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CONSTRUCTION OF BOTH GEOMETRIC RELATIONSHIPS OF ELLIPSES AND PARABOLA-BOUNDED REGIONS IN GEOMETRIC PLACEMENT PROBLEMS

Mykola I. Hil
GilMI@i.ua
ORCID: 0000-0003-0381-0925

Volodymyr M. Patsuk
vmpatsuk@gmail.com
ORCID: 0000-0003-3350-4515

A. Podgorny Institute
of Mechanical Engineering
Problems of NASU
2/10, Pozharskyi St., Kharkiv,
61046, Ukraine

Currently, there is a significant growth of interest in the practical problems of mathematically modeling the placement of geometric objects of various physical natures in given areas. When solving such problems, there is a need to build their mathematical models, which are implemented through the construction of analytical conditions for the relations of the objects being placed and placement regions. The problem of constructing conditions for the mutual non-intersection of arbitrarily oriented objects whose boundaries are formed by second-order curves is widely used in practice and, at the same time, much less studied than a similar problem for simpler objects. A fruitful and worked out method of representing such conditions is the construction of Stoyan's Φ-functions (further referred to as phi-functions) and quasi-phi-functions. In this article, considered as geometric objects are an ellipse and a parabola-bounded region. The boundaries of the objects under study allow both implicit and parametric representations. The proposed approach to modeling the geometric relationships of ellipses and parabola-bounded regions is based on coordinate transformation, reduction of an ellipse equation to a circle equation with the use of a canonical transformation. In particular, constructed are the conditions for the inclusion of an ellipse in a parabola-bounded region, as well as the conditions for their mutual non-intersection. The conditions for the relationships between the geometric objects under study are constructed on the basis of the canonical equations of the ellipse and parabola, taking into account their placement parameters, including rotations. These conditions are presented in the form of a system of inequalities, as well as in the form of a single analytical expression. The presented conditions can be used in constructing adequate mathematical models of optimization problems of placing corresponding geometric objects for an analytical description of feasible regions. These models can be used further in the formulation of mathematical models of packing and cutting problems, expanding the range of objects and / or increasing solution accuracy and decreasing time to solution.

Keywords: *ellipse, parabola, non-intersection, inclusion, phi-function.*

Introduction

The most important part of solving problems associated with the modeling of the placement of geometric objects in given regions is the construction of adequate mathematical models of corresponding optimization problems. The main component of such mathematical models is an analytical representation of the conditions for the interaction of the geometric objects being placed and placement regions, including the conditions of mutual non-intersection of geometric objects, as well as the conditions of their inclusion in the placement region.

For most of the classes of two-dimensional geometric objects, including circles, polygons, and objects obtained by their combination on the basis of unions and intersections, such conditions are realized on the basis of phi-functions [1] and quasi-phi-functions [2]. In the two-dimensional case, phi-functions are constructed for objects whose boundaries consist of segments of straight lines as well as convex and concave arcs of circles [3].

At the same time, the construction of the conditions of mutual non-intersection and inclusion for objects whose boundaries are described by other types of curves is much more complicated, and there are fewer results in the area. So, in [4], for the optimal packing of ellipses, an approach using quasi-phi-functions is used. In [5], E. Birgin et al. successfully apply a special transformation of space for packing optimization problems involving ellipses, simplifying these problems. In [6–8], the authors apply the approximation of ellipses by a set of circles constructed in a certain way. A common method is also the approximation of object boundaries with 2D broken lines and 3D polyhedral surfaces. Due to the complexity of constructing phi-functions for the above mentioned objects, quasi-phi-functions were proposed, which helped solve a number of problems, in particular, for ellipses and ellipsoids [6, 9, 10].

In this article, considered as geometric objects are ellipses and a parabola-bounded region.

Conditions for the Inclusion of an Ellipse in a Parabola-bounded Region

In the following presentation, we will identify such concepts: an ellipse as a curve line and an ellipse as a region bounded by this curve.

Let, with respect to some coordinate system $\bar{x}o\bar{y}$, there be a region $D\{\bar{u}_1, \bar{\vartheta}_1\}$ bounded by a parabola $S_1\{\bar{u}_1, \bar{\vartheta}_1\}$ and an ellipse $S_2\{\bar{u}_2, \bar{\vartheta}_2\}$, where $\bar{u}_i = \{\bar{x}_i, \bar{y}_i\}$ and $\bar{\vartheta}_i$, $i = 1,2$ are the placement parameters of an object S_i (the position of the origin and angle of rotation of its own coordinate system associated with the object S_i relative to the coordinate system $\bar{x}o\bar{y}$). In their own coordinate systems xoy and $X'OY'$, both the parabola and the ellipse are given by the equations $y = px^2$ ($p > 0$) and $B^2X'^2 + A^2Y'^2 - A^2B^2 = 0$ ($A > B$), respectively. Note that $S_2\{\bar{u}_2, \bar{\vartheta}_2\}$ is the inclusion in the region $D\{\bar{u}_1, \bar{\vartheta}_1\}$ if $S_2\{\bar{u}_2, \bar{\vartheta}_2\} \cap D_2\{\bar{u}_2, \bar{\vartheta}_2\} = S_2\{\bar{u}_2, \bar{\vartheta}_2\}$.

We choose the xoy coordinate system as the main one, relative to which we have the region $D\{u_1, \vartheta_1\}$ and the ellipse $S_2\{u_2, \vartheta_2\}$, where $u_1 = (0,0)$, $\vartheta_1 = 0$, $u_2 = (x_0, y_0)$, $\vartheta_2 = (\bar{\vartheta}_2 - \bar{\vartheta}_1)$. Here,

$$x_0 = (\bar{x}_2 - \bar{x}_1)\cos\bar{\vartheta}_1 - (\bar{y}_2 - \bar{y}_1)\sin\bar{\vartheta}_1,$$
$$y_0 = (\bar{x}_2 - \bar{x}_1)\sin\bar{\vartheta}_1 + (\bar{y}_2 - \bar{y}_1)\cos\bar{\vartheta}_1. \quad (1)$$

We introduce a new coordinate system $x'oy'$ rotated through an angle of ϑ_2 relative to xoy. Taking into account the transformation formulas in this coordinate system, the parabola and ellipse equations have the forms $x'\sin\vartheta_2 + y'\cos\vartheta_2 = p[x'\cos\vartheta_2 - y'\sin\vartheta_2]^2$ and $B^2(x' - x'_0)^2 + A^2(y' - y'_0)^2 - A^2B^2 = 0$, respectively, where

$$x'_0 = x_0\cos\vartheta_2 + y_0\sin\vartheta_2,$$
$$y'_0 = -x_0\sin\vartheta_2 + y_0\cos\vartheta_2,$$

x_0, y_0 are determined from (1).

We carry out compression in the direction of the ox' axis in accordance with the transformation formulas $x' = \frac{A}{B}\bar{X}$, $y' = \bar{Y}$. Then in the new coordinate system $\bar{X}O\bar{Y}$ the parabola is described by the equation

$$\frac{A}{B}\bar{X}\sin\vartheta_2 + \bar{Y}\cos\vartheta_2 = p[\frac{A}{B}\bar{X}\cos\vartheta_2 - \bar{Y}\sin\vartheta_2]^2,$$

and the ellipse turns into a circle whose equation has the form

$$(\bar{X} - \bar{X}_0)^2 + (\bar{Y} - \bar{Y}_0)^2 - B^2 = 0,$$

where $\bar{X}_0 = \frac{B}{A}[x_0\cos\vartheta_2 + y_0\sin\vartheta_2)]$, $\bar{Y}_0 = -x_0\sin\vartheta_2 + y_0\cos\vartheta_2$.

We write the parabola equation as

$$a_{11}\bar{X}^2 + 2a_{12}\bar{X}\bar{Y} + a_{22}\bar{Y}^2 + 2a_{13}\bar{X} + 2a_{23}\bar{Y} + a_{33} = 0, \quad (2)$$

where

$$a_{11} = pA^2\cos^2\vartheta_2,\ a_{22} = pB^2\sin^2\vartheta_2,\ a_{12} = -pAB\sin\vartheta_2\cos\vartheta_2,$$
$$a_{13} = -\frac{1}{2}AB\sin\vartheta_2,\ a_{23} = -\frac{1}{2}B^2\cos\vartheta_2,\ a_{33} = 0. \tag{3}$$

It is known [11] that the equation of form (2) (if we introduce a new coordinate system XOY by rotating through an angle of φ that satisfies the equation $\mathrm{tg}\,2\varphi = \frac{2a_{12}}{a_{11}-a_{22}}$) is reduced to the canonical parabola equation $X = \frac{1}{2p'}Y^2$, where $p' = \frac{1}{J}\sqrt{-\frac{\Delta}{J}}$, $J = a_{11} + a_{22}$, $\Delta = \det\lfloor a_{ij}\rfloor$, $i, j = 1,2,3$ ($a_{ij} = a_{ji}$).

With account taken of (3), after simple transformations

$$p' = \frac{AB^2}{2p(A^2\cos^2\vartheta_2 + B^2\sin^2\vartheta_2)^{3/2}}. \tag{4}$$

Thus, in the XOY coordinate system, we have the region $\overline{D}\{0,0,0\}$ bounded by the parabola $X = \frac{1}{2p'}Y^2$ and the circle $\overline{S}\{X_0, Y_0\}$ bounded by the circumference $(X - X_0)^2 + (Y - Y_0)^2 - B^2 = 0$, where

$$X_0 = \frac{B}{A}(x_0\cos\vartheta_2 + y_0\sin\vartheta_2)\cos 2\varphi + (x_0\sin\vartheta_2 - y_0\cos\vartheta_2)\sin 2\varphi,$$
$$Y_0 = -\frac{B}{A}(x_0\cos\vartheta_2 + y_0\sin\vartheta_2)\sin 2\varphi + (x_0\sin\vartheta_2 - y_0\cos\vartheta_2)\cos 2\varphi, \tag{5}$$
$$\sin 2\varphi = -2\frac{AB\sin\vartheta_2\cos\vartheta_2}{A^2\cos^2\vartheta_2 + B^2\sin^2\vartheta_2},\ \cos 2\varphi = \frac{A^2\cos^2\vartheta_2 - B^2\sin^2\vartheta_2}{A^2\cos^2\vartheta_2 + B^2\sin^2\vartheta_2}.$$

Then the conditions for the inclusion of the ellipse $S_2\{\overline{u}_2, \overline{\vartheta}_2\}$ in the region $D\{\overline{u}_1, \overline{\vartheta}_1\}$ are reduced to the conditions for the inclusion of the circle $\overline{S}\{X_0, Y_0\}$ in the region $\overline{D}\{0,0,0\}$. We denote by $\overline{S}^\gamma$ the circle of radius γB with the center at the point (X_0, X_0). It can be argued that the circle $\overline{S}\{X_0, Y_0\}$ is the inclusion in the region $\overline{D}\{0,0,0\}$ if there is a point (X^*, Y^*) satisfying the conditions:

a) the point (X^*, Y^*) is not an internal point of the circle $\overline{S}\{X, Y\}$;

b) the center of the circle (X_0, Y_0) is inside the region $\overline{D}\{0,0,0\}$;

c) the point (X^*, Y^*) is in the positive half-plane bounded by a straight line $X - \overline{X} = 0$, where $\overline{X}$ is the abscissa of the point of contact of the parabola $X = \frac{1}{2p'}Y^2$ and the circle of radius B with center at the point $(\overline{X}_0, 0)$;

d) the point (X^*, Y^*) belongs to the parabola $X = \frac{1}{2p'}Y^2$ and the circle $\overline{S}^\gamma$;

e) the angular coefficients of the tangents to the parabola $X = \frac{1}{2p'}Y^2$ and the circle $\overline{S}^\gamma$ at the point (X^*, Y^*) are equal.

The value $\overline{X}$ is determined from the system of equations

$$\begin{cases} \overline{X} - \dfrac{1}{2p'}\overline{Y}^2 = 0, \\ (\overline{X} - \overline{X}_0)^2 + \overline{Y}^2 - B^2 = 0, \\ p'\overline{Y} + \overline{Y}(\overline{X} - \overline{X}_0) = 0, \end{cases}$$

which implement the conditions for the point $\overline{S}\{\overline{X}_0, 0\}$ to belong to the circle $(\overline{X}, \overline{Y})$ and parabola $X = \frac{1}{2p'}Y^2$, as well as the equality of the angular coefficients of the tangents to the circle and parabola at the point $(\overline{X}, \overline{Y})$. The solution to this system is $\overline{X} = \frac{1}{2p'}(B^2 - p'^2)$, $\overline{Y} = \pm\sqrt{B^2 - p'^2}$, provided $p' < B$ (the radius of curvature of the parabola at the point $(0,0)$ is less than the radius of the circle). Otherwise, $\overline{X} = \overline{Y} = 0$.

Condition e) in this case is represented as

$$\phi(X_0, Y_0, Y^*) \equiv p'(Y^* - Y_0) + Y^*(\frac{1}{2p'}Y^{*2} - X_0) = 0. \tag{6}$$

Thus, the conditions for the inclusion of the circle $\overline{S}\{X_0, Y_0\}$ in the region $\overline{D}\{0,0,0\}$ are reduced to the fulfillment of the system of inequalities

$$\begin{cases} f_2(X_0, X_0, Y^*) \equiv (\dfrac{1}{2p'}Y^{*2} - X_0)^2 + (Y^* - Y_0)^2 - B^2 \geq 0, \\ f_1(X_0, X_0) \equiv X_0 - \dfrac{1}{2p'}Y_0^2 \geq 0, \\ h(Y^*) \equiv \dfrac{1}{2p'}Y^{*2} - \overline{X} > 0, \end{cases}$$

where p' is determined from (4); X_0, Y_0 are determined from (5), Y^* is one of the solutions to equation (6).

The conditions for the inclusion of the circle $\overline{S}(X_0, Y_0)$ in the region $\overline{D}\{0,0,0\}$ can be considered as conditions for the non-intersection of the objects $R^2 \setminus \operatorname{int} \overline{D}\{0,0,0\}$ and $\overline{S}(X_0, Y_0)$, i.e., presented in the form of a phi-function

$$\Phi(X_0, Y_0, Y^*) = \max_{Y_i^*} \min \{f_2(X_0, Y_0, Y^*), f_1(X_0, Y_0), h(Y^*)\},$$

where Y_i^*, $i = 1,2,...$ are the roots of equation (6), $Y_i^* \in \left[\sqrt{B^2 - p'^2}, \sqrt{2p'X_0}\right]$, if $Y_0 \geq 0$, $Y_i^* \in \left[-\sqrt{2p'X_0}, -\sqrt{B^2 - p'^2}\right]$, if $Y_0 < 0$.

Non-intersection Conditions for an Ellipse and a Parabola-bounded Region

Let in the coordinate system $\overline{xoy}$ there be a region $D\{\overline{u}_1, \overline{\vartheta}_1\}$ bounded by a parabola $S_1\{\overline{u}_1, \overline{\vartheta}_1\}$ and an ellipse $S_2\{\overline{u}_2, \overline{\vartheta}_2\}$, which in their own coordinate systems xoy and $X'OY'$ are described by the equations $y - px^2 = 0$ ($p > 0$) and $B^2X'^2 + A^2Y'^2 - A^2B^2 = 0$ ($A > B$).

We will further understand the non-intersection of objects $\Upsilon_1, \Upsilon_2 \in R^2$ as the non-intersection of their interiors $\operatorname{int}\Upsilon_1 \cap \operatorname{int}\Upsilon_2 = \varnothing$ and allow touching, i.e., the intersection of boundaries.

If we choose the xoy coordinate system as the main one, then relative to it we have the region $D\{u_1, \vartheta_1\}$ and the ellipse $S_2\{u_2, \vartheta_2\}$, where $u_1 = (0,0)$, $\vartheta_1 = 0$, $u_2 = (x_0, y_0)$, $\vartheta_2 = \overline{\vartheta}_2 - \overline{\vartheta}_1$. Here,

$$x_0 = (\bar{x}_2 - \bar{x}_1)\cos\vartheta_1 - (\bar{y}_2 - \bar{y}_1)\sin\vartheta_1,$$
$$y_0 = (\bar{x}_2 - \bar{x}_1)\sin\vartheta_1 + (\bar{y}_2 - \bar{y}_1)\cos\vartheta_1. \quad (7)$$

With the transformation formulas

$$X' = (x - x_0)\cos\vartheta_2 + (y - y_0)\sin\vartheta_2,$$
$$Y' = -(x - x_0)\sin\vartheta_2 + (y - y_0)\cos\vartheta_2$$

the ellipse equation $S_2\{x_0, y_0, \vartheta_2\}$ in the xoy coordinate system takes the form

$$B^2[(x - x_0)\cos\vartheta_2 + (y - y_0)\sin\vartheta_2]^2 +$$
$$A^2[-(x - x_0)\sin\vartheta_2 + (y - y_0)\cos\vartheta_2]^2 - A^2B^2 = 0. \quad (8)$$

Let (x^*, y^*) be some point of the parabola $y - px^2 = 0$. The equation of the tangent to the parabola at the point (x^*, y^*) has the form

$$F(x, y) \equiv 2px^* x - y - px^{*2} = 0. \quad (9)$$

We denote by (x_{q_i}, y_{q_i}), $i = 1,2$, the corresponding points of ellipse (8) at which the tangents are parallel to tangent (9) at the point (x^*, y^*). The angular coefficient of the tangent to ellipse (8) at the point (x_q, y_q) is

$$-\frac{R(x_q - x_0) + L(y_q - y_0)}{L(x_q - x_0) + S(y_q - y_0)},$$

where

$$R = B^2\cos^2\vartheta_2 + A^2\sin^2\vartheta_2,$$
$$S = B^2\sin^2\vartheta_2 + A^2\cos^2\vartheta_2, \quad (10)$$
$$L = (B^2 - A^2)\sin\vartheta_2\cos\vartheta_2.$$

To determine the coordinates x_q, y_q we use two conditions:

a) the point (x_q, y_q) is an ellipse point, i.e., it satisfies equation (8);

b) the angular coefficients of the tangents to the parabola and to the ellipse at the corresponding points (x^*, y^*) and (x_q, y_q) are equal.

Condition b) in this case takes the form

$$2px^*[L(x_q - x_0) + S(y_q - y_0)] + R(x_q - x_0) + L(y_q - y_0) = 0,$$

where x_0 y_0 are derived from (7).

Thus, we have a system of two equations

$$\begin{cases} B^2[(x_q - x_0)\cos\vartheta_2 + (y_q - y_0)\sin\vartheta_2]^2 + \\ A^2[-(x_q - x_0)\sin\vartheta_2 + (y_q - y_0)\cos\vartheta_2]^2 - A^2B^2 = 0, \\ 2px^*[L(x_q - x_0) + S(y_q - y_0)] + R(x_q - x_0) + L(y_q - y_0) = 0. \end{cases}$$

The solution to this system is the vector with the values of required coordinates

$$x_{q_{1,2}} = x_0 \pm \frac{AB}{\sqrt{B^2(\cos\vartheta_2 + D\sin\vartheta_2)^2 + A^2(D\cos\vartheta_2 - \sin\vartheta_2)^2}},$$

$$y_{q_{1,2}} = y_0 \pm \frac{DAB}{\sqrt{B^2(\cos\vartheta_2 + D\sin\vartheta_2)^2 + A^2(D\cos\vartheta_2 - \sin\vartheta_2)^2}},$$

where $D = -\dfrac{2pLx^* + R}{2pSx^* + L}$, L, R, S are derived from (10).

It can be argued that if:

– the points (x_{q_1}, y_{q_1}) and (x_{q_2}, y_{q_2}) have non-negative deviations relative to tangent (9);

– the point (x_0, y_0) has a positive deviation relative to tangent (9),

then the ellipse $S_2\{u_2, \vartheta_2\}$ and the region $D\{u_1, \vartheta_1\}$ (and, therefore, $S_2\{\overline{u}_2, \overline{\vartheta}_2\}$ and $D\{\overline{u}_1, \overline{\vartheta}_1\}$) do not intersect.

In the analytical representation, these conditions are expressed as a system of inequalities

$$\begin{cases} 2px^* x_{q_1} - y_{q_1} - px^{*2} \ge 0, \\ 2px^* x_{q_2} - y_{q_2} - px^{*2} \ge 0, \\ 2px^* x_0 - y_0 - px^{*2} > 0, \end{cases}$$

where (x_0, y_0) have the form (7).

Thus, the conditions for the non-intersection of the ellipse $S_2\{\overline{u}_2, \overline{\vartheta}_2\}$ and the region $D\{\overline{u}_1, \overline{\vartheta}_1\}$ bounded by the parabola $S_1\{\overline{u}_1, \overline{\vartheta}_1\}$ can be represented in an analytical form as follows:

$$\max_{x^*} \min \{2px^* x_{q_1} - y_{q_1} - px^{*2},\ 2px^* x_{q_2} - y_{q_2} - px^{*2},\ 2px^* x_0 - y_0 - px^{*2}\} \ge 0,$$

where (x_0, y_0) are derived from (7).

The conditions for the mutual non-intersection of ellipses

Let ellipses $S_i\{\overline{u}_i, \overline{\vartheta}_i\}$, $i = 1,2$, whose boundaries in their own coordinate systems xoy and $X'OY'$ are described by the equations $b^2x^2 + a^2y^2 - a^2b^2 = 0$ and $B^2X'^2 + A^2Y'^2 - A^2B^2 = 0$, respectively, be given in some coordinate system $\overline{x}o\overline{y}$. In the coordinate system xoy, chosen as the main one, we have ellipses $S_i\{u_i, \vartheta_i\}$, $i = 1,2$, where $u_1 = (0,0)$, $\vartheta_1 = 0$, $u_2 = (x_0, y_0)$, $\vartheta_2 = \overline{\vartheta}_2 - \overline{\vartheta}_1$. The equations of the ellipses $S_1\{0,0,0\}$ and $S_2\{x_0, y_0, \vartheta_2\}$ have the form $b^2x^2 + a^2y^2 - a^2b^2 = 0$ and (8), respectively.

Let (x^*, y^*) be an arbitrary ellipse point $S_1\{0,0,0\}$.

The equation of the tangent to the ellipse $S_1\{0,0,0\}$ at the point (x^*, y^*), taking into account the fact that $x = a\cos\varphi$, $y = b\sin\varphi$, is represented as

$$F(x, y, \varphi^*) \equiv b\cos\varphi^* \cdot x + a\sin\varphi^* \cdot y - ab = 0. \qquad (11)$$

Denote by (x_{q_i}, y_{q_i}), $i = 1,2$, the points of the ellipse $S_2\{x_0, y_0, \vartheta_2\}$ whose tangents are parallel to tangent (11) at the point (x^*, y^*). To determine the coordinates (x_q, y_q), as in the previous case, we have a system of two equations

$$\begin{cases} B^2[(x_q - x_0)\cos\vartheta_2 + (y_q - y_0)\sin\vartheta_2]^2 + \\ A^2[-(x_q - x_0)\sin\vartheta_2 + (y_q - y_0)\cos\vartheta_2]^2 - A^2B^2 = 0, \\ (Ra\sin\varphi^* - Lb\cos\varphi^*)(x_q - x_0) + (La\sin\varphi^* - Sb\cos\varphi^*)(y_q - y_0) = 0, \end{cases}$$

where the second equation of the system fulfills the equality of the angular coefficients of the tangent to the ellipse $S_1\{0,0,0\}$ at the point (x^*, y^*) and the tangent to the ellipse $S_2\{x_0, y_0, \vartheta_2\}$ at the point (x_q, y_q). Here, (x_0, y_0) are derived from (7), and R, S, L are derived from (10).

The solution to this system is the vector with the values of required coordinates

$$x_{q_{1,2}} = x_0 \pm \frac{AB}{\sqrt{B^2(\cos\vartheta_2 + \overline{D}\sin\vartheta_2)^2 + A^2(\overline{D}\cos\vartheta_2 - \sin\vartheta_2)^2}},$$

$$y_{q_{1,2}} = y_0 \pm \frac{\overline{D}AB}{\sqrt{B^2(\cos\vartheta_2 + \overline{D}\sin\vartheta_2)^2 + A^2(\overline{D}\cos\vartheta_2 - \sin\vartheta_2)^2}},$$

where $\overline{D} = \dfrac{Lb\cos\varphi^* - Ra\cos\varphi^*}{La\sin\varphi^* - Sb\cos\varphi^*}$.

It is easy to verify that if the conditions:

– the points (x_{q_1}, y_{q_1}) and (x_{q_2}, y_{q_2}) have non-negative deviations relative to tangent (11);

– the points $(0,0)$ and (x_0, y_0) are located on the opposite sides of tangent (11),

are met, then the ellipses $S_1\{0,0,0\}$ and $S_2\{x_0, y_0, \vartheta_2\}$ (and, therefore, $S_1\{\overline{u}_1, \overline{\vartheta}_1\}$ and $S_2\{\overline{u}_2, \overline{\vartheta}_2\}$) do not intersect. In analytical form, these conditions represent a system of inequalities

$$\begin{cases} b\cos\varphi^* \cdot x_{q_1} + a\sin\varphi^* \cdot y_{q_1} - ab \ge 0, \\ b\cos\varphi^* \cdot x_{q_2} + a\sin\varphi^* \cdot y_{q_2} - ab \ge 0, \\ b\cos\varphi^* \cdot x_0 + a\sin\varphi^* \cdot y_0 - ab > 0, \end{cases}$$

where (x_0, y_0) are determined from (7).

Thus, the conditions for the non-intersection of the ellipses $S_i\{\overline{x}_i, \overline{y}_i, \overline{\vartheta}_i\}$, $i = 1,2$, can be represented as

$$\max_{\varphi^*} \ \min\ \{b\cos\varphi^* \cdot x_{q_1} + a\sin\varphi^* \cdot y_{q_1} - ab, \quad b\cos\varphi^* \cdot x_{q_2} + a\sin\varphi^* \cdot y_{q_2} - ab,$$
$$b\cos\varphi^* \cdot x_0 + a\sin\varphi^* \cdot y_0 - ab\} \ge 0.$$

Conclusions

Conditions for the mutual non-intersection and inclusion for geometric objects with boundaries defined by the equations of second-order curves, in particular, an ellipse and a parabola, are constructed. Such curves describe a rather large class of practical problems.

The obtained inclusion and mutual non-intersection conditions in the form of systems of inequalities can be used in constructing adequate mathematical models of optimization problems of placing the corresponding geometric objects for an analytical description of feasible regions.

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Побудова геометричних співвідношень еліпсів та областей, обмежених параболою, в задачах розміщення геометричних об'єктів

М. І. Гіль, В. М. Пацук

Інститут проблем машинобудування ім. А. М. Підгорного НАН України,
61046, Україна, м. Харків, вул. Пожарського, 2/10

На цей час значно зростає інтерес до практичних задач математичного моделювання розміщення геометричних об'єктів різної фізичної природи в заданих областях. Під час розв’язання таких задач виникає необхідність в побудові їхніх математичних моделей, які реалізуються через побудову аналітичних умов відношень розміщуваних об'єктів і областей розміщення. Задача побудови умов взаємного неперетину довільно орієнтованих об'єктів, межі яких утворені кривими другого порядку, має широке застосування на практиці і водночас досліджена значно менше, ніж аналогічна задача для більш простих об'єктів. Плідним і відпрацьованим методом опису таких умов є побудова Ф-функцій і квазі-Ф-функцій. У даній статті як геометричні об'єкти розглядаються еліпс і область, обмежена параболою. Межі об'єктів, що розглядаються, допускають як неявне, так і параметричне зображення. Запропонований підхід до моделювання геометричних відношень еліпсів і областей, обмежених параболами, ґрунтується на перетворенні координат, приведенні рівняння еліпса до рівняння кола з використанням канонічного перетворення. Зокрема, побудовані умови включення еліпса в область, обмежену параболою, а також умови їх взаємного неперетину. Побудова умов взаємовідношень об'єктів, що розглядаються, здійснена на основі канонічних рівнянь еліпса і параболи з урахуванням їх параметрів розміщення, включаючи обертання. Ці умови зображені у вигляді системи нерівностей, а також у вигляді єдиного аналітичного виразу. Зображені умови можуть бути використані під час побудови адекватних математичних моделей оптимізаційних задач розміщення відповідних геометричних об'єктів для аналітичного опису областей допустимих розв’язків. Ці моделі можуть використовуватися далі в формулюванні математичних моделей практичних задач упаковки та розкрою, розширюючи коло об'єктів та/або підвищуючи точність і знижуючи час отримання розв’язання.

***Ключові слова**: еліпс, парабола, неперетин, включення, Ф-функція.*

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METHODOLOGY TO SOLVE OPTIMAL PLACEMENT PROBLEMS FOR 3D OBJECTS

Yurii H. Stoian
stoayn@ipmach.kharkov.ua
ORCID: 0000-0002-1476-1818

Andrii M. Chuhai
chugay.andrey80@gmail.com
ORCID: 0000-0002-4079-5632

A. Podgorny Institute of Mechanical Engineering Problems of NASU
2/10, Pozharskyi St., Kharkiv, 61046, Ukraine

This paper is devoted to solving optimization problems of packing 3D objects both by constructing exact mathematical models and by developing approaches based on the application of non-linear optimization methods and modern solvers. Developed are constructive tools for both mathematical and computer modeling of relations between oriented and non-oriented 3D objects, whose boundaries are formed by cylindrical, conical, and spherical surfaces and planes in the form of new classes of both Stoyan's Φ-function (further referred to as phi-functions) and quasi-phi-functions. Based on the developed mathematical modeling tools, constructed and investigated is the basic mathematical model of the problem of optimally packing 3D objects, whose boundaries are formed by cylindrical, conical, and spherical surfaces and planes, as well as the model's various implementations, which cover a wide class of scientific and applied problems of packing 3D objects. Developed is the methodology for solving the problems of packing 3D objects that allow both continuous rotations and translations at the same time. Proposed are strategies, methods and algorithms for solving the optimization problems of packing 3D objects with taking into account technological constraints (minimum admissible distances, prohibited zones, the possibility of continuous translations and rotations). On the basis of the proposed mathematical modeling tools, mathematical models, methods, and algorithms, developed is the software that uses parallel computing technology to automatically solve the optimization problems of packing 3D objects. The results obtained can be used for solving problems of optimizing layout solutions; for computer modeling in materials science, powder metallurgy, and nanotechnologies; in optimizing the 3D printing process for the SLS technology of additive production; in information and logistics systems that optimize transportation and storage of goods.

Keywords: *packing, 3D objects, geometric design, phi-functions, mathematical modeling, continuous rotations, nonlinear optimization.*

Introduction

Today, in many fields of science and technology, among the problems that have been intensely solved in recent decades, we can distinguish the computer modeling problems of the optimal placement of 3D objects of different nature. These problems are becoming highly demanded because the replacement of in situ experiments with computer modelling can significantly save both material resources and time. Therefore, it requires the development of models, methods, and algorithms to solve relevant problems.

Possible areas of the practical application of the problems of optimal packing of 3D objects can be conditionally classified as follows: problems of optimization of layout solutions; 3D modeling in materials science, powder metallurgy and nanotechnologies; optimization of the 3D printing process for the SLS additive manufacturing technology; information-logistic systems that provide the optimization of transportation and storage of cargos.