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DEVELOPMENT OF A MODEL FOR THE DYNAMICS OF PROBABILITIES OF STATES OF SEMI-MARKOV SYSTEMS

The **subject** is the study of the dynamics of probability distribution of the states of the semi-Markov system during the transition process before establishing a stationary distribution. The **goal** is to develop a technology for finding analytical relationships that describe the dynamics of the probabilities of states of a semi-Markov system. The **task** is to develop a mathematical model that adequately describes the dynamics of the probabilities of the states of the system. The initial data for solving the problem is a matrix of conditional distribution laws of the random duration of the system's stay in each of its possible states before the transition to some other state. **Method.** The traditional method for analyzing semi-Markov systems is limited to obtaining a stationary distribution of the probabilities of its states, which does not solve the problem. A well-known approach to solving this problem is based on the formation and solution of a system of integral equations. However, in the general case, for arbitrary laws of distribution of the durations of the stay of the system in its possible states, this approach is not realizable. The desired result can only be obtained numerically, which does not satisfy the needs of practice. To obtain the required analytical relationships, the Erlang approximation of the original distribution laws is used. This technique significantly increases the adequacy of the resulting mathematical models of the functioning of the system, since it allows one to move away from overly obligatory exponential descriptions of the original distribution laws. The formal basis of the proposed method for constructing a model of the dynamics of state probabilities is the Kolmogorov system of differential equations for the desired probabilities. The solution of the system of equations is achieved using the Laplace transform, which is easily performed for Erlang distributions of arbitrary order. **Results.** Analytical relations are obtained that specify the desired distribution of the probabilities of the states of the system at any moment of time. The method is based on the approximation of the distribution laws for the durations of the stay of the system in each of its possible states by Erlang distributions of the proper order. A fundamental motivating factor for choosing distributions of this type for approximation is the ease of their use to obtain adequate models of the functioning of probabilistic systems. Conclusions. A solution is given to the problem of analyzing a semi-Markov system for a specific particular case, when the initial distribution laws for the duration of its sojourn in possible states are approximated by second-order Erlang distributions. Analytical relations are obtained for calculating the probability distribution at any time.

Keywords: Semi-Markov system; model of dynamics of probabilities of states; approximating Erlang distributions; analytical calculation of probabilities of states.

Introduction

Traditional technologies for constructing models of the functioning of complex systems are based on the use of Markov theory. The corresponding models constructively use the Markov property of the behavior of such systems [1]. The simplicity and efficiency of such models are a natural consequence of the fact that in Markov systems the distribution density of the duration of the stay of the system in any particular state is determined only by this state, but does not depend on when and how the analyzed system got into this state. This circumstance makes it possible, using simple Kolmogorov differential equations [2, 3], to solve the problem of finding the distribution of the probabilities of the stay of the system on the set of possible states with an analytical description of the dynamics of this distribution. Failure to fulfill the Markov property leads to a significant complication of the problem of analyzing the behavior of systems. Difficulties in solving the corresponding problems make the problem of developing special methods for constructing behavior models of semi-Markov systems relevant.

Literature analysis

The semi-Markov process, as it is known [4, 5], differs from the Markov process in that the distribution law of the random duration of stay in each of the possible states is not exponential. There are several alternative ways to define a semi-Markov process [6]. The least demanding of them in terms of the amount of information used is as follows. Sets of possible states E and transitions between them, as well as the matrix $(Q_{ij}(t))$ of independent distribution functions of the time spent by the process in state I before the transition to state j, i∈E, j∈E, are given. Moreover, if tij is the random duration of stay in i before the transition to j, so $Q_{ij}(t) = P(t_{ij} < t)$. Then the probability of transition Pij(t) from I to j is the probability that no transition to any other state occurs during this time. This probability is equal to

$$P_{ij}(t) = \int_0^t \prod_{k \neq j} (1 - Q_{ik}(\tau)) dQ_{ij}(\tau) = P(\zeta(t) = j, \\ t_{ij} < t / \zeta(0) = i), i \neq j. \quad (1)$$

The set of functions $P_{ij}(t)$ together with the initial state also uniquely define a semi-Markov process. The probability $P_{ij}(\infty)$ of transition from i to j in an unlimited time is

$$P_{ij} = P_{ij}(\infty) = \int_0^\infty \prod_{k \neq j} (1 - Q_{ik}(\tau)) dQ_{ij}(\tau) \quad (2)$$

and determines the probability of transition of the Markov chain embedded in the process. Wherein $\sum_{j \neq i} P_{ij} = 1.$

Next, a conditional distribution function of the duration of stay in *I* before the transition to *j* is introduced, which is equal to

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$$F_{ij}(t) = P(t_{ij} < t / \zeta(0) = i, \zeta(t) = j). \qquad (3)$$

Then

$$P_{ij}(t) = F_{ij}(t) P_{ij}. \qquad (4)$$

The matrix of transition probabilities of the embedded Markov chain $P=(P_{ij})$ together with the matrix of conditional distribution functions $F(t)=(F_{ij}(t))$ and the initial state determine the third way of specifying the Markov process [6].

The fourth way of setting this process is implemented as follows [7]. The unconditional distribution function of the duration of stay in i before leaving to some other state is determined

$$F_{ij}(t) = P(t_i < t) = \sum_{j \neq i} P_{ij} F_{ij}(t) = \sum_{j \neq i} P_{ij}(t). \qquad (5)$$

The same function can be defined through the original matrix (Qij(t)):

$$F_{ij}(t) = 1 - \prod_{j \neq i} (1 - Q_{ij}(t)). \qquad (6)$$

The expression corresponding to (5) for the distribution density of the duration of stay in i before leaving has the form

$$f_i(t) = \sum_{j \neq i} p_{ij} \frac{dF_{ij}(t)}{dt} = \sum_{j \neq i} p_{ij} f_{ij}(t). \qquad (7)$$

Next, the matrix of conditional transition probabilities is introduced

$$q_{ij}(t) = P(\zeta(t) = j / t_{ij} = t, \zeta(0) = i).$$

Herewith

$$p_{ij}(t) = P(\zeta(t) = j / t_{ij} = t, \zeta(0) = i) = \int_0^t q_{ij}(\tau) f_i(\tau) d\tau.$$

All of the above methods for specifying a semi-Markov process are equivalent.

In the works on the analysis of semi-Markov systems, the results of solving the following two important problems are used. The first task is to calculate the average durations of the stay of the system in each of their states, which is solved using the standard ratio

$$\overline{\tau}_i = \int_0^\infty t f_i(t) dt = \sum_{j \neq i} P_{ij} \int_0^\infty t f_{ij}(t) dt = \sum_{j \neq i} P_{ij} \overline{\tau}_{ij}. \qquad (7)$$

The second task is to obtain the final distribution of the probabilities of the system states, which is solved as follows. First, using the transition probability matrix P=(P_{ij}), the stationary distribution of state probabilities for the Markov chain embedded in the semi-Markov process is found. The vector $\pi=(\pi_1, \pi_2, \ldots, \pi_n)$ of these probabilities is found by solving the system of equations

$$\pi = \pi P, \qquad \sum_{i=1}^{n} \pi_i = 1. \qquad (8)$$

Now, as shown in [6, 7], the desired final distribution of state probabilities is determined by the relations

$$P_i = \frac{\pi_i \overline{\tau}_i}{\sum_{i=1}^{n} \pi_i \overline{\tau}}, \; i = 1, 2, \ldots, n. \qquad (9)$$

The obtained results (7) and (9) are very useful in solving practical problems. However, in many cases, for example, when solving control problems for semi-Markov systems, it is necessary to have an analytical description of the dynamics of state probabilities. In addition, it is important to know the dependences of the values of the components of the probability distribution of the system states on the numerical values of the system parameters.

In this regard, the purpose of the study is to develop a technology for finding the dynamics of the probability distribution of the states of a semi-Markov system.

Main result. Calculation of the probability distribution of the states of a semi-Markov system. Consider a system with its possible states and build a model of the functioning of such systems.

Let us introduce, using (3), $F_{ij}(t) = P(\tau_{ij} < t)$ – the probability that the random duration τij of the stay of the system at *i* before the transition to state *j* will be less than *t*. Then the probability of no transition to state *j* on the interval [*t*, *t*+ *τ*] is defined as follows:

$$Q_{ij}(t+\tau) = Q_{ij}(t) Q_{ij}(t, t+\tau);$$

$$Q_{ij}(t, t+\tau) = \frac{Q_{ij}(t+\tau)}{Q_{ij}(t)}. \qquad (10)$$

In this case, the probability of transition from *i* to *j* on this interval will be equal to

$$W_{ij}(t, t+\tau) = 1 - Q_{ij}(t, t+\tau) = 1 - \frac{Q_{ij}(t+\tau)}{Q_{ij}(t)} = \frac{Q_{ij}(t) - Q_{ij}(t+\tau)}{Q_{ij}(t)} = \frac{(1 - Q_{ij}(t+\tau)) - (1 - Q_{ij}(t))}{Q_{ij}(t)} = \frac{F_{ij}(t+\tau) - F_{ij}(t)}{1 - F_{ij}(t)}. \qquad (11)$$

Let the distribution law of the duration of stay at *i* before going to *j* be the Erlang distribution law of order m. Wherein [8]

$$F_i(t) = 1 - \sum_{m=0}^{m-1} \frac{(\lambda_{ij} t)^m}{m!} e^{-\lambda ij t}.$$

If *m*=2, so

$$F_{ij}(t) = 1 - (1 + \lambda_{ij} t) e^{-\lambda ij t}. \qquad (12)$$

Now, substituting (12) into (11), we obtain the probability of transition from *i* to *j* on the interval [*t*, *τ*+*t*].

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$$W_{ij}(t,\, t+\tau) = \frac{[1-(1+\lambda_{ij}\, t+\tau)e^{-\lambda ij t+\tau}] - [1-(1+\lambda_{ij}\, t)e^{-\lambda ij t}]}{1-(1+\lambda_{ij}\, t)e^{-\lambda ij t}} = \\ = \frac{(1+\lambda_{ij}\, t)-(1+\lambda_{ij}\, t+\tau)e^{-\lambda ij \tau}}{1+\lambda_{ij}\, t} = 1 - \frac{1+\lambda_{ij}\,(t+\tau)}{1+\lambda_{ij}\, t} e^{-\lambda ij \tau}. \quad (13)$$

Then the distribution density of the duration of stay at i before the transition to j on the interval $[t,\ t+\tau]$ has the form:

$$f(t,\tau) = \frac{\lambda_{ij}^{\ 2}\,(t+\tau)}{1+\lambda_{ij}\, t} e^{-\lambda ij \tau}. \quad (14)$$

It is clear that at $t = 0$ relation (13) is reduced to formula (12), which determines the Erlang distribution law of order 2 for an interval of length t, and relation (14) takes the form

$$f(t) = \lambda_{ij}^{\ 2}\, \tau e^{-\lambda ij \tau}, \quad (15)$$

usual for the second-order Erlang distribution density. It follows from relation (14) that the distribution density of the random duration of stay in state i before transition to j on the interval $[t,\ t+\tau]$ depends on the value of the parameter τ. This value can be random. Consider, for example, a two-channel queuing system with an incoming flow of claims, in which the distribution density of the interval between claims is described, for example, by the second-order Erlang law (15). A request received at the moment when both channels are free takes one of them and begins to be served. Let the next request arrive at a random time interval. If the servicing of the previous customer by the first channel has not been completed by this moment τ, then the next customer occupies the free second channel. Let the distribution laws of the random value of the interval between incoming customers and the duration of service of each of the channels have the form, respectively [9]

$$F(\tau) = 1-(1+\lambda\tau)e^{-\lambda\tau}, \\ \mathrm{G}(\tau) = 1-(1+\mu\tau)e^{-\mu\tau}. \quad (16)$$

Then the probability of completion of servicing by the first channel on the interval $[T,\ T+\tau]$ in accordance with (13) is

$$F_1(T,\, T+\tau) = 1 - \frac{1+\mu(T+\tau)}{1+\mu T} e^{-\mu\tau}. \quad (17)$$

The corresponding distribution density is described by the formula

$$f_1(\mathrm{T},\mathrm{T}+\tau) = \frac{\mu^2\,(T+\tau)}{1+\mu t} e^{-\mu\tau}. \quad (18)$$

In addition, in accordance with (16), the distribution law for the duration of servicing a customer by the second channel is determined by the relation

$$\mathrm{G}_2(\tau) = 1-(1+\mu\tau)e^{-\mu\tau}, \quad (19)$$

and the distribution density of this duration is

$$f_2(\tau) = \mu^2 \tau e^{-\mu\tau}. \quad (20)$$

The random moment of the end of servicing by any of the two busy channels is specified by the minimum of the durations of servicing claims by these channels. The corresponding distribution density is determined by the densities $f_1(\mathrm{T},\mathrm{T}+\tau 1)$ and $f2(\tau 2)$ as follows. Let $x1 = T + \tau 1$ and $x2 = \tau 2$ determine the service duration values for the first and second channels. We introduce $u = min\{x1, x2\}$. The distribution function of this random variable u has the form

$$G(u) = 1-(1-G_1(u)(1-G_2(u)), \quad (21)$$

The corresponding distribution density, found by differentiating (21) with respect to u, is equal to

$$g(u) = f_1(u)(1-G_2(u)) + f_2(u)(1-G_1(u)). \quad (22)$$

Let us substitute functions (17) - (20) into (22), bearing in mind that at the time of service by any of the channels $\tau_1=\tau_2=\tau$. In this case, the random variable u takes on the value $T+\tau$ if the first channel completed servicing earlier than the second, and corresponds to the value of τ if the second channel finished servicing earlier than the first. That's why

$$g(\mathrm{T},\tau) = \frac{\mu^2\,(T+\tau)}{1+\mu T} e^{-\mu\tau}(1+\mu\tau)e^{-\mu\tau} + \mu^2\tau e^{-\mu\tau}\frac{1+\mu(T+\tau)}{1+\mu T}e^{-\mu\tau} = \\ = \frac{\mu^2}{1+\mu T}\Big[(T+\tau)(1+\mu\tau)e^{-2\mu\tau} + \tau(1+\mu(T+\tau)\Big]e^{-2\mu\tau} = \\ = \frac{\mu^2}{1+\mu T}\big[(T+\tau)(1+\mu\tau)+\tau(1+\mu(T+\tau)\big]e^{-2\mu\tau} = \\ = \frac{\mu^2}{1+\mu T}(T + T\mu\tau + \tau + \mu\tau^2 + \tau + \mu\tau^2 + \mu T\tau)e^{-2\mu\tau} = \\ = \frac{\mu^2}{1+\mu T}(\tau + (T+\tau) + 2\mu T\tau + 2\mu\tau^2))e^{-2\mu\tau} = \\ = \frac{\mu^2}{1+\mu T}(\tau + (T+\tau) + 2\mu\tau(T+\tau))e^{-2\mu\tau} = \\ = \frac{\mu^2}{1+\mu T}(\tau + (T+\tau)(1+2\mu\tau))e^{-2\mu\tau}. \quad (23)$$

It follows from this relation that the analytical description of the distribution density of the duration of the stay of the system in the state when both channels are occupied until the moment one of them is released contains a random parameter T. The fundamental complexity of the situation arising in this case is as follows. Relation (23) describes not a single density, as it happens when all the parameters of the distribution density are known constants. The random nature of the parameter T gives rise to a family of densities, each of which corresponds to a specific value of this parameter. It is clear that in this situation a direct solution to the problem of analyzing such a system is impracticable. An approximate solution can be obtained if the parameter T is set by its mean value $\bar{T}$. In this case, relation (23) takes the form

$$g(\bar{T},\tau) = \frac{\mu^2}{1+\mu\bar{T}}(\tau + (\bar{T}+\tau)(1+2\mu\tau))e^{-2\mu\tau}. \quad (24)$$

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This relationship makes it possible, using numerical integration, to calculate the average value of the duration of the system's stay in the busy state of both channels, and from this to obtain an estimate of the intensity $\hat{\mu}$ of the system's exit from this state.

The above distributions and calculations (23) - (24) allow us to analyze the process of functioning of the considered two-channel queuing system. Let us introduce a set *{Ei}*, *i* = 0,1,2, of system states corresponding to the number of occupied channels. We find the distribution of the probabilities of the states of the system *Pk(t)*, *k*=0,1,2, by solving the Kolmogorov system of differential equations [10]

$$\frac{dP_k(t)}{dt} = \sum_{j\in Z_k^+} \lambda_{jk} P_j(t) - P_k(t)\sum_{j\in Z_k^-}\lambda_{kj},\ k=0,1,2. \quad (25)$$

where, λ_{jk} – intensity of system transition from state *j* to state *k*; Z_k^+ – the set of system states, from which a direct transition to the *k*-th state is possible; Z_k^- – the set of states of the system into which a direct transition from the *k*-th state is possible.

We solve the resulting system of equations using the Laplace transform. As is known, the Laplace transform of the function *u(t)* is the function

$$L(u(t)) = F(s) = \int_0^\infty u(t)e^{-st}dt. \quad (26)$$

The Laplace transform of the derivative *u'(t)* of the function *u(t)* is defined by the relation

$$L(u'(t)) = \int_0^\infty u'(t)e^{-st}dt. \quad (27)$$

Integrating (27) by parts, we obtain

$$L(u'(t)) = \int_0^\infty e^{-st}u'(t)dt =$$

$$= u(t)e^{-st}\Big|_0^\infty + s\int_0^\infty u(t)e^{-st}dt = sL(u(t)) - u(0).$$

Transforming (25) according to Laplace, we obtain

$$s\pi_k(s) - P_k(0) = \sum_{j\in Z_k^+}\lambda_{jk}\pi_j(s) - P_k(s)\sum_{j\in Z_k^-}\lambda_k, \quad (28)$$

where $\pi_k(s) = L(P_k(t))$.

In the considered task we have $Z_0^+=1;\ Z_0^-=1;\ Z_1^+=\{0;2\};\ Z_1^-=\{0;2\};\ Z_2^+=1;\ Z_2^-=1;\ P_0(0)=1;\ P_1(0)=P_2(0)=0$. Therefore, the system of equations (28) is simplified to the form

$$\begin{aligned} &s\pi_0(s) = \lambda_{10}\pi_1(s) - \lambda_{01}\pi_0(s) + 1 = 0,\\ &s\pi_1(s) = \lambda_{01}\pi_0(s) + \lambda_{21}\pi_2(s) - (\lambda_{10}+\lambda_{12})\pi_1(s),\\ &s\pi_2(s) = \lambda_{12}\pi_1(s) - \lambda_{21}\pi_2(s). \end{aligned}$$

After reducing similar terms, we get:

$$\begin{aligned} &(s+\lambda_{01})\pi_0(s) - \lambda_{10}\pi_1(s) = 1,\\ &\lambda_{01}\pi_0(s) - (s+\lambda_{10}+\lambda_{12})\pi_1(s) + \lambda_{21}\pi_2(s) = 0,\\ &s\lambda_{12}\pi_1(s) - (s+\lambda_{21})\pi_2(s) = 0. \end{aligned} \quad (29)$$

The system of linear algebraic equations (29) is solved in a standard way. By Cramer's rule, we have:

$$\pi_i(s) = \frac{D_i(s)}{D(s)},\ i=0,1,2, \quad (30)$$

where

$$D = \det\begin{pmatrix} (s+\lambda_{01}) & -\lambda_{10} & 0\\ \lambda_{01} & -(s+\lambda_{10}+\lambda_{12}) & \lambda_{21}\\ 0 & \lambda_{12} & (s+\lambda_{21}) \end{pmatrix},$$

$$D_0 = \det\begin{pmatrix} 1 & -\lambda_{10} & 0\\ 0 & -(s+\lambda_{10}+\lambda_{12}) & \lambda_{21}\\ 0 & \lambda_{12} & (s+\lambda_{21}) \end{pmatrix},$$

$$D_1 = \det\begin{pmatrix} (s+\lambda_{01}) & 1 & 0\\ \lambda_{01} & 0 & \lambda_{21}\\ 0 & 0 & (s+\lambda_{21}) \end{pmatrix},$$

$$D_2 = \det\begin{pmatrix} (s+\lambda_{01}) & -\lambda_{10} & 1\\ \lambda_{01} & -(s+\lambda_{10}+\lambda_{12}) & 0\\ 0 & \lambda_{12} & 0 \end{pmatrix},$$

Note that when solving the resulting system of equations, it is necessary to take into account the normalization condition $P_0(t)+P_1(t)+P_2(t)=1$, which after the Laplace transform has the form

$$\sum_{k=0}^{2}\pi_k(s) = \frac{1}{s} \ \text{ or } \ s\sum_{k=0}^{2}\pi_k(s) = 1.$$

As a result of performing the necessary operations in accordance with (30), we obtain

$$\pi_0(s) = \frac{A_{00}+A_{01}s+A_{02}s^2}{B_0+B_1s+B_2s^2+B_3s^3}, \quad (31)$$

$$\pi_1(s) = \frac{A_{10}+A_{11}s}{B_0+B_1s+B_2s+B_3s}, \quad (32)$$

$$\pi_2(s) = \frac{A_{20}+A_{21}s}{B_0+B_1s+B_2s+B_3s}. \quad (33)$$

We carry out the inverse Laplace transform by expanding the fractional rational functions (31) - (33) into elementary fractions. To do this, it is necessary to find the roots of the polynomial in the denominator of the functions being expanded by solving the equation

$$B_0+B_1s+B_2s^2+B_3s^3=0\,.$$

Let these roots be equal S_1, S_2, S_3.

The technology of further operations depends on the nature of these roots. In this case, in the general case, the following options are possible.

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a) All roots are real, different. Then

$$\frac{A_{00}+A_{01}s+A_{02}s^2}{B_0+B_1s+B_2s^2+B_3s^3}=\frac{a_{00}+a_{01}s+a_{02}s^2}{(s-s_1)(s-s_2)(s-s_3)}= \\ =\frac{\alpha_1}{s-s_1}+\frac{\alpha_2}{s-s_2}+\frac{\alpha_3}{s-s_3},\ a_{0i}=\frac{A_{0i}}{B_3},\ i=0,1,2,3. \quad (34)$$

The unknown coefficients α_1, α_2, α_3 are found after reducing (34) to a common denominator and equating the coefficients at the same powers of s to the left and right of the equal sign.

b) Roots are real, multiples. Then

$$\frac{a_{00}+a_{01}s+a_{02}s^2}{(s-s_1)^{k1}(s-s_2)^{k2}(s-s_3)^{k3}}=\frac{\alpha_{11}}{s-s_1}+\frac{\alpha_{12}}{(s-s_1)^2}+\frac{\alpha_{13}}{(s-s_1)^3}+ \\ +\frac{\alpha_{21}}{s-s_2}+\frac{\alpha_{22}}{(s-s_2)^2}+\frac{\alpha_{23}}{(s-s_2)^3}+ \quad (35) \\ +\frac{\alpha_{31}}{s-s_3}+\frac{\alpha_{32}}{(s-s_3)^2}+\frac{\alpha_{33}}{(s-s_3)^3}.$$

In this case, the total number of terms in relation (35) cannot exceed three. The method for finding the unknown coefficients in (35) is the same as above.

c) Among the roots there are complex. Then

$$\frac{a_{00}+a_{01}s+a_{02}s^2}{(s-s_1)(s^2-ps+q)}=\frac{\alpha_1}{s-s_1}+\frac{\beta s+\gamma}{s^2-ps+q}.$$

Unknown coefficients are found in the same way as before. Further, according to the correspondence table of the originals and their Laplace transforms, unknown functions $P_0(t)$, $P_1(t)$, $P_2(t)$, are found, which specify the desired distribution of the probabilities of the system states.

The task becomes much more complicated if the number of channels in the system is more than two. In this case, a complicating circumstance manifests itself even if the number of channels is only three. The point is as follows. In a two-channel system, the transition from the E_1 to E_2 state occurs only if the service of the previous request by the first channel is not completed. In a three-channel system, the situation is different. The fact is that before the transition to E_3, the system could wander for an unpredictable long time on the set of states E_0, E_1, E_2. Therefore, at the time of the transition of the system to E_3, it is impossible to estimate the duration of continuous operation of the two channels occupied by this moment, which excludes the possibility of implementing the technology under consideration. Apparently, the only real way to solve the problem of analyzing an n-channel semi-Markov queuing system is to construct and solve a system of interval-transition equations [11] with respect to the dynamics of the probability distribution of the states of the system.

A possible direction for further research consists in the development of methods for solving the problem in the case when the initial data are given indistinctly [12] or inaccurately [13]. To solve these problems, the methods proposed in [14] and [15] can be useful.

Conclusions

A technology for constructing a model of the dynamics of probabilities of states of a semi-Markov system is proposed. When solving the problem, the Erlang approximation of the conditional distribution laws of the random duration of the stay of the system in each of the possible states of the system before the transition to another state is used. The problem is reduced to a system of differential equations for the sought state probabilities. The solution of the system of equations is achieved using the Laplace transform.

References

1. Dynkin, E. B. (1963), *Markov processes* [*Markovskiye protsessy*], Moscow : Fizmatgiz, 583 p.
2. Barucha-Reed, A. T. (1969), *Elements of the theory of Markov processes and their applications* [*Elementy teorii markovskikh protsessov i ikh prilozheniya*], Moscow : Nauka, 248 p.
3. Kemeny, J., Snell, J. (1970), *Markov's ultimate goals* [*Konechnyye tseli Markova*], Moscow : Nauka, 208 p.
4. Silvestrov, D. S. (1971), *Semi-Markov processes with a discrete set of states* [*Polumarkovskiye protsessy s diskretnym mnozhestvom sostoyaniy*], Kyiv : KNU, 186 p.
5. Barlow, R. E. (1962), "Applications of semi-Markov processes to counter problems", *Stud. appl. prob. Stanford*, Calif. Univ. Press, P. 34–62.
6. Korolyuk, V. S. (1967), "Semi-Markov processes and their applications" ["Polumarkovskiye protsessy i ikh prilozheniya"], Cybernetics, No. 5, P. 58–65.
7. Korolyuk, V. S., Brody, S. M., Turbin, A. F. (1974), "Semi-Markov processes and their application" ["Polumarkovskiye protsessy i ikh primeneniye"], *Results of science and technology. Ser. Theor. ver. Mat. Stat.*, Vol. II, P. 47–97.
8. Ventzel, E. S., Ovcharov, L. A. (1983), *Applied Problems of Probability Theory* [*Prikladnyye zadachi teorii veroyatnostey*], Moscow : Radio and Communication.
9. Pugachev, V. S. (1962), *The theory of random functions* [*Teoriya sluchaynykh funktsiy*], Moscow : Fizmatgiz, 659 p.
10. Kofman, A., Kruon, R. (1965), *Mass service, theory and applications* [*Massovoye obsluzhivaniye, teoriya i prilozheniya*], Moscow : MIR, 302 p.
11. Raskin, L. G. (1988), *Mathematical methods for researching operations and analyzing complex systems* [*Matematicheskiye metody issledovaniya operatsiy i analiza slozhnykh sistem*], Kharkiv : VIRTA, 178 p.
12. Zadeh, L. (1965), "Fuzzy Sets", *Information and Control*, Vol. 8, P. 338–353.

13. Pawlak, Z. (1997), "Rough Sets approach to knowledge–based decision support", *European Journal of Operation Research*, Vol. 99, No. 1, P. 48–57.
14. Raskin, L., Sira, O. (2016), "Fuzzy models of rough mathematics", *Eastern-European Journal of Enterprise Technologies*, Vol. 6, Issue 4, P. 53–60. DOI: 10.15587/1729-4061.2016.86739
15. Raskin, L., Sira, O. (2016), "Method of solving fuzzy problems of mathematical programming", *Eastern-European Journal of Enterprise Technologies*, Vol. 5, Issue 4, P. 23–28. DOI: 10.15587/1729-4061.2016.81292

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РОЗРОБКА МОДЕЛІ ДИНАМІКИ ІМОВІРНОСТЕЙ СТАНІВ НАПІВМАРКОВСЬКИХ СИСТЕМ

Предмет – дослідження динаміки розподілу ймовірностей станів напівмарковських системи протягом перехідного процесу до встановлення стаціонарного розподілу. **Мета** – розробка технології відшукання аналітичних співвідношень, що описують динаміку ймовірностей станів напівмарковських системи. **Завдання** розробка математичної моделі, адекватно описує динаміку ймовірностей станів системи. Вихідні дані для вирішення завдання - матриця умовних законів розподілу випадкової тривалості перебування системи в кожному з можливих її станів до переходу в будь-яке інше стан. **Метод.** Традиційний метод аналізу напівмарковських систем обмежується отриманням стаціонарного розподілу ймовірностей її станів, що не вирішує поставлену задачу. Відомий підхід до вирішення цього завдання заснований на формуванні та вирішенні системи інтегральних рівнянь. Однак в загальному випадку для довільних законів розподілу тривалостей перебування системи в можливих своїх станах цей підхід не реалізуємо. Шуканий результат може бути отриманий тільки чисельно, що не задовольняє потреби практики. Для отримання необхідних аналітичних співвідношень використовується ерланговський апроксимація вихідних законів розподілу. Цей прийом істотно підвищує адекватність одержуваних при цьому математичних моделей функціонування системи, так як дозволяє відійти від надмірно зобов'язують експоненційних описів вихідних законів розподілу. Формальна основа запропонованого методу побудови моделі динаміки ймовірностей станів - система диференціальних рівнянь Колмогорова щодо шуканих ймовірностей. Рішення системи рівнянь досягається з використанням перетворення Лапласа, яке легко здійснимо для ерланговський розподілів довільного порядку. **Результати.** Отримано аналітичні

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співвідношення, які визначають шуканий розподіл ймовірностей станів системи на будь-який момент часу. Метод заснований на апроксимації законів розподілу тривалостей перебування системи в кожному з можливих своїх станів розподілами Ерланга належного порядку. Принциповим мотивуючим обставиною для вибору з метою апроксимації розподілів саме цього типу є простота їх використання для отримання адекватних моделей функціонування імовірнісних систем. **Висновки.** Наведено рішення задачі аналізу напівмарковських системи для конкретного окремого випадку, коли вихідні закони розподілу тривалості її перебування в можливих станах апроксимуються розподілами Ерланга другого порядку. Отримано аналітичні співвідношення для розрахунку розподілу ймовірностей на будь-який момент часу.

Ключові слова: напівмарковських система; модель динаміки ймовірностей станів; апроксимуючий розподіл Ерланга; аналітичний розрахунок ймовірностей станів.

РАЗРАБОТКА МОДЕЛИ ДИНАМИКИ ВЕРОЯТНОСТЕЙ СОСТОЯНИЙ ПОЛУМАРКОВСКИХ СИСТЕМ

Предмет – исследование динамики распределения вероятностей состояний полумарковской системы в течение переходного процесса до установления стационарного распределения. **Цель** – разработка технологии отыскания аналитических соотношений, описывающих динамику вероятностей состояний полумарковской системы. **Задача** – разработка математической модели, адекватно описывающей динамику вероятностей состояний системы. Исходные данные для решения задачи – матрица условных законов распределения случайной продолжительности пребывания системы в каждом из возможных ее состояний до перехода в какое-либо другое состояние. **Метод.** Традиционный метод анализа полумарковских систем ограничивается получением стационарного распределения вероятностей ее состояний, что не решает поставленную задачу. Известный подход к решению этой задачи основан на формировании и решении системы интегральных уравнений. Однако в общем случае для произвольных законов распределения продолжительностей пребывания системы в возможных своих состояниях этот подход не реализуем. Искомый результат может быть получен только численно, что не удовлетворяет потребности практики. Для получения требуемых аналитических соотношений используется эрланговская аппроксимация исходных законов распределения. Этот прием существенно повышает адекватность получаемых при этом математических моделей функционирования системы, так как позволяет отойти от чрезмерно обязывающих экспоненциальных описаний исходных законов распределения. Формальная основа предложенного метода построения модели динамики вероятностей состояний – система дифференциальных уравнений Колмогорова относительно искомых вероятностей. Решение системы уравнений достигается с использованием преобразования Лапласа, которое легко выполнимо для эрланговских распределений произвольного порядка. **Результаты.** Получены аналитические соотношения, задающие искомое распределение вероятностей состояний системы на любой момент времени. Метод основан на аппроксимации законов распределения продолжительностей пребывания системы в каждом из возможных своих состояний распределениями Эрланга надлежащего порядка. Принципиальным мотивирующим обстоятельством для выбора в целях аппроксимации распределений именно этого типа является простота их использования для получения адекватных моделей функционирования вероятностных систем. **Выводы.** Приведено решение задачи анализа полумарковской системы для конкретного частного случая, когда исходные законы распределения продолжительности ее пребывания в возможных состояниях аппроксимируются распределениями Эрланга второго порядка. Получены аналитические соотношения для расчета распределения вероятностей на любой момент времени.

Ключевые слова: полумарковская система; модель динамики вероятностей состояний; аппроксимирующие распределения Эрланга; аналитический расчет вероятностей состояний.

Бібліографічні описи / Bibliographic descriptions

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