**Y. G. Stoyan**, Associate Member of the NASU
**A. M. Chugay,** Cand. Sc. (Engineering)

A. Podgorny Institute of Mechanical Engineering Problems of NASU, Kharkiv, Ukraine
e-mail: stoyan@ipmach.kharkov.ua

**Keywords**: *packing, homothetic polytopes, rotations, optimization, phi-functions.*

UDC 519.85

# PACKING CONVEX HOMOTHETIC POLYTOPES INTO A CUBOID

*A mathematical model of the problem of packing homothetic polytopes into a cuboid of minimal volume based on the fi-function for two convex polytopes is presented. A number of features of the mathematical model are noted. Based on the peculiarities, an approach for constructing starting points, a fast algorithm for finding local minima, and an exhaustive search for local minima to obtain a good approximation to the global extremum are proposed. Numerical results are given.*

## Introduction

3D packing problems are classical optimization problems which have extensive engineering applications. At present, the interest in finding effective solutions to the problems is growing rapidly.

These problems combine a large set of various practical problems related with finding the optimal packings of geometric objects of one type into the objects of another. In most cases, when solving 3D packing problems, it is necessary to pack all the given objects into containers of minimum sizes.

Such problems arise in various fields of science and engineering where full-scale experiments are replaced by computer simulation, which significantly saves time and material resources. For example, an actual application of the problems is the 3D simulation of microstructures of different materials (including nanomaterials) [1, 2]. Recent advances in the field have been related with the development of the computer technology of the 3D X-ray tomography analysis of mineral particles [2].

There also exists a wide spectrum of problems in modern biology, mineralogy, medicine, materials science, nanotechnology, robotics, pattern recognition systems, control systems, space apparatus control systems, as well as in chemical industry, power engineering, mechanical engineering, shipbuilding, aircraft construction, civil engineering, etc.

It is well known that the 3D object packing problem is NP-complete. Because of its NP complexity, the problem is difficult to solve satisfactorily. This is why in order to find its approximate solution a lot of researchers use a very wide variety of techniques, including heuristics (heuristics based on different approximation rules [3], genetic algorithms [4], simulated annealing algorithm [5], artificial bee colony algorithm [6]), extended pattern search [7], traditional optimization methods [8, 9] and different mixed approaches utilizing heuristics and methods of non-linear mathematical programming [10].

As mentioned in [3] solution processes consist of the following loop procedures: 1) ordering a sequence of objects; 2) applying geometric procedures to the objects according to their position in the sequence; 3) calculating an objective function.

The geometric procedures can be implemented by methods which differ by: the path of object movement, complexity of rotation modeling and whether an intersection object is allowed during solution phases.

In the majority of papers, either orientation changes of 3D objects are not allowed or only discrete changes in the orientation for given angles (45 or 90 degrees) are allowed. For example, in [11] only the parallel translation algorithm is used for packing convex polytopes. In [12] the authors propose the HAPE3D algorithm which can be applied to an arbitrarily shaped polyhedral which, in its turn, can rotate around each coordinate axis at eight different angles only.

In [13] the authors point out that for 3D packing problems making calculations of 0 to 360 degrees orientations of objects with respect to each axis is impossible.

At present, due to the difficulty in constructing adequate mathematical models, there are few works that solve 3D packing problems provided that continuous rotations of geometric objects are allowed. The solutions to the problems are considered in [8, 9, 14, 15, 16]. In [8, 9, 14] continuous and differentiable nonlinear programming models and algorithms for packing ellipsoids in the 3-dimensional space are introduced. In paper [16] the problem of packing different convex 3D objects is solved.

This work is devoted to solving the packing problem of homothetic convex polytopes with continuous angles of rotation. Our approach is based on the mathematical simulation of relations between geometric objects reducing the problem solution to nonlinear programming. To this end, we use the phi-function technique [9, 15] for the analytic description of object interaction and allocation in a container, taking into account continuous rotations and parallel translations.

## 1. Problem statement

Let there be homothetic polytopes $P_i$, $i \in I = \{1,2,\ldots,n\}$ and a cuboid

$$C = \{X \in R^3, w_1 \le x \le w_2, l_1 \le x \le l_2, \eta_1 \le x \le \eta_2\}, \tag{1}$$

where $w_1, w_2, l_1, l_2, \eta_1$ and $\eta_2$ are variable, whence the vector $\Upsilon = (w_1, w_2, l_1, l_2, \eta_1, \eta_2)$ determines the sizes of $C$.

The polytopes $P_i$ are specified by the vertices

$$p_{io} = h_i^{\nabla}(p_{io}^1, p_{io}^2, p_{io}^3), i \in I, o \in O = \{1,2,\ldots,\varepsilon\}, \tag{2}$$

where $h_i^{\nabla}$ is the homothetic coefficient of $P_i$.

We suppose without loss of generality that

$$1 = h_1^{\nabla} \ge h_2^{\nabla} \ge \ldots \ge h_n^{\nabla} \tag{3}$$

and at least one inequality is strict.

On the ground of the vertices we construct the inequality systems

$$\{\chi_{ik}(X) = a_k x + b_k y + c_k z + h_i^{\nabla} d_k \ge 0, i \in I, k \in K = \{1,2,\ldots,\kappa\}, \tag{4}$$

which specify the polytopes $P_i$, $i \in I$, respectively. Let $F_{ik}$ be the planes specified by the equations $\chi_{ik}(X) = 0$, $k \in K$.

It follows from (1) that $C$ is described by the inequality system

$$\{\varpi_r(X, \Upsilon) \ge 0, r \in \{1,2,\ldots,6\}, \tag{5}$$

where $\varpi_1(X,\Upsilon) = x - w_1$, $\varpi_2(X,\Upsilon) = w_2 - x$, $\varpi_3(X,\Upsilon) = y - l_1$, $\varpi_4(X,\Upsilon) = l_2 - y$, $\varpi_5(X,\Upsilon) = z - \eta_1$, and $\varpi_6(X,\Upsilon) = \eta_2 - z$.

The location of the polytope $P_i$ in the Euclidean 3D arithmetic space $R^3$ is defined by the translation vector $v_i = (x_i, y_i, z_i)$ and rotation angles $\theta_i = (\varphi_i, \psi_i, \omega_i)$, $i \in I$. Thus, the motion vector $u_i = (v_i, \theta_i) = (x_i, y_i, z_i, \varphi_i, \psi_i, \omega_i)$ defines the location of $P_i$ in $R^3$, i.e. the vector $u = (u_1, u_2, \ldots, u_n) \in R^m$, where $m = 6n$, defines the location of $P_i$, $i \in I$, in $R^3$. Thus, the coordinates of vertices (2) take the form

$$p_{io}(u_i) = (p_{io}^1(u_i), p_{io}^2(u_i), p_{io}^3(u_i)) = (h_i^{\nabla} g_i (p_{ioj}^0)^T + x_i, h_i^{\nabla} l_i (p_{io}^0)^T + y_i, h_i^{\nabla} q_i (p_{io}^0)^T + z_i), i \in I, o \in O, \tag{6}$$

where

$$R_i = \begin{vmatrix} g_i \\ l_i \\ q_i \end{vmatrix} = \begin{vmatrix} g_{i1} & g_{i2} & g_{i3} \\ r_{i1} & r_{i2} & r_{i3} \\ q_{i1} & q_{i2} & q_{i3} \end{vmatrix}, \tag{7}$$

$$\begin{aligned} g_{i1} &= \cos\psi_i \cos\omega_i - \cos\phi_i \sin\psi_i \sin\omega_i, \\ g_{i2} &= -\cos\psi_i \sin\omega_i - \cos\phi_i \sin\psi_i \cos\omega_i, \\ g_{i3} &= \sin\phi_i \sin\psi_i, \end{aligned} \tag{8}$$

$$\begin{aligned} r_{i1} &= \sin\psi_i \cos\omega_i + \cos\phi_i \cos\psi_i \sin\omega_i, \\ r_{i2} &= -\sin\psi_i \sin\omega_i + \cos\phi_i \cos\psi_i \cos\omega_i, \\ r_{i3} &= -\sin\phi_i \cos\psi_i, \end{aligned}$$

$$q_{i1} = \sin\varphi_i \sin\omega_i, q_{i2} = \sin\varphi_i \cos\omega_i, q_{i3} = \cos\varphi_i.$$

The inequality systems (4) take the form

$$\{\zeta_{ik}(X, u_i, \Upsilon) = \chi_{ik}((R_i X^T + v_i), \Upsilon) \geq 0, i \in I, k \in K. \quad (9)$$

In what follows, the polytope $P_i$ subjected to the translation of $v_i$ and rotations $\varphi_i, \psi_i$, and $\omega_i$ is denoted by $P_i(u_i)$ and the cuboid $C$ with variable sizes $\Upsilon$ as $C(\Upsilon)$.

Here, we consider the packing problem in the following setting.

Problem. Define the vector $u \in R^m$, which insures the arrangement $P_i(u_i)$, $i \in I$, without their mutual overlapping in the cuboid $C(\Upsilon)$ so that the cuboid volume $H(\Upsilon) = (w_2 - w_1)(l_2 - l_1)(\eta_2 - \eta_1)$ will reach its minimal value.

## 2. Mathematical model and its characteristics

In order to construct a mathematical model of the problem stated, it is necessary to describe the interaction of polytopes depending on their location in $R^3$ in analytical form. An adapted tool for such a description is phi-functions [19].

Based on the phi-functions, a mathematical model of the problem has the form

$$H(R^*) = \min H(R) \text{ s.t.} (u, R) \in \Lambda \subset R^{m+6}, \quad (10)$$

where

$$\Lambda = \{(u, \Upsilon) \in R^{m+6} : \Phi_{ij}(u_i, u_j) \geq 0, 0 < i < j \in I, \Phi_i(u_i, \Upsilon) \geq 0, i \in I, \quad (11)$$

$$w_1 \geq 0, l_1 \geq 0, \eta_1 \geq 0, w_2 - w_1 \geq 0, l_2 - l_1 \geq 0, \eta_2 - \eta_1 \geq 0\}.$$

Here, the inequality $\Phi_{ij}(u_i, u_j) \geq 0$ insures the non-overlapping of $P_i$ and $P_j$ and the inequality $\Phi_i(u_i, \Upsilon) \geq 0$ provides the containment of $P_i$ in $C(\Upsilon)$. Note that $\Phi_i(u_i, \Upsilon)$ is the phi-function for $P_i$ and the set $A(\Upsilon) = R^3 \setminus intC(\Upsilon)$, where $intC(\Upsilon)$ is the interior of $C$.

Let us consider a number of characteristics of the mathematical model.

1. Since each of the phi-functions [19] for $P_i$ and $P_j$ has the form

$$\Phi_{ij}(u_i, u_j) = \max\{\Psi_{ij}^{1t}(u_i, u_j), \Psi_{ij}^{2d}(u_i, u_j), t, d \in \mathrm{K}, \Psi_{ij}^{3l}(u_i, u_j), l \in L = \{1, 2, ..., \upsilon^2\}\}, \quad (12)$$

where $\upsilon = \varepsilon + \kappa - 2$ is the number of edges of $P_1$, $\Phi_{ij}(u_i, u_j) \geq 0$ if at least one of the inequalities

$$\Psi_{ij}^{1t}(u_i, u_j) \geq 0, t \in T, \Psi_{ij}^{2d}(u_i, u_j) \geq 0, d \in T, \Psi_{ij}^{3l}(u_i, u_j) \geq 0, l \in L \quad (13)$$

is fulfilled. Thus, the number of the inequalities is $2\kappa + \upsilon^2$.

2. Since $\Psi_{ij}^{1t}(u_i, u_j) = \min\{\varphi_{ij}^{1to}(u_i, u_j),\ o \in O\} \geq 0$, $\Psi_{ij}^{2d}(u_i, u_j) = \min\{\varphi_{ij}^{2do}(u_i, u_j),\ o \in O\} \geq 0$ and $\Psi_{ij}^{3l}(u_i, u_j) = \min\{\varphi_{ij}^{3lb}(u_i, u_j),$ and $b \in M = \{1, 2, ..., 5\}\}$ (see [14] and Appendix 1) $\Psi_{ij}^{1t}(u_i, u_j) \geq 0$, $\Psi_{ij}^{2d}(u_i, u_j) \geq 0$, and $\Psi_{ij}^{3l}(u_i, u_j) \geq 0$ are equivalent to the inequality systems

$$\begin{cases} \varphi_{ij}^{1t1}(u_i, u_j) \geq 0 \\ \varphi_{ij}^{1t2}(u_i, u_j) \geq 0 \\ \ldots\ldots\ldots\ldots \\ \varphi_{ij}^{1t\varepsilon}(u_i, u_j) \geq 0 \end{cases}, \quad \begin{cases} \varphi_{ij}^{2d1}(u_i, u_j) \geq 0 \\ \varphi_{ij}^{2d2}(u_i, u_j) \geq 0 \\ \ldots\ldots\ldots\ldots \\ \varphi_{ij}^{2d\varepsilon}(u_i, u_j) \geq 0 \end{cases}, \quad \begin{cases} \varphi_{ij}^{3l1}(u_i, u_j) \geq 0 \\ \varphi_{ij}^{3l2}(u_i, u_j) \geq 0 \\ \ldots\ldots\ldots\ldots \\ \varphi_{ij}^{3ls}(u_i, u_j) \geq 0 \end{cases}, \quad (14)$$

respectively, where

$$\varphi_{ij}^{1to}(u_i, u_j) = \zeta_{ik}(p_{jo}(u_j), u_i), \varphi_{ij}^{2ko}(u_i, u_j) = \zeta_{jk}(p_{io}(u_i), u_j). \quad (15)$$

3. It follows from items1 and 2 that the polytopes $P_i$, $i \in I$, do not overlap if at least one of the inequality systems

$$G_\tau(u) = \begin{cases} \Psi^\tau_{12}(u_1, u_2) \ge 0 \\ \Psi^\tau_{13}(u_1, u_2) \ge 0 \\ \dots\dots\dots\dots \\ \Psi^\tau_{23}(u_2, u_3) \ge 0 \\ \dots\dots\dots\dots \\ \Psi^\tau_{n(n-1)}(u_{n-1}, u_n) \ge 0 \end{cases}, \tau = \Xi = \{0,1,\dots, \zeta = (2\kappa + \upsilon^2)^\sigma + 1\}, \quad (16)$$

where $\Psi^\tau_{ij}(u_i, u_j)$ or $\Psi^{1t}_{ij}(u_i, u_j)$, or $\Psi^{2d}_{ij}(u_i, u_j) \ge 0$, or $\Psi^{3l}_{ij}(u_i, u_j)$, $\sigma = \dfrac{n(n-1)}{2}$ is fulfilled.

4. Since $\Phi_i(u_i, \Upsilon) = \min\{\Psi^r_i(u_i, \Upsilon), r \in R\}$, where $\Psi^r_i(u_i, \Upsilon) = \min\{\phi^r_{io}(u_i, \Upsilon) = \varpi_r(p_{io}(u_i), \Upsilon), o \in O\}$ (see (5) and (6)), $\Phi_i(u_i, \Upsilon) \ge 0$ is the equivalent to the inequality system

$$\{\Psi^r_i(u_i, \Upsilon) \ge 0, r \in R. \quad (17)$$

5. It follows from the relation (10) and items 2, 3 and 4 that the point $(u, \Upsilon) \in \Lambda$ if at least one of the inequality systems

$$F_\tau(u, \xi) = \begin{cases} G_\tau(u) \ge 0 \\ \Phi_i(u_i, \Upsilon) \ge 0, i \in I \\ w_1 \ge 0, l_1 \ge 0, \eta_1 \ge 0 \\ w_2 - w_1 \ge 0 \\ l_2 - l_1 \ge 0 \\ \eta_2 - \eta_1 \ge 0 \end{cases} = \begin{cases} \phi_{1\tau}(\xi_1) \ge 0 \\ \phi_{2\tau}(\xi_2) \ge 0 \\ \dots\dots\dots \\ \phi_{\mu\tau}(\xi_\mu) \ge 0 \end{cases}, \tau \in \Xi, \quad (18)$$

where $\xi_t, t \in (1,2,\dots,\mu)$ or $(u_i, u_j)$, or $(u_i, w_1)$, or $(u_i, w_2)$, or $(u_i, l_1)$, or $(u_i, l_2)$, or $(u_i, \eta_1)$, or $(u_i, \eta_2)$, or $(w_1, w_2)$, or $(l_1, l_2)$, or $(\eta_1, \eta_2)$, or $w_1$, or $l_1$, or $\eta_2$ is fulfilled.

Thus, the feasible region $\Lambda$ can be presented as

$$\Lambda = \bigcup_{\tau=0}^{\zeta} \Lambda_\tau, \quad (19)$$

where $\Lambda_\tau$ is defined by the inequality system $F_\tau(u, \xi) \ge 0$.

6. The relation (17) permits to theoretically find the global minimum point of the problem (10)–(11) as a result of solution to the problem

$$H(\Upsilon^*) = \min\{H(\Upsilon^{*\tau}), \tau \in \Xi\},$$

where

$$H(\Upsilon^{*\tau}) = \min H(\Upsilon) \text{ s.t.} (u, \Upsilon) \in \Lambda_\tau \subset R^{m+6}, \tau \in \Xi.$$

7. The problem is *NP*-hard.

The characteristics of the model show that the process of solving the problem must include the construction of starting points, computation of local minima and non-exhausted search for local minimum points to obtain a good approximation to the global minimum.

**3. Construction of starting points**

In order to diminish the time for solving similar problems, it is desirable to obtain starting points belonging to the feasible region. At present, the construction of starting points is executed by either greedy or heuristic algorithms. The algorithms do not permit to generate all possible starting points, which results in a significant restriction of the corresponding local minimum points. Furthermore, the algorithms demand essential time expenditures. The above facts determine the necessity to develop new approaches to deriving starting points. We offer an approach which allows us to construct the starting points belonging to the feasible region without any limits. To realize the approach we cover the polytopes $P_i$, $i \in I$ with spheres of minimal radii and pack the latter into a cuboid $C$ of a minimal volume. Then we take the center coordinates of the spheres and sizes of $C$ obtained as a starting point and compute the local minimum point of the problem (10)–(11).

### 3.1. Packing spheres

Firstly, we cover the polytope $P_1$ with a sphere $S_1$ of minimal radius $r_1^0$. To that end, we solve the problem (see (6))

$$\rho_1^0 = \min \rho_1 \text{ s.t.} (\rho_1, v) \in \Gamma, \ \Gamma = \{(\rho_1, v) \in R^4, \|v_0 - p_{io}^0\| \le \rho_1, o \in O\}, \tag{20}$$

where $v_0 = (x_0, y_0, z_0)$ is the sphere center. In what follows, we suppose that the origin $O_1$ of the eigen coordinate system of $P_1$ and the center of $S_1$ coincide. Thus, the spheres $S_i(\rho_i^0)$ of minimal radii $\rho_i^0 = \rho_1^0 h_i^\nabla$ cover the polytopes $P_i$, $i \in I$.

Now, we give $\Upsilon = \Upsilon^0 = (w_1^0, w_2^0, l_1^0, l_2^0, \eta_1^0, \eta_2^0,)$ so that $0 < w_1^0 < w_2^0$, $0 < l_1^0 < l_2^0$ and $0 < \eta_1^0 < \eta_2^0$. Next we suppose that the radii $\rho_i$ of the spheres $S_i$, $i \in I$, are variable and form the vector $\rho = (\rho_1, \rho_2, ..., \rho_n) \in R^n$. Thus, the vector of all the variables is $X = (v, \rho) \in R^{4n}$. Having given $v_i^\bullet$, $i \in I$, randomly, so that $v_i^\bullet \in C(\Upsilon^0)$ and assumed $\rho = 0$, we obtain the point $X^\bullet = (v^\bullet, 0) \in R^{4n}$. Then for the starting point $X^\bullet$ we calculate the local maximum point $X^0 = (v^0, \rho^0)$ of the problem

$$\Pi(\rho^0) = \max \Pi(\rho) = \max \sum_{i=1}^{n} \rho_i \ \text{ s.t. } X \in \Omega \subset R^{4n}, \tag{21}$$

where

$$\Omega = \{X \in R^{4n}, \Phi_{ij}^{SS}(v_i, v_j, \rho_i, \rho_j) \ge 0, 0 < i < j \in I, \Phi_i^S(v_i, \rho_i) \ge 0, i \in I, s_i(\rho_i) = \rho_i - \rho_i^\nabla \ge 0, i \in I\}, \tag{22}$$

$$\Phi_{ij}^{SS}(v_i, v_j, \rho_i, \rho_j) = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 - (\rho_i + \rho_j)^2,$$

$$\Phi_i^S(v_i, \rho) = \min\{x_i - \rho_i - w_2^0, y_i - \rho_i - l_2^0, z_i - \rho_i - \eta_2^0, w_1^0 - x_i - \rho_i, l_1^0 - y_i - \rho_i, \eta_1^0 - z_i + \rho_i\}.$$

The problem (21)-(22) possesses the following features:

1. If $X^0 = (v^0, \rho^0)$ is the local maximum point of the problem and $\Pi(\rho^0) = \sum_{i=1}^{n} \rho_i^0 = \sum_{i=1}^{n} \rho_i^\nabla = d$, then the spheres $S_i(v_i^0)$, $i \in I$ are packed into the cuboid $C(\Upsilon^0)$. This means that in this case the point $X^0$ is the global maximum point of the problem (20).

2. If the global maximum point $X^0 = (v^0, \rho^0)$ is such that at least one of the components of $\rho^0$ is strictly less than the appropriate component of $\rho^\nabla = (\rho_1^\nabla, \rho_2^\nabla, ..., \rho_n^\nabla)$, then the spheres $S_i$, $i \in I$ are not packed into the cuboid $C(\Upsilon^0)$.

The point $X^0 = (v^0, \rho^0)$ is, evidently, the global maximum point such that $\Pi(\rho^0) = d$ can be easily obtained if we provide $\Upsilon^0$ with an arrangement of $S_i$, $i \in I$ within the cuboid $C(\Upsilon^0)$.

### 3.2. Starting points

We now take the polytopes $P_i(v_i^0)$ instead of the spheres $S_i(v_i^0)$ and fix both $\varphi_i = \dot{\varphi}_i, \psi_i = \dot{\psi}_i$ and $\omega_i = \dot{\omega}_i$, $i \in I$. Since in this case the rotation angles are constants, the vector of all the variables is $(v, \Upsilon) \in R^{3n+6}$. Consequently, the inequality system of the form (16) consists of linear inequalities. Thus, the problem (10)–(11) takes the form

$$H(\Upsilon^*) = \min H(\Upsilon) \ \text{ s.t.} (v, \Upsilon) \in Q \subset R^{3n+6}, \tag{23}$$

where

$$Q=\{(v,\Upsilon)\in R^{3n+6}:\Phi_{ij}(v_i,v_j)\ge 0, 0<i<j\in I, \Phi_i(v_i,\Upsilon)\ge 0, i\in I,$$
$$w_1\ge 0, l_1\ge 0, \eta_1\ge 0, w_2-w_1\ge 0, l_2-l_1\ge 0, \eta_2-\eta_1\ge 0\}. \quad (24)$$

It follows from (19) that

$$Q=\bigcup_{\tau=0}^{\zeta} Q_\tau,$$

where $Q_\tau$ is specified by a linear inequality system. This means that the computation of local minimum points of the problem (23)-(24) can be reduced to solving a sequence of sub-problems of optimizing linear programming whose solution spaces are specified by an essentially smaller number of inequalities than one of the problems (10)–(11). Let us consider the solution approach.

Let $X^0=(v^0,\rho^0)$ be the global maximum point of the problem (20). Now we derive the vector $(v^0,\Upsilon^0)$ (it is evident that $(v^0,\Upsilon^0)\in Q$) and select the sub-region of $Q_\tau$, $\tau\in\Xi$, which contains the point $(v^0,\Upsilon^0)\in Q$. Let $(v^0,\Upsilon^0)\in Q_0\subset Q$ and $Q_0$ be specified by the linear inequality system (see (16)) of the form

$$B_0(v,\Upsilon)=F_0(v,\tilde{\theta},\Upsilon)=\begin{cases}\phi_{10}(\xi_1)\ge 0\\ \phi_{20}(\xi_2)\ge 0\\ \dots\dots\dots\dots\\ \phi_{\mu 0}(\xi_\mu)\ge 0\end{cases}, \quad (25)$$

where $\varphi_{k0}(\xi_k)\ge 0$ is an inequality from either the systems (14) (see item 5) or systems $w_1\ge 0, l_1\ge 0, \eta_1\ge 0, w_2-w_1\ge 0$, $l_2-l_1\ge 0$, $\eta_2-\eta_1\ge 0$, $\xi_t$, or $(v_i,v_j)$, or $(v_i,w_1)$, or $(v_i,w_2)$, or $(v_i,l_1)$, or $(v_i,l_2)$, or $(v_i,\eta_1)$, or $(v_i,\eta_2)$, or $(w_1,w_2)$, or $(l_1,l_2)$, or $(\eta_1,\eta_2)$, or $w_1$, or $l_1$, or $\eta_2$, the number $\mu$ depending on the number of the inequalities forming $\Psi_{ij}^{\tau}(v_i,v_j)\ge 0$ in $G_\tau(v)$.

In order to decrease the time loss when solving the problem (23) – (24) we reduce solving the problem to solving a sequence of sub-problems whose feasible regions are specified by a considerably smaller number of inequalities. To this end, we make use of the following property: all the spheres cannot simultaneously touch both each other and the same facet of the set $A(\Upsilon^0)$. This means that the values of many left-hand parts of the inequalities in the equality system $B_0(v,\Upsilon)\ge 0$ at the point $(v^0,\Upsilon^0)\in Q_0$ are greater than some small $\rho_1^\nabla>\delta>0$. This makes it possible to single out the inequality subsystem $B_{00}(v,\Upsilon)\ge 0$ from $B_0(u,\Upsilon)\ge 0$ which consists of a considerably smaller number of inequalities. For extracting the subsystem $B_{00}(u,\Upsilon)\ge 0$ from $B_0(v,\Upsilon)\ge 0$ we use the spheres $S_i$ with the radii of $\rho_i^\nabla$, $i\in I$ (see (20)). For this purpose we compute

$$\left\|v_i^0-v_j^0\right\|-(\rho_i^\nabla+\rho_j^\nabla)\le\delta, i\in J_1, j\in J_2,$$
$$-w_1^0+x_i^0-\rho_i^\nabla\le\frac{1}{2}\delta, i\in I_1, w_2^0-x_i^0-\rho_i^\nabla\le\frac{1}{2}\delta, i\in I_2, \quad (26)$$
$$-l_1^0+y_i^0-\rho_i^\nabla\le\frac{1}{2}\delta, i\in I_3,$$
$$-l_2^0+y_i^0-\rho_i^\nabla\le\frac{1}{2}\delta, i\in I_4, -\eta_1^0+z_i^0-\rho_i^\nabla\le\frac{1}{2}\delta, i\in I_5, -\eta_2^0+z_i^0-\rho_i^\nabla\le\frac{1}{2}\delta, i\in I_6.$$

Next we compute $\Phi_{ij}(v_i^0, v_j^0) = \Psi_{ij}^{1t_{ij}}(v_i^0, v_j^0)$, $i \in J_1 \subset I$, $j \in J_2 \subset I$, $\Phi_{ij}(v_i^0, v_j^0) = \Psi_{ij}^{2d_{ij}}(v_i^0, v_j^0)$, $i \in J_1$, $j \in J_2$, $\Phi_{ij}(v_i^0, v_j^0) = \Psi_{ij}^{3l_{ij}}(v_i^0, v_j^0)$, $i \in J_1$, $j \in J_2$, and $\Phi_i(v_i^0, \Upsilon^0) = \min\{\Psi_i^r(v_i^0, \Upsilon^0), r \in R\}$, $i \in \sum_{r=1}^{6} I_r = K_\delta$, $I_r \subset I$. Based on the above inequalities, we derive the inequality system

$$B_{00}(v,\Upsilon) = F_{00}(v,\tilde{\theta},\Upsilon) = \begin{cases} \Psi_{ij}^{1t_{ij}}(v_i,v_j) \ge 0, i \in J_1, j \in J_2, \\ \Psi_{ij}^{1d_{ij}}(v_i,v_j) \ge 0, i \in J_1, j \in J_2, \\ \Psi_{ij}^{1l_{ij}}(v_i,v_j) \ge 0, i \in J_1, j \in J_2, \\ \Psi_i^r(u_i,\Upsilon) \ge 0, i \in K_\delta, r \in R, \\ w_1 \ge 0, l_1 \ge 0, \eta_1 \ge 0, \\ w_2 - w_1 \ge 0, l_2 - l_1 \ge 0, \eta_2 - \eta_1 \ge 0, \\ -\frac{\sqrt{2}}{4}\delta \le x_i^0 - x_i \le \frac{\sqrt{2}}{4}\delta, i \in I, \\ -\frac{\sqrt{2}}{4}\delta \le y_i^0 - y_i \le \frac{\sqrt{2}}{4}\delta, i \in I, \\ -\frac{\sqrt{2}}{4}\delta \le z_i^0 - z_i \le \frac{\sqrt{2}}{4}\delta, i \in I, \end{cases} = \begin{cases} \phi_{0i_1}(\xi_{i_1}) \ge 0 \\ \phi_{0i_2}(\xi_{i_2}) \ge 0 \\ \ldots\ldots\ldots\ldots \\ \phi_{0i_q}(\xi_{i_q}) \ge 0 \\ -\frac{\sqrt{2}}{4}\delta \le x_i^0 - x_i \le \frac{\sqrt{2}}{4}\delta, i \in I, \\ -\frac{\sqrt{2}}{4}\delta \le y_i^0 - y_i \le \frac{\sqrt{2}}{4}\delta, i \in I, \\ -\frac{\sqrt{2}}{4}\delta \le z_i^0 - z_i \le \frac{\sqrt{2}}{4}\delta, i \in I. \end{cases}, \quad (27)$$

which describes the sub-region $Q_{00}$. It should be emphasized that any point $(v,\Upsilon) \in Q_{00}$ in accordance with the inequalities $-\frac{\sqrt{2}}{4}\delta \le x_i^0 - x_i \le \frac{\sqrt{2}}{4}\delta$, $-\frac{\sqrt{2}}{4}\delta \le y_i^0 - y_i \le \frac{\sqrt{2}}{4}\delta$, $-\frac{\sqrt{2}}{4}\delta \le z_i^0 - z_i \le \frac{\sqrt{2}}{4}\delta$, $i \in I$ belongs to $Q_0$, i.e. $Q_{00} \subset Q_0$. In addition, $q$ is essentially smaller than $\mu$ in (18).

Having taken the point $(v^0, \Upsilon^0)$ as a starting point and solved the problem

$$H(\Upsilon^{0*}) = \min H(\Upsilon) \text{ s.t. } (v,Y) \in Q_{00} \subset R^{3n+6},$$

we find the local minimum point $(v^{1*}, \Upsilon^{1*})$. After that we define the sets $J_1, J_2$ and $K_\delta$ in (26) at the point $(v^{1*}, \Upsilon^{1*})$, forming a new linear system $B_{01}(u,\Upsilon) \ge 0$ of the form (27), and for the starting point $(v^{1*}, \Upsilon^{1*})$ solve the problem

$$H(\Upsilon^{2*}) = \min H(\Upsilon) \text{ s.t. } (v,\Upsilon) \in Q_{10} \subset R^{3n+6},$$

where $Q_{10}$ is specified by $B_{01}(u,\Upsilon) \ge 0$.

The iterative process continues until the equality $(v^{\gamma*}, \Upsilon^{\gamma*}) = (v^{(\gamma+1)*}, \Upsilon^{(\gamma+1)*})$ is reached. This means that $H(\Upsilon^{\gamma*}) = \min H(\Upsilon)$ s.t. $(v,\Upsilon) \in Q_0$.

Making use of active inequalities at the point $(u^{\gamma*}, \Upsilon^{\gamma*})$, we can execute a transition to a new sub-region $Q_1$. Let the inequalities $\varphi_{1i_{ji}}(\xi_{i_j}) \ge 0$, $j \in \Gamma_0$ be active at the point $(u^{\gamma*}, \Upsilon^{\gamma*}) \in Q_0$. We single out the inequality subsystems $\Psi_{ij}^0(v_i, v_j) \ge 0$, $i \in I_{10}$, $j \in I_{20}$ (see (14)) which contain the active inequalities $\varphi_{1i_{ji}}(\xi_{i_j}) \ge 0$, $j \in \Gamma_0$. Next we choose the inequalities $\Phi_{ij}(v_i, v_j) \ge 0$ from the inequality system (11) which incorporate the inequality subsystems $\Psi_{ij}^0(v_i, v_j) \ge 0$, $i \in I_{10}$, and $j \in I_{20}$ (see (12)) and compute $\Phi_{ij}(v_i^{0*}, v_j^{0*}) = \chi_{ij}^0$, $i \in I_{10}$, and $j \in I_{20}$. This permits us to select the functions $\Psi_{ij}^a(v_i, v_j)$ such that $\Phi_{ij}(v_i^{0*}, v_j^{0*}) = \Psi_{ij}^a(v_i^{0*}, v_j^{0*}) = \chi_{ij}^0 > 0$, $i \in I_{10}^0 \subset I_{10}$, $j \in I_{20}^0 \subset I_2$. We can now form a new inequality system specifying the new feasible sub-region $Q_1$

by substituting the inequality subsystems $\Psi_{ij}^{0}(v_i,v_j)\ge 0$ from the system specifying $\Lambda_0$ for the inequality subsystems $\Psi_{ij}^{a}(v_i,v_j)>0$, $i\in I_{10}^{0}$, $j\in I_{20}^{0}$, $a\ne 0$ (see item 3 from Section 2). It is evident that $(v^{\gamma *},\Upsilon^{\gamma *})\in Q_1\subset Q$. So, having taken the starting point $(v^{0*},\Upsilon^{0*})=(v^{\gamma *},\Upsilon^{\gamma *})$ and solved the problem

$$H(\Upsilon^{1*})=\min H(\Upsilon)\ \text{s.t.}\,(v,\Upsilon)\in Q_1\subset R^{3n+6},$$

we obtain the local minimum point $(v^{1*},\Upsilon^{1*})$ and so on. Thus, searching for a local minimum point of the problem (23)-(24) is reduced to solving a sequence of sub-problems

$$H(\Upsilon^{\theta *})=\min H(\Upsilon)\ \text{s.t.}\,(v,\Upsilon)\in Q_\lambda\subset R^{3n+6},\lambda=1,2,\ldots\ . \tag{28}$$

The computational process continues until the equality $Q_\lambda=Q_{\lambda+1}$ is reached, i.e. we will be unable to move from the feasible sub-region $Q_\gamma$ to a new feasible sub-region $Q_{\gamma+1}$. In this case we take the point $(v^{\lambda *},\Upsilon^{\lambda *})$ as the local minimum point of the problem (23)-(24). Thus, the point $(v^{\lambda *},\tilde\theta,\Upsilon^{\lambda *})$ where $\tilde\theta$ is given randomly, can be taken as a starting point for the problem (10)–(11).

**4. Searching for a local minimum point**

Let the angular parameters $\theta_i=(\varphi_i,\psi_i,\omega_i)$, $i\in I$ be variable. Then the sets $Q_\tau$ are transformed into the sets $\Lambda_\tau$, $\tau\in\Upsilon$. Having taken both $\tilde\Upsilon=\Upsilon^{0*}$ and $\theta=\tilde\theta$, we derive the point $(\tilde u,\tilde\Upsilon)=(v^{\lambda *},\tilde\theta,\Upsilon^{\lambda *})\in\Lambda_\lambda$ where $\Lambda_\lambda$ is specified by $F_\lambda(u,\Upsilon)\ge 0$ (see (18)). For the sake of convenience, we suppose $\Lambda_0=\Lambda_\lambda$ and $F_0(u,\Upsilon)=F_\lambda(u,\Upsilon)$. To search for the local minimum point of the problem (10)–(11) we modify the approach offered in Subsection 3.2. The modification consists of the following steps.

Each of the subsystems $\Psi_{ij}^{1t_{ij}}(u_i,u_j)\ge 0$, $i\in J_1^0$, $j\in J_2^0$, $\Psi_{ij}^{2d_{ij}}(u_i,u_j)\ge 0$, $i\in J_3^0$, $j\in J_4^0$, and $\Psi_i^{r}(u_i,\Upsilon)\ge 0$, $i\in K_\delta$, $r\in R$ in the system $F_{00}(u,\Upsilon)\ge 0$ consists of $\varepsilon$ inequalities where $\varepsilon$ is the number of polytope vertices (see (2)). Due to the inequalities $\left\|v_i^0-v_i\right\|\le\frac{1}{2}\delta$, $i\in I$, the values of the angles $\varphi_i$, $\psi_i$ and $\omega_i$, $i\in I$ are, as a rule, smaller than $\frac{\pi}{2}$. So the number of inequalities from the systems $\Psi_{ij}^{1t_{ij}}(u_i,u_j)\ge 0$, $i\in J_1^0$, $j\in J_2^0$, $\Psi_{ij}^{21d_{ij}}(u_i,u_j)\ge 0$, $i\in J_3^0$, $j\in J_4^0$ and $\Psi_i^{r}(u_i,\ddot y)\ge 0$, $i\in K_\delta$, $r\in R$ (14) can be omitted. In order to exclude the inequalities from the system $F_{00}(u,\Upsilon)\ge 0$ we compute

$$\begin{aligned}
&\max\{\varphi_{ij}^{1to}(\dot u_i,\dot u_j),o\in O\}=a_{ij}^{t},i\in J_1^0,j\in J_2^0,\\
&\max\{\varphi_{ij}^{2do}(\dot u_i,\dot u_j),o\in O\}=b_{ij}^{d},i\in J_3^0,j\in J_4^0,\\
&\max\{\varphi_{io}^{r}(\dot u_i,\ddot y),o\in O\}=a_{ir},r\in R,i\in K_{\delta 0}\subset K_\delta
\end{aligned} \tag{29}$$

and single out the inequalities

$$\begin{aligned}
&\phi_{ij}^{1t_{ij}o}(\tilde u_i,\tilde u_j)\le\frac{1}{q}a_{ij}^{o},i\in J_1^0,j\in J_2^0,o\in O_{ij}\subset O,\\
&\phi_{ij}^{2d_{ij}o}(\tilde u_i,\tilde u_j)\le\frac{1}{q}b_{ij}^{o},i\in J_3^0,j\in J_4^0,o\in O_{ij}\subset O,\\
&\phi_r(\tilde u_i,\tilde\Upsilon)\le\frac{1}{q}a_{ir},r\in R,i\in K_\delta,
\end{aligned} \tag{30}$$

where $q\ge 2$. Then the inequalities

$$\varphi_{ij}^{1t_{ij}o}(u_i,u_j)\ge 0, i\in J_1^0, j\in J_2^0, o\in O_{ij}\subset O, \tag{31}$$

$$\varphi_{ij}^{2d_{ij}o}(u_i,u_j)\ge 0, i\in J_3^0, j\in J_4^0, o\in O_{ij}\subset O,$$

$$\phi_r(u_i,\tilde{\Upsilon})\ge 0, r\in R, i\in K_\delta$$

together with the inequalities $w_1\ge 0$, $l_1\ge 0$, $\eta_1\ge 0, w_2-w_1\ge 0$, $l_2-l_1\ge 0$, $\eta_2-\eta_1\ge 0, \left\|v_i^0-v_i\right\|\le\frac{1}{2}\delta,$ $i\in I$ form the inequality subsystem

$$F_{00}^0(u,\Upsilon)=\begin{cases}\phi_{0i_1}(\xi_{i_1})\ge 0\\ \phi_{0i_2}(\xi_{i_2})\ge 0\\ \cdots\cdots\cdots\cdots\\ \phi_{0i_\varsigma}(\xi_{i_q})\ge 0\\ \left\|v_i^0-v_i\right\|\le\frac{1}{2}\delta, i\in I,\end{cases}, \tag{32}$$

which specifies the set $N_0$.

Having taken $(\tilde{u},\tilde{\Upsilon})$ as a starting point, we compute the local minimum point $(u^{0*},\Upsilon^{0*})$ of the problem

$$H(\Upsilon^{0*})=\min H(\Upsilon)\ \text{s.t.}\,(u,\Upsilon)\in N_0. \tag{33}$$

On the ground of the point $(u^{0*},N^{0*})$, we derive a new inequality system $F_1(u,\Upsilon)\ge 0$ of the form (27). Next, for the point $(u^{0*},\Upsilon^{0*})$ we define $J_1,J_2,...,I_8$ in (26) and derive the system $F_{11}(u,\Upsilon)\ge 0$ of the form (27) (see Subsection 3.2) which describes the set $\Lambda_{11}$. Then for the starting point $(u^{0*},\Upsilon^{0*})$ we find $J_1^0,J_2^0,J_3^0,J_4^0$ and $K_{\delta 0}$ and form the system $F_{11}^0(u,\Upsilon)$ of the form (32) specifying the set $N_1$. For the starting point $(u^{0*},\Upsilon^{0*})$ we solve the problem $H(\Upsilon^{1*})=\min H(\Upsilon)$ s.t. $(u,\Upsilon)\in N_1$, providing that the local minimum point $(u^{1*},\Upsilon^{1*})$ is determined. Then, based on the point $(u^{1*},\Upsilon^{1*})$, we construct a new inequality system $F_2(u,\Upsilon)\ge 0$ of the form (27) specifying the sub-region $\Lambda_2$, define $J_1,J_2,...,I_8$ in (26), and generate the system $F_{22}(u,\Upsilon)\ge 0$ of the form (27) (see Subsection 3.2), which specifies the set $\Lambda_{22}$. After that, for the starting point $(u^{1*},\Upsilon^{1*})$ we find $J_1^0,J_2^0,J_3^0,J_4^0$ and $K_{\delta 0}$, and construct the system $F_{22}^0(u,\Upsilon)$ of the form (32) specifying the set $N_2$. We solve the problem $H(\Upsilon^{2*})=\min H(\Upsilon)$, s.t. $(u,\Upsilon)\in R_2$ for the starting point $(u^{1*},\Upsilon^{1*})$ and define the local minimum point $(u^{2*},\Upsilon^{2*})$, and so on. Thus, searching for the local minimum point of the problem (10)–(11) is reduced to solving a sequence of sub-problems

$$H(\Upsilon^{\rho*})=\min H(\Upsilon)\ \text{s.t.}\,(u,\Upsilon)\in N_\rho, \rho=1,2,\ldots. \tag{34}$$

The process is finished if $H(\Upsilon^{\rho*})=H(\Upsilon^{(\rho-1)*})$. Then $(u^{0*},\Upsilon^{0*})=(u^{\rho*},\Upsilon^{\rho*})$ is taken as the local minimum point of the problem (10)–(11).

We can draw the following conclusions.

The smaller $\delta$, the quicker the computation of the local minimum point of one problem from the sequence (34) and the greater number of $p$ iterations is necessary to compute the local minimum point of the problem (10)–(11). The greater $\delta$, the more time expenditures are necessary when tackling one problem from the sequence (34). Consequently, there has to be a compromise which will ensure a good degree of convergence to the local minimum point of the problem (10)–(11) when choosing $\delta$. The value of $\delta$ is defined experimentally. We take $\delta=\frac{1}{2}\rho_1^0$.

## 5. Transition from one local extremum to another

In order to compute a new local minimum point it is necessary to build a new starting point. If the starting point is randomly generated, then the objective value at the appropriate local minimum point may be either improved or made worse or invariable. We suggest a way of a transition from the local minimum point to a new local minimum point at which the objective value is either improved or invariable.

### 5.1. Auxiliary problem

Let $(u^{0*}, \Upsilon^{0*})$ be the local minimum point of the problem (10)–(11). This point has a corresponding arrangement of polytopes $P_i(u^{0*})$, $i \in I$. It can be found that around some polytope $P_i$ there is a "free space" which allows to place a larger polytope $P_j \subset P_i$ instead of the polytope $P_i$. If we can detect the polytopes, then it will permit us to "switch" between the polytopes without their overlapping in the packing $P_i(u^{0*})$, $i \in I$ so that the value of the objective will not worsen. The new placement of the polytopes has a corresponding point $(u^0, \Upsilon^0) \in \Lambda$. If we take the point as a starting one for the problem (10)–(11) we compute a new local minimum point $(u^{1*}, \Upsilon^{1*})$. It is evident that $H(\Upsilon^{1*}) \le H(\Upsilon^{0*})$. In order to execute the approach, we assume that homothetic coefficients $h_i$, $i \in I$ are variables and form the vector $h = (h_1, h_2, ..., h_n) \in R^n$, i.e. the vector of all the variables is $Y = (u, \Upsilon, h) \in R^\tau$, where $\tau = m + n + 6$. Consequently, the phi-function for $P_i(u_i, h_i)$ and $P_j(u_j, h_j)$ depends on $h_i$ and $h_j$ as well, i.e. they take the form $\Phi_{ij}(u_i, u_j, h_i, h_j)$, and the phi-function for $P_i(u_i, h_i)$ and $A(\Upsilon)$ depends on $h_i$, i.e. it has the form $\Phi_i(u_i, h_i, \Upsilon)$. Thus, we can enlarge and diminish the sizes of polytopes by changing the values of their homothetic coefficients $h_i$, $i \in I$. Inasmuch as for any $h_i \in (0, \infty)$ the polytopes $P_i(u_i, h_i)$, $i \in I$ are homothetic, then the phi-functions $\Phi_{ij}(u_i, u_j, h_i, h_j)$ for $P_i(u_i, h_i)$ and $P_j(u_j, h_j)$, and phi-functions $\Phi_i((u_i, h_i, \Upsilon)$ for $P_i(u_i, h_i)$ and $A(\Upsilon)$ have the same form for any $h_i \in (0, \infty)$.

Now we state the following auxiliary problem

$$H(\Upsilon) = \min H(\Upsilon) \ \text{s.t.} \ Y = (u, \Upsilon, h) \in \Delta \subset R^\tau, \tag{35}$$

where

$$\Delta = \{Y \in R^\tau, \Phi_{ij}(u_i, u_j, h_i, h_j) \ge 0, 0 < i < j \in I, \Phi_i(u_i, \Upsilon, h_i) \ge 0, \tag{36}$$

$$h_i \ge 0, i \in I, \sum_{i=1}^{n} (h_i - h_{i1}^{\nabla}) \ge 0, \ w_1 \ge 0, l_1 \ge 0, \eta_1 \ge 0,$$

$$w_2 - w_1 \ge 0, l_2 - l_1 \ge 0, \eta_2 - \eta_1 \ge 0, H(\Upsilon^{0*}) - H(\Upsilon) \le \varepsilon\},$$

$$\varepsilon = (\frac{1}{2})^t \, 0.1 H(\Upsilon^{0*}), t = 0.1.2..... \tag{37}$$

We form the starting point $Y^{0*} = (u^{0*}, \Upsilon^{0*}, h^{\nabla})$, where $h^{\nabla} = (h_1^{\nabla}, h_2^{\nabla}, ..., h_n^{\nabla})$, and compute the local minimum point $Y^{1*} = (u^{1*}, \Upsilon^{1*}, h^{1*})$ of the problem. Since restrictions $h_i - h_i^{\nabla} \ge 0$, $i \in I$ are absent, the values of some of $h_i^{1*}$, $i \in I$ may become either smaller than 1 or greater than 1. Let $h_i^{1*} > h_i^{\nabla}$, $i \in k_1 = \{1, 2, ..., p\} \subset I$, and $h_i^{1*} \le h_i^{\nabla}$, $i \in k_2 = \{1, 2, ...q\} \subset I$. In addition, the greater $\varepsilon$, the greater number of homothetic coefficients $h_i$, $i \in I$ essentially alter their values (pay attention to the inequality $H(\Upsilon^{0*}) - H(\Upsilon) \le \varepsilon$). Thus, some polytopes 'become larger', i.e. the exists a free space around them, whereas some polytopes 'become smaller', i.e. there exists a lack of free space around them. Thus, the vector $h^{1*} = (h_1^{1*}, h_2^{1*}, ..., h_2^{1*})$ can be used to point out which can be changed so that the value of $H(\Upsilon)$ will not deteriorate.

### 5.2. Jump algorithm

The algorithm proposed in this section is a modification of the Jump algorithm (JA) that was suggested in [18]. The modification of the JA is executed in the following way.

Uppermost we form the descending sequence

$$h_{i_1}^{1*} \geq h_{i_2}^{1*} \geq .... \geq h_{in}^{1*} \quad (38)$$

If $i_j = i, \ i \in I$ for all the indices, then the point $(u^{0*}, \Upsilon^{0*})$ is accepted as an approximation to the global minimum point of the problem (10)-(11). If $i_j \neq i$ at least for two indices, then having taken $h_j^{1*} = h_{i_j}^{1*}, \ j \in I,$ we derive the sequence

$$h_1^{1*} \geq h_2^{1*} \geq .... \geq h_n^{1*}. \quad (39)$$

Since $h_i^{1*} > h_i^{\nabla}$ may occur for some $i \in I$, we compute $h_j^0 = \min\{h_j^{1*}, h_j^{\nabla}\}, \ j \in I.$ It insures the inequality $\sum_{i=1}^{n} h_i^0 < n.$ We now construct the points $\tilde{X} = (\tilde{u}, \tilde{h}),$ where $\tilde{u}_j = u_j^{1*}, \ \tilde{h}_j = h_j^{1*}, \ j \in I$ (see (33)-(35) and $\tilde{\tilde{X}} = (\tilde{\tilde{u}}, \tilde{\tilde{h}})$, where $\tilde{\tilde{u}}_j = u_j^{1*}, \ \tilde{\tilde{h}} = h_j^0, \ j \in I$. Note that if $X^{1*} \neq \tilde{X}$, then the arrangement corresponding to the point $\tilde{X}$ is obtained from the arrangement corresponding to the point $X^{1*}$ as a result of "rearrangements" of some polytopes.

If $X^{1*} \neq \tilde{X}$, then $\sum_{i=1}^{n} \tilde{h}_i < n,$ i.e. some polytopes $P_i(\tilde{h}_i), \ i \in I,$ are reduced (shrunk). Therefore, it is necessary to increase the sizes of the polytopes to their initial ones under fixed $\Upsilon = \Upsilon^{1*}.$ To this end, we solve the helper problem

$$\Pi(h^*) = \max \Pi(h) = \max \sum_{i=1}^{n} h_i \ \text{ s.t. } X = (u, h) \in D \subset R^{\tau - 6}, \quad (40)$$

where

$$\Pi(h^*) = \max \Pi(h) = \max \sum_{i=1}^{n} h_i \ \text{ s.t. } X = (u, h) \in D \subset R^{\tau - 6}, \quad (41)$$

$$\phi_i(h_i) = h_i^{\nabla} - h_i \geq 0, h_i \geq 0, i \in I\}.$$

Note that the problem enables us to increase the homothetic coefficients to their initial values (see the inequalities $\phi_i(h_i) = h_i^{\nabla} - h_i \geq 0, \ \ h_i \geq 0, \ \ i \in I$ ). It is evident that $\tilde{\tilde{X}} \in D$.

Having taken the starting point $\tilde{\tilde{X}}$ and solved the problem (40)–(41), we find the local maximum point $\hat{X} = (\hat{u}, \hat{h})$.

Note that the problem possesses characteristics similar to those of the problems (10)–(11). We remind the following properties:

1. If $\Pi(\hat{h}) = \sum_{i=1}^{n} \hat{h}_i = \sum_{i=1}^{n} h_i^{\nabla} = n$, then $\hat{h} = h^{\nabla}$ and the polytopes $P_i, \ i \in I$ are packed into $P(\Upsilon^g)$. This means that the point $\hat{X}$ is the global maximum point of the problem (40)–(41).

2. If at least one of $\hat{h}_i, \ i \in I$ is strictly less than $h_i^{\nabla}$ at the global maximal point $\hat{X}$, then the polytopes $P_i, \ i \in I$ cannot be packed into $P(\Upsilon^{1*})$.

Thus, two cases are possible: $\Pi(\hat{h}) = n$ and $\Pi(\hat{h}) < n.$

It is evident that if $\Pi(\hat{h}) = n$, then $(\hat{u}, \Upsilon^{1*}) \in \Lambda$ and $(\hat{u}, \Upsilon^{1*})$ is not, in the general case, the local minimal point of the problem (10)–(11). Therefore, we take the starting point $(\hat{u}, \Upsilon^{1*})$, solve the problem (10)–(11), and obtain a new local minimal point $(u^{0*}, \Upsilon^{0*})$. After that we construct a new starting point $Y^{0*} = (u^{0*}, \Upsilon^{0*}, h^{\nabla})$ and for this very point solve the problem (35)–(36). As a result, a new local minimum point $Y^{1*} = (u^{1*}, \Upsilon^{1*}, h^{1*})$ is computed. On the basis of the point $Y^{1*}$ and the sequences (38)–(39) we generate a new point $X^{1*} = (u^{1*}, h^{1*})$. Having taken the point as a starting one, we solve the problem (40)–(41) and so on until $\Pi(\hat{h}) < n$ is reached.

If $\Pi(\hat{h}) < n$, we then increase *t* by 1 in (37) and for the starting point $Y^{0*} = (u^{0*}, \Upsilon^{0*}, h^{\nabla})$ solve the problem (35)–(36) and so on. The iterative process is repeated until $\varepsilon \le 10^3 H(\Upsilon^{0*})$ is fulfilled.

**6. General solution scheme**

In this section a general scheme of solving the problem (10)–(11) is illustrated.

1. Compute the radii of spheres $S_i$ circumscribed around the polytopes $P_i$, $i \in I$ (see problem (20)).

2. Find the global maximum point of the problem (21)–(22) of the packing spheres $S_i$, $i \in I$ into the cuboid *C*.

3. Give the values of angles $\phi_i = \tilde{\phi}_i, \psi_i = \tilde{\psi}_i$ and $\omega_i = \tilde{\omega}_i$, $i \in I$ randomly and fix them.

4. Take the polytopes $P_i$ instead of the spheres $S_i$, $i \in I$, solve the sequence of the linear programming problems (23)–(24), and find the local minimum point $(v^{0*}, \Upsilon^{0*})$.

5. Derive a point $(\tilde{u}, \tilde{\Upsilon}) = (v^{0*}, \tilde{\theta}, \Upsilon^{0*}) \in \Lambda$, where $\tilde{v} = v^{0*}$, $\phi_i = \tilde{\phi}_i, \psi_i = \tilde{\psi}_i, \omega_i = \tilde{\omega}_i$, $i \in I$ and $\tilde{\Upsilon} = \Upsilon^{0*}$.

6. Take the starting point $(\tilde{u}, \tilde{\Upsilon})$ and compute the local minimum point $(u^{0*}, \Upsilon^{0*})$ of the problem (10)–(11) (see Section 4).

7. Form a point $Y^{0*} = (u^{0*}, \Upsilon^{0*}, h^{\nabla})$, where $h^{\nabla} = (h_1^{\nabla}, h_2^{\nabla}, ..., h_n^{\nabla})$.

8. Compute the local maximum point $Y^{1*} = (u^{1*}, \Upsilon^{1*}, h^{1*})$ of the problem (35)-(36) for the starting point $Y^{0*}$.

9. Obtain the sequences (38)–(39), compute $h_j^0 = \min\{h_j^{1*}, h_j^{\nabla}\}$, $j \in I$, and construct the points $\tilde{X} = (\tilde{u}, \tilde{h})$, where $\tilde{u}_j = u_j^{1*}$, $\tilde{h}_j = h_j^{1*}$, $j \in I$ and $\tilde{\tilde{X}} = (\tilde{\tilde{u}}, \tilde{\tilde{h}})$, where $\tilde{u}_j = u_j^0$, $\tilde{\tilde{h}} = h_j^0$, $j \in I$.

10. Find the local maximum point $\hat{X} = (\hat{u}, \hat{h})$ of the problem (40)–(41) for the starting point $\tilde{\tilde{X}}$.

11. Generate a point $(\hat{u}, \Upsilon^{1*})$ if $\Pi(\hat{h}) = n$, compute the local minimum point $(u^{0*}, \Upsilon^{0*})$ of the problem (10)–(11) for the starting point $(\hat{u}, \Upsilon^{1*})$ and return to item 7.

12. Increase *t* by 1 in (37) if $\Pi(\hat{h}) = n$, and return to item 8 if $\varepsilon > 10^3 H(\Upsilon^{0*})$.

13. Take the local minimum point $(u^{0*}, \Upsilon^{0*})$ as an approximation to the global minimum point of the problem (10)–(11) if $\varepsilon > 10^3 H(\Upsilon^{0*})$.

**7. Numerical results**

At present, we do not know any publications which present the results of solving the problem under consideration. So, in order to demonstrate the usefulness and performance capabilities of our mathematical model and solution approach we solve a number of problems for different dimensions. In our cases we consider a collection of polytopes from 10 to 500 and use polytopes with 4, 5 and 16 vertices.

We have run our experiments on an Intel Core i5- 750 CPU computer. For local optimization we used the IPOPT code (https://projects.coin-or.org/Ipopt).

The table below presents the computational efforts and results of solving our problems.

***Table. Computational results***

| Number of homothetic polytopes | Number of vertices of polytopes | Runtime | Result of packing polytopes |
|---|---|---|---|
| 10 | 4 | 20 min | Fig.1 |
| 20 | 16 | 1 hour | Fig.2 |
| 50 | 4 | 3 hours | Fig.3 |
| 100 | 5 | 8 hours | Fig.4 |
| 300 | 4 | 14 hours | Fig.5 |
| 400 | 16 | 30 hours | Fig.6 |
| 500 | 4 | 34 hours | Fig.7 |

Figs. 1 a – 1 g depict the results of packing polytopes.

The obtained results show that despite the complexity of the mathematical model, the developed solution approach makes it possible to solve large dimension problems.

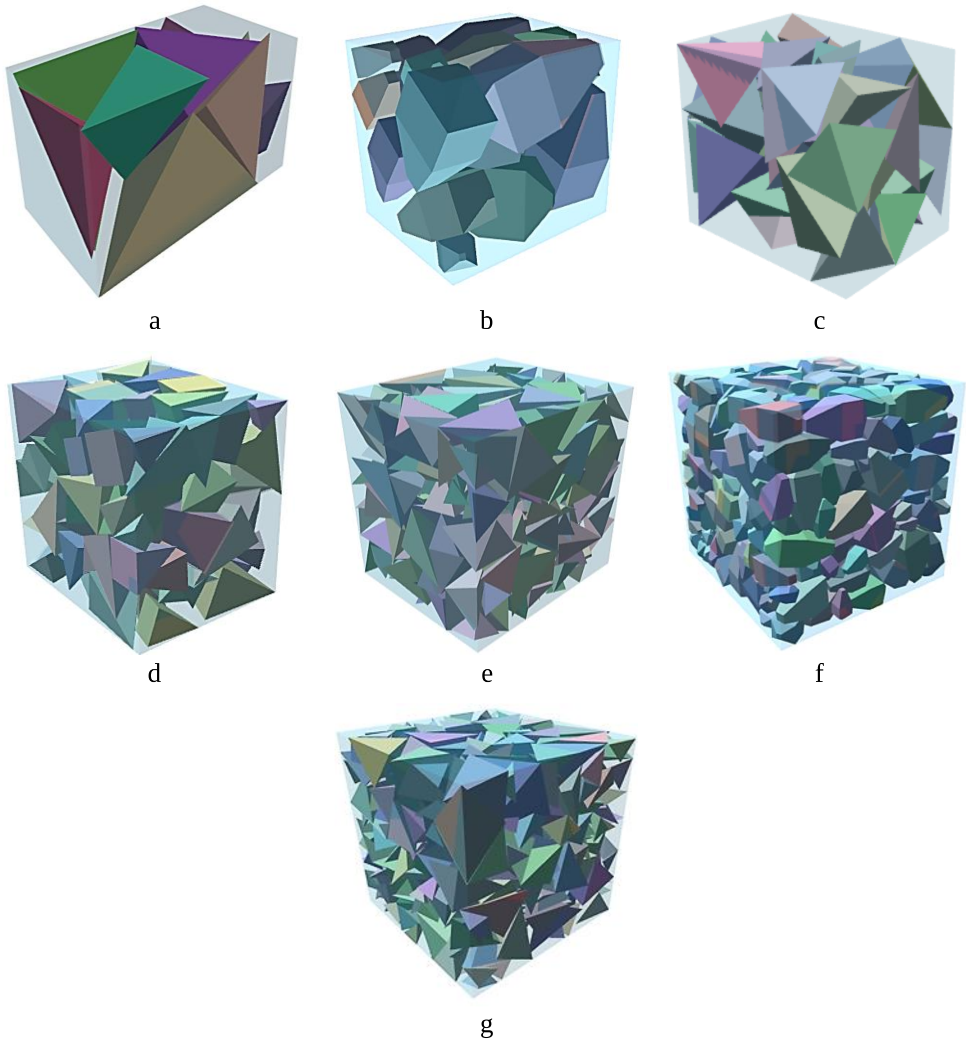

***Fig. 1. Result of packing:***
a – 10 polytopes; b – 20 polytopes; c – 50 polytopes;
d – 100 polytopes; e – 300 polytopes; f – 400 polytopes; g – 500 polytopes

**Conclusion**

Covering polytopes with spheres of minimal radii makes it possible to generate arbitrary starting points.

The hypothesis that the homothetic coefficients of polytopes are variable allows us to develop a new way of generating local extremum points. This approach simplifies the solution process and increases the speed of obtaining results.

The modification of the JA executes smooth transitions between local maximum points in the helper problem, ensuring an increase in the objective value. The algorithm is especially effective if the adjacent homothetic coefficients of polytopes in the sequence (2) are slightly different.

The reduction of the problem (10)–(11) to solving a sequence of sub-problems and decreasing the number of inequalities specifying the feasible region allows us to considerably reduce computational time.

The optimization approaches and the solution algorithms worked out can be applied when tackling optimization packing problems of any homothetic 3D solids.

The proposed approach with some modifications can be applied for solving the problem of packing different convex polytopes.

**References**


1. Petrov M. S., Gaidukov V. V., Kadushnikov R. M. Numerical method for modelling the microstructure of granular materials. *Powder Metallurgy and Metal Ceramics*. 2004. No. 43 (7–8). P. 330–335.
2. Wang Y., Lin C. L., Miller J. D. 3D image segmentation for analysis of multisize particles in a packed particle bed. *Powder Techn*. 2016. No. 301. 160–168.
3. Verkhoturov M., Petunin A., Verkhoturova G., Danilov K., Kurennov D. The 3D Object Packing Problem into a Parallelepiped Container Based on Discrete-Logical Representation. *IFAC-PapersOnLine*. 2016. No. 49 (12). P. 1–5.
4. Karabulut K. A., İnceoğlu M. Hybrid Genetic Algorithm for Packing in 3D with Deepest Bottom Left with Fill Method. *Advances in Inform. Systems*. 2004. No. 3261. P. 441–450.
5. Cao P., Fan Z., Gao R., Tang J. Complex Housing: Modelling and Optimization Using an Improved Multi-Objective Simulated Annealing Algorithm. *Proc. ASME*. 2016. No. 60563, V02BT03A034.
6. Guangqiang L. A., Fengqiang Z., Rubo Z.; Du, Jialu Du., Chen G.,Yiran Z. Parallel Particle Bee Colony Algorithm Approach to Layout Optimization. *J. Computational and Theoretical Nanoscience*. 2016. No. 13 (7). P. 4151–4157.
7. Torczon V., Trosset M. From evolutionary operation to parallel direct search: Pattern search algorithms for numerical optimization. *Computing Sci. and Statistics*. 1998. No. 29. P. 396–401.
8. Birgin E.G., Lobato R. D., Martinez J. M. Packing ellipsoids by nonlinear optimization. *J. Global Optimization*. 2016. No. 65. P. 709–743.
9. Stoyan Y. ,Pankratov A., Romanova T. Quasi-phi-functions and optimal packing of ellipses. *J. Global Optimization*. 2016. No. 65 (2). P. 283–307.
10. Fasano G. A. Global optimization point of view to handle non-standard object packing problems. *J.Global Optimization*. 2013. No. 55 (2). P. 279 –299.
11. Egeblad J., Nielsen B.K., Brazil M. Translational packing of arbitrary polytopes. *Computational Geometry*: *Theory and Appl*. 2009. No. 42 (4). P. 269–288.
12. Liu X,. Liu J., Cao A., Yao Z. HAPE3D - a new constructive algorithm for the 3D irregular packing problem. *Frontiers of Information Techn. & Electronic Eng*. 2015. No. 16 (5). P. 380–390.
13. Youn-Kyoung, Joung, Sang Do. Noh Intelligent 3D packing using a grouping algorithm for automotive container engineering. *J. Computational Design and Eng*. 2014. No. 1 (2). P. 140–151.
14. Kallrath J. Packing ellipsoids into volume-minimizing rectangular boxes. *J. Global Optimization*. No. 67 (1–2). P. 151–185.
15. Stoyan Y.G. , Chugay A.M. Packing different cuboids with rotations and spheres into a cuboid *Advances in Decision Sci*. 2014. Availabel at https://www.hindawi.com/journals/ads/2014/571743.
16. Stoyan Y.G. , Semkin V.V., Chugay A.M. Modeling Close Packing of 3D Objects. *Cybernetics and Systems Analysis*. 2016. No. 52 (2). P. 296–304.
17. Pankratov O., Romanova T., Stoyan Y., Chuhai A. Optimization of packing polyhedra in spherical and cylindrical containers. *Eastern European J. Enterprise Techn*. 2016. Vol. 1. No. 4 (79). P. 39–47.
18. Stoyan Y. G., Chugay A. M. Mathematical modeling of the interaction of non-oriented convex polytopes. *Cybernetic Systems Analysis*. 2012. No. 48. P. 837–845.
19. Stoyan Y. Yaskov G. Packing unequal circles into a strip of minimal length with a jump algorithm *Optimization Letters*. 2014. No. 8 (3). P. 949–970.

**Appendix 1**

It follows from paper [19] that the function of the form $\Psi_{ij}^{3l}(u_i,u_j)$ defines interaction between two edges of polytopes $P_i$ and $P_j$. Let $[p_{io}(u_i),p_{ik}(u_i)]$ be an edge of $P_i$ which lies on a straight line formed by the intersection of planes $\zeta_{ir}(X,u_i)=0$ and $\zeta_{i,r+1}(X,u_i)=0$ and $[p_{js}(u_j),p_{jv}(u_j)]$ − an edge of $P_j$ which lies on a straight line formed by the intersection of planes $\zeta_{jl}(X,u_i)=0$ and $\zeta_{j,l+1}(X,u_i)=0$. The planes form dihedral angles $\Delta_{ir}$ and $\Delta_{jl}$ so that $P_i\subset\Delta_{ir}$ and $P_j\subset\Delta_{jl}$. It is evident that if the dihedral angles are not intersected then $P_i\cap P_j=\varnothing$. So let us derive the condition for the non-overlapping of the dihedral angles in analytical form.

Having translated the edge $[p_{js}(u_j),p_{jv}(u_j)]$ by the vector $\overline{p_{ik}(u_i)-p_{js}(u_j)}$ and the edge $[p_{io}(u_i),p_{ik}(u_i)]$ by the vector $\overline{p_{js}(u_j)-p_{ik}(u_i)}$, we obtain

$$[p_{io}(u_j)-p_{js}(u_i)+p_{ik}(u_j),p_{js}(u_j)]=[p_{io}(u_j),a_{ij}^{ab}(u_i,u_j)],$$

where $a_{ij}^{ab}(u_i,u_j)=(a_{ij}^{1ab}(u_i,u_j),a_{ij}^{2ab}(u_i,u_j),a_{ij}^{3ab}(u_i,u_j))$, $a$ and $b$ are the edge numbers, and $[p_{io}(u_j)-p_{js}(u_i)+p_{ik}(u_j),p_{jv}(u_j)]=[a_{ij}^{ba}(u_i,u_j),p_{io}(u_j)]$, respectively.

Next we construct an equation of plane $Q_{ij}^{ab}$ passing through the points $p_{io}(u_j),p_{iv}(u_j)$ and $a_{ij}^{ab}(u_i,u_j)$

$$F_{ij}^{ab}(X,u_i,u_j)=(x-p_{io}^1(u_j))(x-p_{iv}^1(u_j))(x-a_{ij}^{ab}(u_j))+(y-p_{io}^2(u_j))(y-p_{iv}^2(u_j))(y-a_{ij}^{ab}(u_j))+$$
$$+(z-p_{io}^3(u_j))(z-p_{iv}^3(u_j))(z-a_{ij}^{ab}(u_j))=0$$

and an equation of plane $Q_{ij}^{ba}$ passing through the points $p_{js}(u_j),p_{jv}(u_j)$ and $a_{ij}^{ba}(u_i,u_j)$

$$F_{ij}^{ba}(X,u_i,u_j)=(x-p_{js}^1(u_j))(x-p_{jv}^1(u_j))(x-a_{ij}^{ba}(u_j))+(y-p_{js}^2(u_j))(y-p_{jv}^2(u_j))(y-a_{ij}^{ba}(u_j))+$$
$$+(z-p_{js}^3(u_j))(z-p_{jv}^3(u_j))(z-a_{ij}^{ba}(u_j))=0.$$

Thus, the plane $\Theta_{ij}^{ab}$ passes through the edges $[p_{io}(u_i),p_{ik}(u_i)]$ and $[p_{ik}(u_j),a_{ij}^{ab}(u_i,u_j)]$ and the plane $\Theta_{ij}^{ba}$ passes through the edges $[p_{js}(u_i),p_{jv}(u_i)]$ and $[a_{ij}^{ba}(u_i,u_j),p_{io}(u_j)]$. Since $[p_{io}(u_i),p_{ik}(u_i)]$ is parallel to $[a_{ij}^{ba}(u_i,u_j),p_{io}(u_j)]$ and $[p_{js}(u_j),p_{jv}(u_j)]$ is parallel to $[p_{ik}(u_j),a_{ij}^{ab}(u_i,u_j)]$, the planes $\Theta_{ij}^{ab}$ and $\Theta_{ij}^{ba}$ are parallel. If the plane $\Theta_{ij}^{ab}$ ($\Theta_{ij}^{ba}$) is a separating plane of the dihedral angles $\Delta_{ir}$ and $\Delta_{jt}$, then the inequalities

$$\min\{\varphi_{ij}^{31ab}(u_i,u_j),\varphi_{ij}^{32ab}(u_i,u_j)\}=\min\{F_{ij}^{ab}(p_{i,k-1}(u_i),u_i,u_j),F_{ij}^{ab}(p_{i,k+1}(u_i),u_i,u_j)\}\ge 0,$$

$$\min\{\varphi_{ij}^{33ba}(u_i,u_j),\varphi_{ij}^{34ba}(u_i,u_j)\}=\min\{F_{ij}^{ba}(p_{j,v-1}(u_j),u_i,u_j),F_{ij}^{ba}(p_{j,v+1}(u_j),u_i,u_j)\}\ge 0,$$

$$\varphi_{ij}^{35ab}(u_i,u_j)=F_{ij}^{ab}(p_{jv}(u_i),u_i,u_j)\ge 0$$

are fulfilled, where $p_{i,k-1}(u_i),p_{i,k+1}(u_i)\in\Theta_{ij}^{ab}$ are the adjacent vertices of the vertex $p_{ik}(u_i)$ and $p_{j,v-1}(u_j),p_{j,v+1}(u_j)\in\Theta_{ij}^{ba}$ are the adjacent vertices of the vertex $p_{jv}(u_i)$.

Thus, if the inequality holds true

$$\Psi_{ij}^{3ab}(u_i,u_j)=\min\{F_{ij}^{ab}(p_{i,k-1}(u_i),u_i,u_j),F_{ij}^{ab}(p_{i,k+1}(u_i),u_i,u_j),F_{ij}^{ba}(p_{j,v-1}(u_j),u_i,u_j),$$
$$F_{ij}^{ba}(p_{j,v+1}(u_j),u_i,u_j),F_{ij}^{ab}(p_{jv}(u_i),u_i,u_j)\}>0,$$

then $\Delta_{ir}\cap\Delta_{jt}=\varnothing$. Whence if at least one inequality of the set $\Psi_{ij}^{3ab}(u_i,u_j)>0$, $a,b\in\{1,2,...,\upsilon=\varepsilon+\rceil-2\}$, then $P_i\cap P_j=\varnothing$. For the sake of convenience we label $\Psi_{ij}^{ab}(u_i,u_j)$, $a,b\in\{1,2,...,\upsilon=\varepsilon+\rceil-2\}$, as $\Psi_{ij}^{3l}(u_i,u_j)=\min\{\varphi_{ij}^{3cl}(u_i,u_j),\ c\in M\}$, $l\in L$ (see items 2 and 3, Subsection 2). If the function $\max\{\Psi_{ij}^{3l}(u_i,u_j),l\in L,\}>0$, then $P_i\cap P_j=\varnothing$.

*Received 17 January 2018*