) in the form $V^{+}(p_{\mu}, j^{\mathrm{B}}_{\mu\nu})V^{-}$.

4. Bosonic spin s=(1,0) multiplet solution of the Foldy – Wouthuysen and Dirac equations

Here as the next step in FB duality investigation we consider the bosonic solution of the Dirac (FW) equation. A bosonic solution of the FW equation (7) is found completely similarly to the procedure of construction of standard fermionic solution. Thus, the bosonic solution is determined by some stationary diagonal complete set of operators of bosonic physical quantities for the spin s=(1,0)-multiplet in the FW representation, e. g., by the set "momentum-spin projection $\underset{\sim}{s}^3$:

$$(\vec{p} = -\nabla,\ \underset{\sim}{s}^3) \qquad (16)$$

where the spin operators $\underset{\sim}{\vec{s}}$ and $\underset{\sim}{s}^3$ for the spin s=(1,0)-multiplet are given in (13). The fundamental solutions of equation (7), which are the common eigen solutions of the bosonic complete set (16), have the form

$$\varphi^{-}_{\vec{k}\mathrm{r}}(t,\vec{x}) = \frac{1}{(2\pi)^{3/2}} e^{-ikx}\mathrm{d}_{\mathrm{r}},$$
$$\varphi^{+}_{\vec{k}\tilde{\mathrm{r}}}(t,\vec{x}) = \frac{1}{(2\pi)^{3/2}} e^{ikx}\mathrm{d}_{\tilde{\mathrm{r}}}, \qquad (17)$$
$$kx = \omega t - \vec{k}\vec{x},$$

where $\mathrm{d}_{\alpha} = \left(\delta^{\beta}_{\alpha}\right)$ are the Cartesian orts in the space $\mathbb{C}^{\otimes 4} \subset \mathbb{H}^{3,4}$, numbers $\mathrm{r} = (1,2),\ \tilde{\mathrm{r}} = (3,4)$ mark the eigen values $(+1,-1,0,\underline{0})$ of the operator $\underset{\sim}{s}^3$ from (13).

The bosonic solutions of equation (7) are the generalized states, belonging to the space ${}^{\times}\mathbb{S}^{3,4}$; they form the complete orthonormalized system of bosonic states. Therefore, any bosonic physical state of the FW field ϕ from the dense in $\mathbb{H}^{3,4}$ manifold $\mathbb{S}^{3,4}$ (the general bosonic solution of the equation (7)) is uniquely presented in the form

$$\phi_{(1,0)}(x) = \frac{1}{(2\pi)^{3/2}} \int d^3k[\xi^{\mathrm{r}}(\vec{k})\mathrm{d}_{\mathrm{r}}e^{-ikx} + \xi^{*\tilde{\mathrm{r}}}(\vec{k})\mathrm{d}_{\tilde{\mathrm{r}}}e^{ikx}], \qquad (18)$$

where $\xi^{\mathrm{r}}(\vec{k}),\ \xi^{\tilde{\mathrm{r}}}(\vec{k})$ are the coefficients of the expansion of bosonic solution of the FW equation (7) with respect to the Cartesian basis $\mathrm{d}_{\alpha} = \left(\delta^{\beta}_{\alpha}\right)$. The relationships of amplitudes $\xi^{\mathrm{r}}(\vec{k}),\ \xi^{\tilde{\mathrm{r}}}(\vec{k})$ with quantum mechanical bosonic amplitudes $b^{\mathrm{r}}(\vec{k}),\ b^{\tilde{\mathrm{r}}}(\vec{k})$ of probability distribution according to the eigen values of the stationary diagonal complete set of operators of quantum mechanical bosonic s=(1,0)-multiplet are given by

$$\xi^1 = b^1,\ \xi^2 = -\frac{1}{\sqrt{2}}(b^3 + b^4),$$
$$\xi^3 = -ib^2,\ \xi^4 = \frac{1}{\sqrt{2}}(b^3 - b^4), \qquad (19)$$

where the 4 amplitudes $b^{1,2,3,4}(\vec{k})$ are the quantum mechanical momentum-spin amplitudes with the eigen values $(+1,-1,0,\underline{0})$ of the quantum mechanical (1,0) multiplet $\underset{\sim}{s}^3$ operator projections, respectively (last eigen value $\underline{0}$ is related to the proper zero spin). And if $\phi_{(1,0)}(x) \in \mathbb{S}^{3,4}$, then the bosonic amplitudes $\xi^{\mathrm{r}}(\vec{k}),\ \xi^{\tilde{\mathrm{r}}}(\vec{k})$ belong to the Schwartz complexvalued test function space too.

Moreover, the set $\Phi^{\mathrm{B}} \equiv \{\phi_{(1,0)}(x)\}$ of solutions (18) is invariant namely with respect to the unitary bosonic representation of the group $\mathcal{P}$, which is determined by the generators (14) and Casimir operators (15). Therefore, the Bargman - Wigner analysis of the Poincare symmetry of the set $\Phi^{\mathrm{B}} \equiv \{\phi_{(1,0)}(x)\}$ of solutions (18) completes the demonstration that it is the set of Bose-states $\phi_{(1,0)}$ of the field ϕ, i. e. the s=(1,0)-

multiplet states. Hence, the existence of bosonic solutions of the FW equation is proved.

In the terms of quantum mechanical momentum-spin amplitudes $b^{\mathrm{r}}(\vec{k}),\ b^{\tilde{\mathrm{r}}}(\vec{k})$ from (19), the bosonic spin (1,0)-multiplet solution $\psi = V^{+}\phi$ of the Dirac equation (8) is given by

$$\psi(x) = \frac{1}{(2\pi)^{3/2}}\int d^3k\{e^{-ikx}[b^1 v_1^- - \frac{1}{\sqrt{2}}\left(b^3 + b^4\right)v_2^-] + e^{ikx}[ib^{*2}v_1^+ + \frac{1}{\sqrt{2}}\left(b^{*3} - b^{*4}\right)v_2^+]\}, \qquad (20)$$

where the 4-component spinors are the same as in the Dirac theory of fermionic doublet

$$v_{\mathrm{r}}^-(\vec{k}) = N\begin{vmatrix}(\widehat{\omega}+m)\mathrm{d}_{\mathrm{r}}\\ \vec{\sigma}\cdot\vec{k}\mathrm{d}_{\mathrm{r}}\end{vmatrix},\ v_{\mathrm{r}}^+(\vec{k}) = N\begin{vmatrix}\vec{\sigma}\cdot\vec{k}\mathrm{d}_{\mathrm{r}}\\ (\widehat{\omega}+m)\mathrm{d}_{\mathrm{r}}\end{vmatrix}, \quad N \equiv \frac{1}{\sqrt{2\widehat{\omega}(\widehat{\omega}+m)}},\ \mathrm{d}_1 = \begin{vmatrix}1\\0\end{vmatrix},\ \mathrm{d}_2 = \begin{vmatrix}0\\1\end{vmatrix}. \qquad (21)$$

The well known (standard) Fermi solution of the Dirac equation for the spin s=1/2 doublet has the form

$$\psi(x) = \frac{1}{(2\pi)^{3/2}}\int d^3k[e^{-ikx}\left(a_+^- v_1^- + a_-^- v_2^-\right) + e^{ikx}\left(a_+^{*+} v_1^+ + a_-^{*+} v_2^+\right)], \qquad (22)$$

where $a_+^-(\vec{k}),\ a_-^-(\vec{k})$ are the quantum mechanical momentum-spin amplitudes of the particle with charge –e and eigen values of spin projection +1/2 and -1/2; $a_-^+(\vec{k}),\ a_+^+(\vec{k})$ are the quantum mechanical momentum-spin amplitudes of the antiparticle with charge +e and eigen values of spin projection -1/2 and +1/2, respectively.

All the above given ***assertions about the Fermi – Bose duality of the spinor field are valid both in FW and PD representation***, i. e. for both equations FW (7) and Dirac (8). It is easily shown with the help of the FW transformation $V^{\pm}$ (6), which transform the FW solutions and operators into the Dirac solutions and operators.

5. The Fermi – Bose conservation laws for the spinor field

At first we must consider the Lagrange approach (L-approach) for the spinor field $\phi(x)$ in the FW representation. Only after that the Noether theorem and the corresponding analysis of CL are determined.

The L-approach for the spinor field has been formulated at first in [8, 9]. The representation of operator $\widehat{\omega} \equiv \sqrt{-\Delta + m^2}$ in the form of the series over the Laplace operator Δ powers has been used. Mathematical correctness was not considered.

In publication [10] the construction of the L approach under consideration in the quantummechanical rigged Hilbert space (both in the coordinate and momentum realizations of this space) has been verified. In momentum realization this space has the form

$$\tilde{\mathbb{S}}^{3,4} \subset \tilde{\mathbb{H}}^{3,4} \subset {}^{\times}\tilde{\mathbb{S}}^{3,4}; \quad \tilde{\mathbb{H}}^{3,4} = \{\phi \equiv (\phi^{\alpha})_{\alpha=1}^{4} :\ \mathrm{R}_{\vec{k}}^{3} \rightarrow \mathbb{C}^{\otimes 4}; \int d^3k\left|\tilde{\phi}(k)\right|^2 < \infty\}. \qquad (23)$$

Here $\mathrm{R}_{\vec{k}}^{3}$ is the momentum operator $\vec{p}$ spectrum, which is canonically conjugated to the coordinate $\vec{x}$ ($[x^j, p^{\ell}] = i\delta^{j\ell}$). The corresponding $\vec{x}$ -realization is connected to (1) by 3-dimensional Fourier transformation. The alternative using of both realizations is based on the principle of heredity with classical and non-relativistic quantum mechanics of single mass point and with the mechanics of continuous media. The Lagrange function and the action (in alternative $\vec{x}$ or $\vec{k}$-realizations) are constructed in complete analogy with their consideration in the classical mechanics of a system with finite number of freedom degrees $q \equiv (q_1, q_2, ...)$. The difference is only in the fact that here the continuous variable $\vec{k} \in \mathrm{R}_{\vec{k}}^{3}$ is the carrier of freedom degrees. In the $\vec{k}$-realization, where this analogy is maximally clear, the Lagrange function has the form

$$L = L(\tilde{\phi}, \tilde{\phi}^{\dagger}, \tilde{\phi},_0, \tilde{\phi}^{\dagger},_0) = \frac{i}{2}\left(\tilde{\phi}^{\dagger}(\tilde{\phi},_0 + i\gamma^0\tilde{\omega}\tilde{\phi}) - (\tilde{\phi}^{\dagger},_0 - i\tilde{\omega}\tilde{\phi}^{\dagger}\gamma^0)\tilde{\phi}\right), \quad \tilde{\omega} \equiv \sqrt{\vec{k}^2 + m^2}, \tag{24}$$

and in the $\vec{x}$-realization this function can be found from (24) by the Fourier transformation. The Euler – Lagrange equations coincide with the FW equation in both realizations.

The well defined L-approach for the FW field becomes essentially actual problem after the construction in [18] of the quantum electrodynamics in the FW representation.

The L-approach under consideration and the Noether theorem lead to the following general formula for the calculation of the CL

$$Q = \int d^3x \phi^{\dagger}(x) iq\phi(x) \tag{25}$$

where q is the arbitrary symmetry generator.

Note briefly about the ***FB conservation laws (CL)*** for the spinor field. It is preferable to calculate them in the FW (not local PD) representation too. In FW representation the Fermi spin $\vec{s} = (s^{\ell n}) = (s^{23}, s^{31}, s^{12})$ from (11) (together with the boost spin) is the independent symmetry operator for the FW equation. The orbital angular momentum and pure Lorentz angular momentum (the carriers of the external statistical degrees of freedom) are in this representations the independent symmetry operators too (one can find the corresponding independent spin and angular momentum symmetries in the PD representation for the Dirac equation too, but the corresponding operators are essentially nonlocal). Hence, one obtains 10 Poincare and 12 additional (3 spin, 3 pure Lorentz spin, 3 angular momentum, 3 pure angular momentum) CL.

Therefore, in the FW representation one can find very easy the 22 fermionic and 22 bosonic CL. The separation into bosonic and fermionic set is caused by the existence of FB symmetries and solutions (see the Sec. 3, 4 above). Indeed, if substitution of bosonic $\mathscr{P}$ generators q^{B} (14) and bosonic solutions $\phi_{(1,0)}$ (18) into the Noether formula (25) is made, then automatically the bosonic CL for s=(1,0)-multiplet are obtained. The standard substitution of corresponding well known fermionic generators and solutions (22) gives fermionic CL.

We illustrate briefly the difference in fermionic and bosonic CL on the example of corresponding spin conservation. For the fermionic spin

$$\vec{s} = (s^{23}, s^{31}, s^{12}) \equiv (s^{\ell}) = \frac{-i}{2}\begin{vmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{vmatrix} \rightarrow s^3 = \frac{-i}{2}\begin{vmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{vmatrix}, \tag{26}$$

and bosonic spin (13) the corresponding CL are given by

$$S^{\mathrm{F}}_{mn} = \int d^3x \phi^{\dagger}(x) is_{mn}\phi(x) = \int d^3k A^{\dagger}(\vec{k}) is_{mn} A(\vec{k}), \tag{27}$$

$$S^{\mathrm{B}}_{mn} = \int d^3x \phi^{\dagger}_{(1,0)}(x) i\underset{\sim}{s}_{mn}\phi_{(1,0)}(x) = \int d^3k B^{\dagger}(\vec{k}) i\underset{\sim}{s}_{mn} B(\vec{k}), \tag{28}$$

where

$$A(\vec{k}) = \begin{vmatrix}a^-_+(\vec{k})\\ a^-_-(\vec{k})\\ a^{*+}_-(\vec{k})\\ a^{*+}_+(\vec{k})\end{vmatrix}, \quad B(\vec{k}) = \begin{vmatrix}b^1(\vec{k})\\ b^2(\vec{k})\\ b^{*3}(\vec{k})\\ b^{*4}(\vec{k})\end{vmatrix}, \tag{29}$$

and the difference between Fermi spin $\vec{s} = (s^{\ell n}) = (s^{23}, s^{31}, s^{12})$ (26) and Bose spin $\vec{\underset{\sim}{s}} = (\underset{\sim}{s}^{\ell n}) = (\underset{\sim}{s}^{23}, \underset{\sim}{s}^{31}, \underset{\sim}{s}^{12})$ (13) is evident.

We present these CL in terms of quantum mechanical Fermi and Bose amplitudes. Such explicit quantum statistical form has all integral conserved quantities.

6. Conclusions

The property of ***the Fermi – Bose duality of the Dirac equation*** (both in the Foldy –

Wouthuysen and the Pauli – Dirac representations), which proof was started in [5], where the bosonic symmetries of this equation were found, is demonstrated here on the next level – on the level of existence of the spin (1,0) bosonic solutions of the equation under consideration and corresponding bosonic conservation laws.

The investigation of the spinor field in the Foldy – Wouthuysen (not in standard Dirac) representation has the independent meaning and purpose. This representation is interest itself in connection with the recent result [18] of V. Neznamov, who developed the formalism of quantum electrodynamics in the Foldy – Wouthuysen representation, see also the results in [19].

The 64 dimensional ERCD and 29 dimensional proper ERCD algebras, which have been put into consideration in [5], are the useful generalizations of standard 16 dimensional CD algebra. Their application enabled us to prove the existence of additional bosonic symmetries, solutions and conservation laws for the spinor field, for the Foldy – Wouthuysen and the Dirac equations. These algebras are our main mathematical tool for the demonstration of Fermi – Bose duality of the spinor field and Dirac equation.

Similarly, the fermionic spin s=1/2 properties for the Maxwell equations both ***with nonzero*** and zero mass can be proved (see e. g. the procedure given in [3]). Hence, such consideration is directly related to the electromagnetic theory too.

The property of the Fermi – Bose duality of the Dirac equation, which was proved in our publications and here, does not have the direct influence on the existence of the Fermi – Bose statistics. Our results do not break the Fermi statistics for the fermions (with the Pauli principle) and Bose statistics for bosons (with Bose condensation). We also never mixed this statistics between each other. Our assertion is following. One can apply with equal success both Fermi and Bose statistics for one and the same Dirac equation and one and the same spinor field, i. e. ***the Dirac equation can describe both the fermionic and bosonic states***. It is the main conclusion from the results, which are presented here.

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В.М. Симулик[1], І.Ю. Кривський[1], І.Л. Ламер[1], Т.М. Заяць[2]
[1]Інститут електронної фізики НАН України, 88017, Ужгород, вул. Університетська, 21
[2]Ужгородський Національний університет, 88000, Ужгород, вул. Капітульна, 13

ПРО ТЕОРЕТИКО-ГРУПОВІ ОСНОВИ ФЕРМІ – БОЗЕ ДУАЛІЗМУ СПІНОРНОГО ПОЛЯ. РОЗВ'ЯЗКИ ТА ЗАКОНИ ЗБЕРЕЖЕННЯ

Властивість Фермі – Бозе дуалізму рівняння Дірака з ненульовою масою доведена на основі узагальненої алгебри Кліффорда – Дірака та теоретико-групового аналізу спінорного поля. В якості стартової позиції для розгляду використано представлення Фолді – Вотхойзена. Доведено, що результати у стандартному формалізмі Дірака є простими наслідками наведеного розгляду. Показано, що рівняння Дірака може описувати не лише ферміонні, але й бозонні стани. Доведення нашого твердження дано на прикладах наявності бозонних симетрій та розв'язків рівняння Дірака. Крім того доведено, що наслідками рівняння Дірака є також бозонні закони збереження.

Ключові слова: спінорне поле, рівняння Дірака, представлення Фолді – Вотхойзена, ферміони, бозони.

В.М. Симулик[1], И.Ю. Кривский[1], И.Л. Ламер[1], Т.М. Заяц[2]
[1]Институт электронной физики НАН Украины, 88000, Ужгород, ул. Университетская, 21
[2]Ужгородский Национальный университет, 88000, Ужгород, ул. Капитульна, 13

О ТЕОРЕТИКО-ГРУППОВЫХ ОСНОВАХ ФЕРМИ – БОЗЕ ДУАЛИЗМА СПИНОРНОГО ПОЛЯ. РЕШЕНИЯ И ЗАКОНЫ СОХРАНЕНИЯ

Свойство Ферми – Бозе дуализма уравнения Дирака с ненулевой массой доказано на основе обобщенной алгебры Клиффорда – Дирака и теоретико-группового анализа спинорного поля. В качестве старта рассмотрения использовано представление Фолди – Вотхойзена. Доказано, что результаты в стандартном формализме Дирака являются простыми следствиями приведенного рассмотрения. Показано, что уравнение Дирака может описывать не только фермионные, но и бозонные состояния. Доказательство нашего утверждения дано на примерах наличия бозонных симметрий и решений уравнения Дирака. Кроме того доказано, что следствиями уравнения Дирака являются также и бозонные законы сохранения.

Ключевые слова: спинорное поле, уравнение Дирака, представление Фолди – Вотхойзена, фермионы, бозоны.