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THE SYSTEM OF THE PARTICLES IN THE MAGNETIC FIELD ON THE BACKGROUND LOBACHEVSKY SPACE

In this paper we have considered kinetic equations for the one-point distribution function on the background Lobachevsky space in hyperbolic cylindrical, cylindrical and hyperbolic horospheric coordinates with their different uniform magnetic fields, that are similar in the flat space. Some solutions of kinetic equations have been found for equilibrium state and they have been compared with solutions for kinetic equation in the flat space.

Key words: Lobachevsky space, kinetic equation, uniform magnetic field, Killing vector

Introduction

It has been discovered recently that there is intergalactic magnetic field. The induction of this magnetic field has been measured experimentally [1]. It might be interesting to study the behavior of particle system in some cosmological models with magnetic field.

We have studied kinetic equations for the one-point distribution function in the uniform magnetic field on the background Lobachevsky space. Kinetic equations have been formulated in hyperbolic cylindrical, cylindrical and hyperbolic horospheric coordinates with different magnetic fields. Solutions for these kinetic equations have been considered for equilibrium state.

We also presume that the results of this research might be applicable for plasma in the nanostructures.

Kinetic equation for collisionless particles in the constant electromagnetic field on the background Lobachevsky space

Let us consider particles which are characterized by the same mass m coordinates x^i and momenta p^j (where Latin indices *i, j* and so on run over the four space-time coordinate labels *1, 2, 3,* 4, with x^4 the time coordinate).

For describing the entirety of particles, we can use the random function which was introduced by Klimontovich [2] and which can be written as

$$\Phi(x^i,\, p^j) = \\ = \sum_{a=1}^{N} \delta^4 (x^i - x^i_{(a)}(s))\delta^4 (p^j - p^j_{(a)}(s))ds, \qquad (1)$$

where *a=1..N* - are numbers of particles, δ^4 - is the four-dimensional delta function, ds - is the space-time interval.

Using a covariant Liouville equation and equations of motion in Lobachevsky space with electromagnetic field, the collisionless kinetic equation for the random function (1) takes the form [3]

$$p^i_{(a)} \frac{\partial \Phi(x^i, p^j)}{\partial x^i} + \frac{\partial \Phi(x^i, p^j)}{\partial p^i} \\ \times\left(- \Gamma^i_{kl} p^l_{(a)} + \frac{e}{c} F^{ip} p^{(a)}_p \right) = 0. \qquad (2)$$

After averaging the random function $\Phi(x^i, p^j)$ across the configurations (ensemble), we obtain the distribution function

$$\left\langle \Phi(x^i, p^j) \right\rangle = N(x^i, p^j). \qquad (3)$$

The invariant volume in Lobachevsky

space-time can be introduced as $V_x = = \int \sqrt{|g|} d^4 x = \int \sqrt{|g|} dx_1 dx_2 dx_3 dx_4 = \int d\Sigma$ and the invariant volume of momentum space is $V_p = = \int \sqrt{|g|} d^4 p = \int dP$, where g – is the determinant of the metric tensor [4].

Let us express one-point distribution function in terms of a random function

$$\rho(x^i, p^j) = \frac{\langle \Phi(x^i, p^j) \rangle}{N}. \quad (4)$$

Note that $\int \rho(x^i, p^j) d\Sigma dP$ is a number of particles in $d\Sigma$ volume with momentum within the small range dP.

The one-point distribution function $\rho(x^i, p^j)$ in four-dimensional space-time can be expressed in terms of a classical distribution function $\rho_{cl}(x^\alpha, p^\beta, t)$ [5]

$$\begin{aligned} &\rho(x^i, p^j) = \\ &= \rho_{cl}(x^i, p^j)\delta(\sqrt{g^{lm} p_l p_m} - mc)\Theta(p_4), \end{aligned} \quad (5)$$

where Greek indices α, β etc., run over the three spatial coordinate labels, taken as *1, 2, 3*, and $\Theta(p_4)$ is the Heaviside function

$$\Theta(p_4) = \begin{cases} 0, \text{ when } p_4 < 0; \\ 1, \text{ when } p_4 > 0. \end{cases}$$

The delta function in (5) provides nonzero value for classical distribution function only on mass-shell $g^{lm} p_l p_m = m^2 c^2$.

Suppose that external electromagnetic field and background metrics are not depended on the state of particles. Hence, (2) after integrating over t and averaging across the configurations (ensemble), takes the form

$$\begin{aligned} &p^4_{(a)} \frac{\partial \rho(x^i, p^\beta)}{\partial x^4} + p^\mu_{(a)} \frac{\partial \rho(x^i, p^\beta)}{\partial x^\mu} + \\ &+ \frac{e}{c} \frac{\partial \rho(x^i, p^\beta)}{\partial p^\mu} \left(F^\mu_\nu p^\nu_{(a)} + F^\mu_4 p^4_{(a)} \right) - \\ &- \frac{\partial \rho(x^i, p^\beta)}{\partial p^\mu} \Gamma^\mu_{kl} p^k_{(a)} p^l_{(a)} = 0. \end{aligned} \quad (6)$$

Kinetic equations in Lobachevsky space in different coordinates

1) We consider a kinetic equation for Robertson-Walker metrics with negative spatial curvature (Lobachevsky space) in hyperbolic cylindrical coordinates. The interval for this metrics takes the form

$$\begin{aligned} &ds^2 = c^2 dt^2 - \\ &- a^2 (\cosh^2 z (dr^2 + \sinh^2 r d\varphi^2) + dz^2), \end{aligned} \quad (7)$$

where a - is a constant scale factor.

External field is a uniform magnetic field, which in cylindrical coordinates can be written in terms of *4* -potential A_i as [6]

$$\begin{aligned} &A_x = 0, \quad A_z = 0, \\ &A_\varphi = -cBa^2(\cosh r - 1), \quad A_t = 0. \end{aligned} \quad (8)$$

Expressing electromagnetic tensor for (8), and Christoffel symbols for (7), kinetic equation (6) can be rewritten as

$$\begin{aligned} &\frac{\varepsilon}{c}\frac{\partial \rho}{\partial t} + p^r \frac{\partial \rho}{\partial r} + p^z \frac{\partial \rho}{\partial z} + p^\varphi \frac{\partial \rho}{\partial \varphi} + \sinh r \frac{\partial \rho}{\partial p^r} \\ &\times \left(\frac{p^\varphi B e}{\cosh^2 z} - \frac{2 \tanh z\, p^z p^r}{\sinh r} + (p^\varphi)^2 \cosh r \right) + \\ &+ \cosh z \sinh z \frac{\partial \rho}{\partial p^z} \left((p^r)^2 + \sinh^2 r (p^\varphi)^2 \right) - \\ &- 2 \coth r \frac{\partial \rho}{\partial p^\varphi} p^r p^\varphi - 2 \coth r \frac{\partial \rho}{\partial p^\varphi} \\ &\times \left(\frac{\tanh z\, p^z p^\varphi}{\coth r} + \frac{B e p^r}{2 \cosh r \cosh^2 z} \right) = 0, \end{aligned} \quad (9)$$

where ε - is the particle energy, ρ - is a classical distribution function which depends on three momentum, time and three spatial coordinates.

2) By analogy with the first case, let us consider the same metrics in cylindrical coordinates. The interval for this case takes the form

$$\begin{aligned} &ds^2 = c^2 dt^2 - \\ &- a^2 (dr^2 + \sinh^2 r\, d\varphi^2 + \cosh^2 r\, dz^2). \end{aligned} \quad (10)$$

In this cylindrical coordinates, the uniform magnetic field is written as

$$A_r = 0, \quad A_z = 0,$$
$$A_\varphi = -cBa^2 \ln(\cosh r), \quad A_t = 0. \tag{11}$$

Then the kinetic equation takes the form

$$\frac{\varepsilon}{c}\frac{\partial\rho}{\partial t} + p^r\frac{\partial\rho}{\partial r} + p^z\frac{\partial\rho}{\partial z} + p^\varphi\frac{\partial\rho}{\partial\varphi} +$$
$$+\frac{\partial\rho}{\partial p^r}(Be_{(a)}\tanh r\, p^\varphi + (p^z)^2) +$$
$$+\frac{\partial\rho}{\partial p^r}\cosh r\sinh r(p^\varphi)^2 - \tag{12}$$
$$-2\frac{\partial\rho}{\partial p^z}\tanh r\, p^r\ p^z -$$
$$-\frac{\partial\rho}{\partial p^\varphi}\left(\frac{Be_{(a)}p^r}{\cosh r\sinh r} + 2p^r p^\varphi\coth r\right) = 0.$$

3) The same calculations we applied to the horospheric coordinates

$$ds^2 = c^2dt^2 -$$
$$- a^2(e^{-2z}(dr^2 + r^2d\varphi^2) + dz^2). \tag{13}$$

The vector potential of the uniform magnetic field in horospheric coordinates takes the form

$$A_r = 0, \quad A_z = 0,$$
$$A_\varphi = \frac{-c\,B\,a^2r^2}{2}, \quad A_t = 0. \tag{14}$$

For this case the kinetic equation is written as

$$\frac{\varepsilon}{c}\frac{\partial\rho}{\partial t} + p^r\frac{\partial\rho}{\partial r} + p^z\frac{\partial\rho}{\partial z} + p^\varphi\frac{\partial\rho}{\partial\varphi} +$$
$$+\frac{\partial\rho}{\partial p^r}\left(Be_{(a)}re^{2z}p^\varphi + r(p^\varphi)^2 + 2p^r p^z\right) -$$
$$-\frac{\partial\rho}{\partial p^z}\left(e^{-2z}(p^r)^2 + r^2e^{-2z}(p^\varphi)^2\right) - \frac{\partial\rho}{\partial p^\varphi} \tag{15}$$
$$\times\left(\frac{Be_{(a)}e^{2z}p^r}{r} - 2p^z p^\varphi + \frac{2p^r p^\varphi}{r}\right) = 0.$$

Note that the magnetic field in the hyperbolic cylindrical coordinates can't be equated with magnetic field in the cylindrical coordinates due to coordinates or gauge transformation [8]. A similar situation is for field in the horospheric coordinates, this field can't be equated with magnetic fields in other two coordinates. However, all these magnetic fields become similar when the curvature is null.

Solution of kinetic equation for equilibrium state

The entropy flux vector for a system of particles can be expressed in terms of distribution function as [9]

$$S^i = -ck_B\int\frac{\sqrt{|g|}p^i}{\varepsilon}\rho(\ln\rho - 1)d^3p. \tag{16}$$

For equilibrium state, we have $S^i_{;i} = 0$, hence we obtain the expression

$$\rho(x^i, p^\mu_{1^i})\rho(x^i, p^\mu_{2^i}) = \rho(x^i, p^\mu_{1^f})\rho(x^i, p^\mu_{2^f}) \tag{17}$$

where momenta with index «1^i», «2^i» mean the initial state, momenta with index «1^f», «1^f» mean the finite state.

Note that contribution from elastic collision to kinetic equation for equilibrium state is vanishing.

A general solution for our case has the form [10]

$$\rho = C\,exp\left\{\mu - b_i(x^k)\left(p^i + \frac{e}{c}A^i(x^k)\right)\right\}, \tag{18}$$

where $b_i(x^k)$ - are arbitrary parameters which depend on coordinates, C - is the normalization parameter, μ - is a constant which can be identified with the chemical potential.

For the metric tensor g_{ij} and external electromagnetic field A_i, vectors b_i must satisfy the conditions

$$\underset{b}{☹}A_i = 0, \quad \underset{b}{☹}g_{ij} = 0, \tag{19}$$

where $\underset{b}{☹}$ is the Lie derivative with respect to b_i. Therefore vectors b_i are Killing vectors for metrics g_{ij} and electromagnetic field A_i.

For homogeneous, isotropic, stationary space and constant external field we always can choose vector b_i in the form $b_i = \{0,0,0,1\}$. Hence, the solutions of equations (9), (12) and (15) can be written as (chemical potential is put in the normalization parameter C)

$$\rho_1 = C\,\mathrm{e}^{-\frac{c\sqrt{\cosh^2 z\left[\left(p^r\right)^2+\sinh^2 r\left(p^\varphi\right)^2\right]+\left(p^z\right)^2+m^2c^2}}{k_B T}} \quad (20)$$

$$\rho_2 = C\,\mathrm{e}^{-\frac{c\sqrt{\left(p^r\right)^2+\sinh^2 r\left(p^\varphi\right)^2+\cosh^2 r\left(p^z\right)^2+m^2c^2}}{k_B T}} \quad (21)$$

$$\rho_3 = C\,\mathrm{e}^{-\frac{c\sqrt{\mathrm{e}^{-2z}\left(p^r\right)^2+\mathrm{e}^{-2z}r^2\left(p^\varphi\right)^2+\left(p^z\right)^2+m^2c^2}}{k_B T}} \quad (22)$$

The number of Killing vectors b_i depends on space symmetry and symmetry of the external field. Space with constant curvature has maximum number of Killing vectors $\frac{n(n+1)}{2}$, for n-dimensional space.

Let us consider only spatial part, Lobachevsky space in hyperbolic cylindrical coordinates without external field

$$dl^2 = (\cosh^2 z(dr^2 + \sinh^2 r d\varphi^2) + dz^2). \quad (23)$$

There are exist six linearly independent Killing vectors, which satisfy (19). In hyperbolic cylindrical coordinates they are

$$\begin{aligned}
br1^\alpha &= \left\{-\cosh r\tanh z\cos\varphi,\ \sinh r\cos\varphi,\ \frac{\tanh z\sin\varphi}{\sinh r}\right\},\\
br2^\alpha &= \left\{-\cosh r\tanh z\sin\varphi,\ \sinh r\sin\varphi,\ -\frac{\tanh z\cos\varphi}{\sinh r}\right\},\\
br3^\alpha &= \{0,\ 0,\ 1\},\\
bt1^\alpha &= \{\sin\varphi,\ 0,\ \cos\varphi\coth r\},\\
bt2^\alpha &= \{\cos\varphi,\ 0,\ -\sin\varphi\coth r\},\\
bt3^\alpha &= \left\{-\sinh r\sqrt{1-\frac{1}{\cosh^2 z}},\ \cosh r,\ 0\right\}.
\end{aligned} \quad (24)$$

When the curvature is null, Killing vectors take the form

$$\begin{aligned}
br1^{\alpha`} &= \left\{-z\cos\varphi,\ r\cos\varphi,\ \frac{z\sin\varphi}{r}\right\},\\
br2^{\alpha`} &= \left\{-z\sin\varphi,\ r\sin\varphi,\ -\frac{z\cos\varphi}{r}\right\},\\
br3^{\alpha`} &= \{0,\ 0,\ 1\},\\
bt1^{\alpha`} &= \left\{\sin\varphi,\ 0,\ \frac{\cos\varphi}{r}\right\},\\
bt2^{\alpha`} &= \left\{\cos\varphi,\ 0,\ -\frac{\sin\varphi}{r}\right\},\\
bt3^{\alpha`} &= \{0,\ 1,\ 0\}.
\end{aligned} \quad (25)$$

Vectors $bt1^{\alpha`}$, $bt2^{\alpha`}$ and $bt3^{\alpha`}$ are provided by translation symmetry, vectors $br1^{\alpha`}$, $br2^{\alpha`}$ and $br3^{\alpha`}$ are provided by rotation symmetry in the flat space.

If we place vectors (24) in general solution (18) we will take six linearly independent solutions of the kinetic equation (9) without external magnetic field.

The external field and metrics for every case that has been considered above have the same symmetry of rotation around axis z. For this symmetry a Killing vector b_i exists that satisfies the conditions (19). Using Killing vectors (24) solutions of kinetics equations (9), (12) and (15) can be written as

$$\rho_{r1} = C\,\mathrm{e}^{-\frac{p^\varphi\cosh^2 z\sinh^2 r+\frac{eB}{c}(\cosh r-1)}{k_B T}}, \quad (26)$$

$$\rho_{r2} = C\,\mathrm{e}^{-\frac{p^\varphi\sinh^2 r+\frac{eB}{c}\ln\left[(\cosh r)\right]}{k_B T}}, \quad (27)$$

$$\rho_{r3} = C\,\mathrm{e}^{-\frac{p^\varphi\mathrm{e}^{-2z}r^2+\frac{eB}{c}\frac{r^2}{2}}{k_B T}}. \quad (28)$$

For a case when the curvature is null, solutions (20)-(22) will become the similar solution of kinetic equation, which describe the one-point distribution function in a flat space. In a flat space, there also exists a solution connected with translation along axis z. For Lobachevsky space a similar solution exists only for the second case that is described by cylindrical coordinates with magnetic field (11). This solution has the form

$$\rho_{z2} = C\,\mathrm{e}^{-\frac{p^z\cosh^2 r}{k_B T}}. \quad (29)$$

For hyperbolic cylindrical and hyperbolic horospheric coordinates with their magnetic fields Killing vectors or linear combination of these vectors, which satisfy (19) except (26) and (28) don't exist.

For flat space, solutions (20)-(22) become the relativistic Maxwell distribution. For a non-relativistic case, solutions (20)-(22) take the form

$$\rho_1 = C\,\mathrm{e}^{-\frac{\cosh^2 z\left[\left(p^r\right)^2+\sinh^2 r\left(p^\varphi\right)^2\right]+\left(p^z\right)^2}{2mk_BT}}, \quad (30)$$

$$\rho_2 = C\,\mathrm{e}^{-\frac{\left(p^r\right)^2+\sinh^2 r\left(p^\varphi\right)^2+\cosh^2 r\left(p^z\right)^2}{2mk_BT}}, \quad (31)$$

$$\rho_3 = C\,\mathrm{e}^{-\frac{\mathrm{e}^{-2z}\left(p^r\right)^2+\mathrm{e}^{-2z}r^2\left(p^\varphi\right)^2+\left(p^z\right)^2}{2mk_BT}}. \quad (32)$$

The expressions (20)-(22) are more convenient for the following physical applications among all solutions which was developed above. They are provides the well known physical limits namely: the relativistic Maxwell distribution and non relativistic Maxwell distributions in the curved and flat spaces. In the expressions (20)-(22) not be included magnetic field manifestly. However evidently the magnetic field arise in these expressions when we take in to account possibility of the magnetic moments of particles.

Conclusion

In this work we have studied some solutions of kinetic equations for one-point distribution function on the background Lobachevsky space with uniform magnetic field for equilibrium state. For each case we have found a solution which will become the relativistic Maxwell distribution when the curvature of space is null.

And for the second case with cylindrical coordinates, we have found two solutions that are connected with symmetry of the magnetic field (translation along axis *z* and rotation around axis *z*). Similar solutions exist in flat space.

For the first and third cases in hyperbolic cylindrical and hyperbolic horospheric coordinates with their magnetic fields, only one solution has been found which corresponds by symmetry of rotation around axis *z*. The solution, which is connected with translation along axis *z*, does not exist in the first and third cases but appears in flat space.

The same symmetric properties appeared in the solution of quantum-mechanical problems [8].

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СИСТЕМА ЧАСТИНОК У МАГНІТНОМУ ПОЛІ НА ФОНІ ПРОСТОРУ ЛОБАЧЕВСЬКОГО

У цій статті розглянуто кінетичні рівняння для одноточкової функції розподілу на фоні простору Лобачевського в гіперболічних, циліндричних, циліндричних і гіперболічних горисферичних координатах з їх різними однорідними магнітними полями, які подібніі в плоскому просторі. Деякі розв'язки кінетичного рівняння були знайдені для рівноважного стану, і вони були порівняні з розв'язками кінетичного рівняння у плоскому просторі.

Ключові слова: простір Лобачевського, кінетичне рівняння, однорідне магнітне поле, вектор Кіллінга.

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СИСТЕМА ЧАСТИЦ В МАГНИТНОМ ПОЛЕ НА ФОНЕ ПРОСТРАНСТВА ЛОБАЧЕВСКОГО

В этой статье рассмотрены кинетические уравнения для одноточечной функции распределения на фоне пространства Лобачевского в гиперболических, цилиндрических, цилиндрических и гиперболических ориосферических координатах с их различными однородными магнитными полями, которые подобны в плоском пространстве. Некоторые решения кинетического уравнения были найдены для равновесного состояния, и они были сравнены с решениями кинетического уравнения в плоском пространстве.

Ключевые слова: пространство Лобачевского, кинетическое уравнение, однородное магнитное поле, вектор Киллинга.