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# METHODOLOGY TO SOLVE OPTIMAL PLACEMENT PROBLEMS FOR 3D OBJECTS

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*This paper is devoted to solving optimization problems of packing 3D objects both by constructing exact mathematical models and by developing approaches based on the application of non-linear optimization methods and modern solvers. Developed are constructive tools for both mathematical and computer modeling of relations between oriented and non-oriented 3D objects, whose boundaries are formed by cylindrical, conical, and spherical surfaces and planes in the form of new classes of both Stoyan's Φ-functions (further referred to as phi-functions) and quasi-phi-functions. Based on the developed mathematical modeling tools, constructed and investigated is the basic mathematical model of the problem of optimally packing 3D objects, whose boundaries are formed by cylindrical, conical, and spherical surfaces and planes, as well as the model's various implementations, which cover a wide class of scientific and applied problems of packing 3D objects. Developed is the methodology for solving the problems of packing 3D objects that allow both continuous rotations and translations at the same time. Proposed are strategies, methods and algorithms for solving the optimization problems of packing 3D objects with taking into account technological constraints (minimum admissible distances, prohibited zones, the possibility of continuous translations and rotations). On the basis of the proposed mathematical modeling tools, mathematical models, methods, and algorithms, developed is the software that uses parallel computing technology to automatically solve the optimization problems of packing 3D objects. The results obtained can be used for solving problems of optimizing layout solutions; for computer modeling in materials science, powder metallurgy, and nanotechnologies; in optimizing the 3D printing process for the SLS technology of additive production; in information and logistics systems that optimize transportation and storage of goods.*

***Keywords:*** *packing, 3D objects, geometric design, phi-functions, mathematical modeling, continuous rotations, nonlinear optimization.*

## Introduction

Today, in many fields of science and technology, among the problems that have been intensely solved in recent decades, we can distinguish the computer modeling problems of the optimal placement of 3D objects of different nature. These problems are becoming highly demanded because the replacement of in situ experiments with computer modelling can significantly save both material resources and time. Therefore, it requires the development of models, methods, and algorithms to solve relevant problems.

Possible areas of the practical application of the problems of optimal packing of 3D objects can be conditionally classified as follows: problems of optimization of layout solutions; 3D modeling in materials science, powder metallurgy and nanotechnologies; optimization of the 3D printing process for the SLS additive manufacturing technology; information-logistic systems that provide the optimization of transportation and storage of cargos.

It is known that the problem of packing 3D objects is NP-complete. Because of this, it is difficult to solve it satisfactorily. Thus, to find its approximate solution, many research works use a wide variety of methods, including different heuristics (heuristics based on different approximation rules [1–3], genetic algorithms [4], the simulated annealing algorithm [5], the artificial bee colony algorithm [6]), the advanced pattern search [7], traditional optimization methods [8, 9], and various mixed approaches that apply heuristics and non-linear mathematical programming methods [10].

In most works, the orientation of 3D objects is not allowed to be changed, or allowed are only discrete orientation changes for given angles. For example, in [11] only the parallel transfer algorithm is used for packing convex polyhedra. In [12], the authors propose a so-called HAPE 3D constructive algorithm that can be applied to an arbitrary-shaped polyhedron that can rotate around each coordinate axis at only eight different angles.

In [13], the authors note that for 3D packing problems, the orientation of objects from 0º to 360º relative to each axis cannot be calculated.

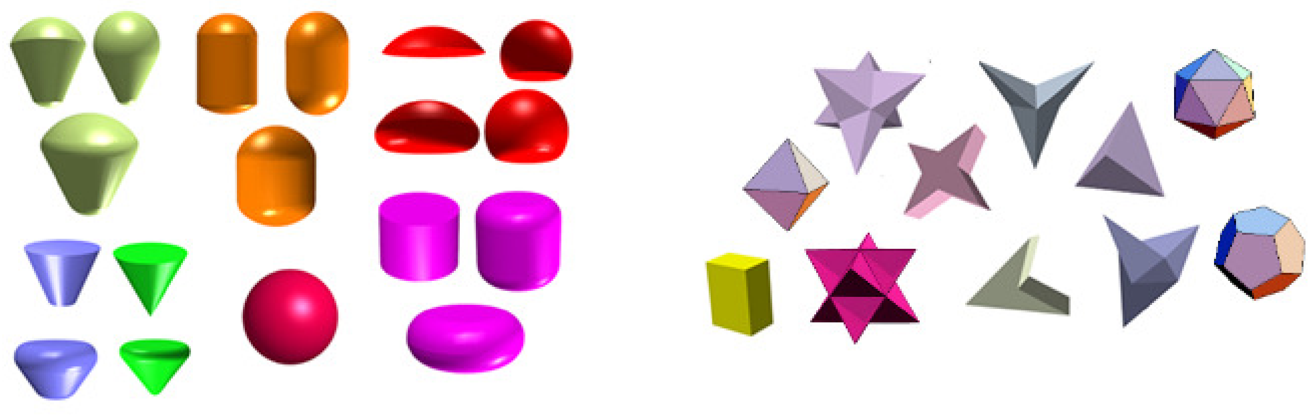
Due to the complexity of constructing adequate mathematical models, there are currently only a few works that solved the problems of 3D packing, provided that continuous rotations of geometric objects are allowed. Solutions to such problems are discussed in [8, 9, 14, 15, 16]. In [8, 9, 14], both continuous and different non-linear programming models, as well as algorithms for packing ellipsoids in 3D are introduced. In [16], the problem of packing various convex 3D objects is solved.

**General Statement of the Problem**

Despite their different formulations, all the optimization packing problems of 3D objects can be described by a general statement, which can be formulated in the following form.

*Problem.* To place a given set of 3D objects $O_i$, $i \in I_n$, (Fig. 1) in a container, taking into account the position constraints of objects, so that the metric characteristics of the container reach the optimal value.

In this paper, the mathematical models of real 3D objects are connected bounded 3D φ-objects (non-empty canonically closed point sets $O \subset R^3$, whose homotopy type of the interior and closures coincide). The whole set of 3D objects that are considered in the paper can be divided into two main groups. The first group consists of those convex 3D objects whose surfaces are formed by cylindrical, conical, and spherical surfaces, as well as their equidistant surfaces (Fig. 1, a). The second group consists of those arbitrary 3D objects that can be approximated by polyhedral objects (Fig. 1, b).



a b

***Fig. 1. Sets of 3D objects:***
a – convex objects whose surfaces are formed by cylindrical, conical, spherical surfaces and planes; b – polyhedral objects

In this study, all the 3D objects $O$ in the first group can be defined by means of a spherical cone $SK_i$, which can be depicted as a convex 3D object $O_i = SC_i = S_{1i} \cup C_i \cup S_{2i}$, where $C_i$ is the truncated cone with the height $2h_i$ and the radii of the upper and lower bases $r_{1i}$ and $r_{2i}$, respectively; $S_{1i}$ is the upper spherosegment with the height $w_{1i}$ and the base radius $r_{1i}$, $S_{2i}$ is the lower spherosegment with the height

$w_{2i}$ and the base radius $r_{2i}$. We denote by $\omega_i = (h_i, r_{1i}, r_{2i}, w_{1i}, w_{2i})$ the vector of the metric characteristics of a spherical cone. By changing values of this vector, we can obtain the following 3D objects: an ordinary cone ($\omega_i = (h_i, r_i, 0,0,0)$), a truncated cone ($\omega_i = (h_i, r_{i1}, r_{i2}, 0,0)$), a circular cylinder ($\omega_i = (h_i, r_i, r_i, 0,0)$), a spherocylinder ($\omega_i = (h_i, r_i, r_i, w_{i1}, w_{i2})$); a spherosegment ($\omega_i = (0, r_i, 0, w_i, 0)$) or ($\omega_i = (0,0, r_i, 0, w_i)$); a spherical disk ($\omega_i = (0, r_i, r_i, w_{i1}, w_{i2})$), and a sphere ($\omega_i = (0, r_i, r_i, r_i, r_i)$).

The 3D objects considered in the paper presuppose congruent and homothetic transformations. Thus, the 3D object $O$ the can be matched with the vector of variables $u_o = (v_o, \theta_o, \lambda_o) \in R^7$, where $v_o = (x_o, y_o, z_o)$ is the translation vector, $\theta_o = (\alpha_o, \beta_o, \gamma_o)$ is the vector of rotation angles, and $\lambda_o$ is the homothetic coefficient. We denote by $O(u)$ the object $O$, which is given in its own coordinate system, translated into the vector $v_o = (x_o, y_o, z_o)$, rotated by angles $\theta_o = (\alpha_o, \beta_o, \gamma_o)$, and subjected to homothetic transformation with coefficient $\lambda_o$, and define it as follows: $O(u) = \{p : p = v_o + \lambda_o \cdot M(\theta_o) \cdot \tilde{p}, \forall \tilde{p} \in O(0,0,0,1)\}$, where $O(0,0,0,1)$ denotes the original object $O$ and $\tilde{p}$ is an arbitrary point of the object $O$ in its own coordinate system.

The placement of 3D objects can be constrained by: object orientation (oriented (of a given non-fixed orientation) and non-oriented (orthogonal orientation change, arbitrary orientation change)); minimum feasible distances; and prohibited zones.

The container $\Omega$, where 3D objects are to be packed, can take the following spatial forms (see Fig. 2): a rectangular parallelepiped, or a sphere, or a straight prism with prohibited zones in the form of cylinders, or cylinder with prohibited zones in the form of straight rectangular prisms.

The objective function can be formulated as follows: to minimize container height, minimize container volume, and maximize the quantity of the objects to be packed into a given container.

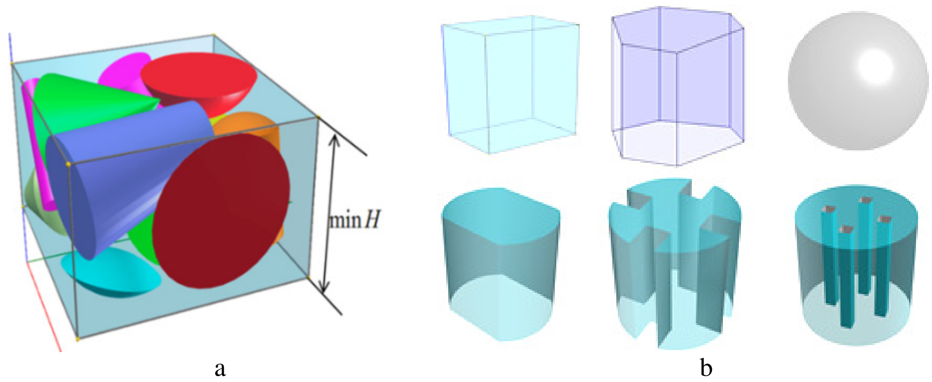



***Fig. 2. Packing of 3D objects:***
a – problem statement; b – examples of spatial forms of the container $\Omega$

**Mathematical Model of the Problem**

Based on the method of phi-functions [9, 17, 18], the mathematical model of the general optimization packing problem of 3D objects can be presented in the following form:

$$F(X^*) = \underset{X \in W}{extr}\, F(X), \tag{1}$$

$$W = \{X \in R^{7n+3n_{qp}+n_\Omega} : \Psi_1(X) \geq 0, \Psi_2(X) \geq 0, \Psi_3(X) \geq 0, \Psi_4(X) \geq 0\}, \tag{2}$$

where $F(X)$ is a continuous twice-differentiable function ; $n$ is the number of 3D objects; $n_{qp} = 0.5(1 - n_q)/n_q$, $n_q$ is the number of 3D objects for which the model uses quasi-phi-functions; $n_\Omega$ is the number of variable metric characteristics of the container $\Omega$; $X = (u_\Omega, u, u_p)$ is the vector of problem variables; $u_\Omega$ is the vector of metric characteristics of the container $\Omega$; $u = (u_1, u_2, ..., u_n)$ is the vector that determines the placement parameters for 3D objects $u_i = (v_i, \theta_i, g_i)$ is the vector that determines the placement parameters for the 3D object $O_i(u_i)$; $i \in I_n$, $v_i = (x_i, y_i, z_i)$ is the translation vector of the 3D object; $\theta_i = (\alpha_i, \beta_i, \gamma_i)$ is the vector of rotation angles for the object $O_i(u_i)$, $i \in I_n$, around the coordinate axes $Ox$, $Oy$, $Oz$, respectively; $g_i$ is the homothetic coefficient of the 3D object $O_i(u_i)$; $u_p = (u_{p_{12}}, u_{p_{13}}, ..., u_{p_{ij}}, ..., u_{p_{qq}})$ is the vector of additional variables that determine the parameters of separating planes (if quasi-phi-functions are used) for each pair of the objects $O_i(u_i)$ and $O_j(u_j)$ $(i, j) \in I_n$;

$$\Psi_1(X) = \min\{\Phi^{O_i\Omega^*}(X), i \in I_n\}, \quad \Psi_2(X) = \min\{\Phi^{O_iO_j}(X), (i, j) \in I_n\},$$

$$\Psi_3(X) = \min\{\Phi^{O_iT_k}(X), i \in I_n, k \in I_z\},$$

$\Phi^{O_i\Omega^*}(X)$ is the phi-function for the objects $O_i$ and $\Omega^* = cl(R^3 \setminus \Omega)$ (describes object placement conditions into the container); $\Phi^{O_iO_j}(X)$ is the phi-function (or quasi-phi-function for the objects $O_i(u_i)$ and $O_j(u_j)$ (describes the conditions of non-intersection or feasible distance for the objects $O_i(u_i)$ and $O_j(u_j)$); $\Phi^{O_iT_j}(X)$ is the phi-function for the object $O_i(u_i)$ and the prohibited zone $T_k$; $\Psi_4(X) \geq 0$ is a system of additional constraints (for example, constraints on the metric characteristics of the placement region or 3D objects to be packed.

For the developed mathematical model of the general optimization packing problem of 3D objects, it is necessary to point out some important features that influenced the development of a general methodology for solving problems. Such features include the following.

1. Model (1)–(2) is the exact mathematical model of the general optimization packing problem of 3D objects. The model is formulated in the form of a mathematical programming problem, and specifies all its global solutions.

2. The feasible region *W* of the problem (1)–(2), in the general case, is a disconnected set, and each of its connected components is a multiply connected ravine set.

3. The inequality $\Psi_1(X) \geq 0$ is a system of continuously differentiable functions.

4. The function $\Psi_2(X)$, depending on the implementation of problem (1)–(2), can be specified by either phi-functions or quasi-phi-functions. In the case of using phi-functions (which are maximin functions). the inequality $\Psi_2(X) \geq 0$ can be represented by a set of inequality systems of continuously differentiable functions.

5. The feasible region is described by a system of inequalities of the functions that include the max and min operators, which is why it can be represented as a union of subregions $W = \bigcup_{q=1}^{\varsigma} W_q$, where each of the subregions $W_q$ is determined by a system of inequalities with continuously differentiable functions. Thus, problem (1)–(2) can be represented as $F(X^*) = extr\{F(X^{*q}), q = 1,2,...,\varsigma\}$, where $F(X^{*q}) = \underset{X \in W_q}{extr}\, F(X)$.

6. In the case, where the feasible region of problem (1)–(2) is given only quasi-phi-fuinctions, it is described by a system of inequalities with continuously differentiable functions.

7. Problem (1)–(2) belongs to the class of NP-hard problems.

Due to the fact that the mathematical model (1)–(2) of the general optimization packing problem of 3D objects is constructed as a mathematical programming problem, the paper develops a common methodology for solving packing problems, with modern methods of nonlinear optimization being used at all stages.

**A Methodology for Solving the Problems of Finding the Optimal Placement of 3D Objects**

The basic idea of the developed methodology is shown schematically in Fig. 3.

As the diagram shows, the proposed methodology is based on the analysis of the input information about the problem to be solved. It uses several approaches whose fundamental difference is the ability to change the orientation of 3D objects in searching for a solution to the problem, since the arbitrary rotations of the objects to be packed make this process much more difficult, and require other methods. Because of this, the methodology uses two main approaches to solving problems: that of packing convex 3D objects and that of packing non-oriented 3D objects.

The ability to arbitrarily change the orientation of objects requires the use of other methods of finding initial placements, and, therefore, use of a different approach. In the case of non-oriented objects, the approach to solving the problem uses two different strategies to find an approximated global optimal solution, the strategies being selected depending on the object shapes. If the objects are convex, a strategy based on homothetical transformations and search for promising starting points is applied. Since the complexity of the problem increases significantly in packing non-convex 3D objects, to solve the problem, a multi-stage multi-start strategy is used, which initially uses the strategy of packing non-oriented convex objects.

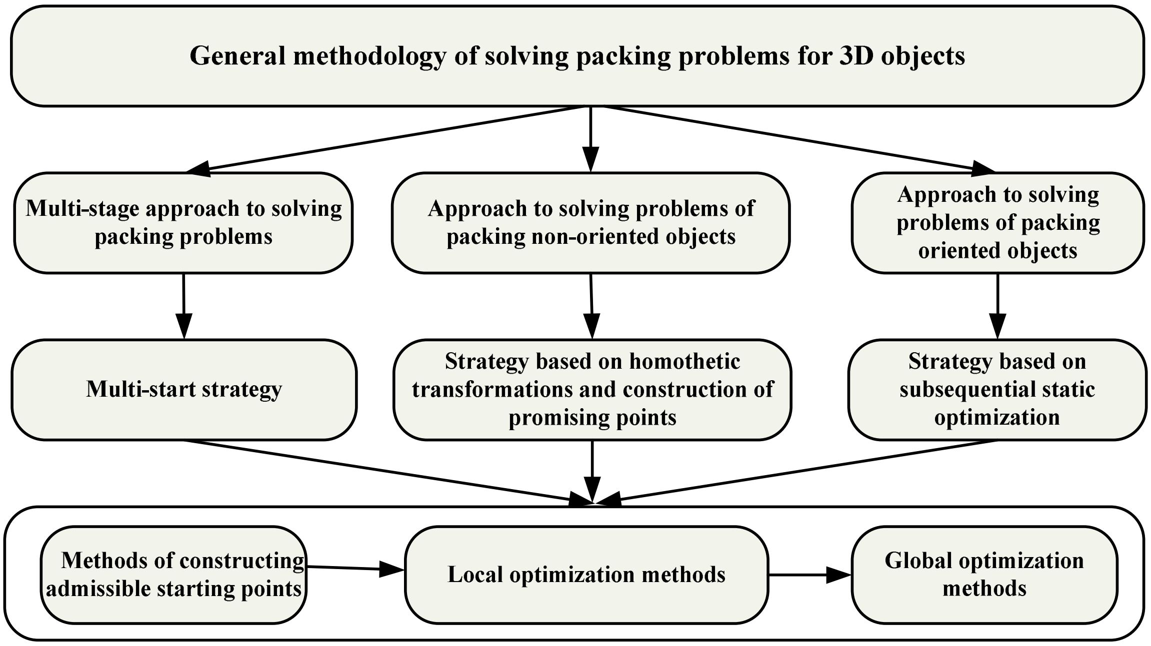



***Fig. 3. Methodology for solving the problems of packing 3D objects***

Each of the proposed strategies is based on the following sequence of stages:

1) construction of feasible starting points from the feasible region;

2) local optimization;

3) global optimization.

For each of the strategies proposed, there has been developed a set of methods, which takes into account the peculiarities of the problems to be solved.

Let us consider each of these strategies in more detail.

*A strategy based on successive statistical optimization*. The main idea of this strategy is to optimize the objective function given on a set of permutations. To construct feasible starting points from the feasible region, we apply methods that use the sequence of placing 3D objects (optimization method by groups of variables) or the sequence of coordinates of their centers (the regular placement method that is focused on the placement of congruent objects). To find local extrema, the modified method of feasible directions was used together with an active set strategy for subregions.

One way to solve multi-extreme problems is to search through local extrema. However, even for a relatively small number of objects, it is impossible to directly search for local extrema. Due to the fact that there is a possiblity for the above problems to establish a correspondence between the permutations of 3D objects and local extrema, a strategy is applied to find an approximation to the global extremum, using a modified decremental neighbourhoods method). This method is a direct random search which focuses on optimizing the functions that are given on a set of permurations.

The decremental neighbourhoods method is based on the properties of the probabilistic distribution of local extrema of the objective function. It allows us to search for sequences of the objects to be placed, and obtain an almost global extremum solution to an approximation to the global extremum solution of the problem in a relatively short time. To implement it, we need to introduce a certain metric in the permutation space. The search for the best values of the objective function is performed in the neighborhoods given on a set of permutations. At each step of the method, the centers and radii of new neighborhoods are selected based on the statistical information accumulated in the process. If the value of the objective function does not improve during the next search stage, then the radii of neighbourgoods decrease.

The implementation of this strategy is considered in [19].

*A strategy based on homothetical transformations and the construction of promising points.* The methods of this strategy use expansion of problem dimention by introducing variable metric characteristics of both objects and homothetic transformations of those objects. The strategy is based on the following sequence of methods:

1) to construct starting points, a homothetic transformation method is used;

2) to find local extrema, both an interior point method and a decomposition strategy are used;

3) to find an approximations to the global extremum, a method of constructing promising placements.

Since mathematical model (1)–(2) is constructed as a classical nonlinear programming problem, various modifications of nonlinear optimization methods can be applied to solve it. However, to apply numerical nonlinear optimization methods, we must have a feasible starting point.

Among the methods used to construct promising starting points, generally used in object placement problems are various modifications of greedy algorithms. However, since the problems of packing 3D objects are NP-hard, the use of greedy algorithms significantly limits the possibility of searching through a huge number of local extrema (whose number exceeds $n!$). In addition, the computational cost of constructing starting points increases significantly if objects allow arbitrary rotations.

Using the phi-function method to construct mathematical model (1)–(2) allows us to apply modern methods of nonlinear optimization at all stages of solving the problem. In this regard, a special approach is proposed to construct feasible starting points, with its main idea being to expand problem dimension by introducing variable metric characteristics of objects and homothetic transformations of those objects. Suppose that objects allow homothetic transformations. For this purpose, we accept the homothetic coefficients to be variable. Then, in order to determine the starting point, a random generation of the coordinates of the objects to be placed in the container is performed. After that, the problem of nonlinear programming is solved, whose purpose is to maximize the sum of homothetic coefficients of all objects. If the solution to this problem results in finding the local maximum point in which all the homothetic coefficients are equal to one, then such a point is taken as the starting point for finding the local extremum of the main problem. It should be noted that, in contrast to the greedy algorithms for constructing starting points, which can yield good, but identical points, the developed method allows us to obtain various starting points by using a method of random generation of object center coordinates.

Since the feasible region is determined by a very large number of inequalities, the direct application of nonlinear optimization methods to find a local extremum will result in considerable computational costs. Therefore, for finding the local extrema of formulated optimization problems, a special decomposition method was developed, which allows us to reduce computational costs by significantly reducing the number of inequalities in the process of seeking local extrema. Since the feasible region is represented as a union of subregions, we can significantly reduce the time for finding the local minimum by reducing it to solving a sequence of subproblems where the feasible region is determined by a much smaller number of inequalities. The key idea of the method allows us to select, at each step, a feasible subregion and generate subsets of the chosen subregion at each step in this way. Based on the starting point analysis, a system of additional constraints on the placement parameters of each object is added into the problem constraint system, which allows movement within an individual container. Then the inequalities for all the pairs of the objects whose individual containers do not intersect are removed. Thus, we reduce both the number of constraints and, in the case of quasi-phi-functions, the number of additional variables. Then, a search for the local minimum point for the constructed subproblem is performed. The resulting local subproblem extremum is used as the starting point for the next iteration. A detailed implementation of the proposed approach is given in [15, 16].

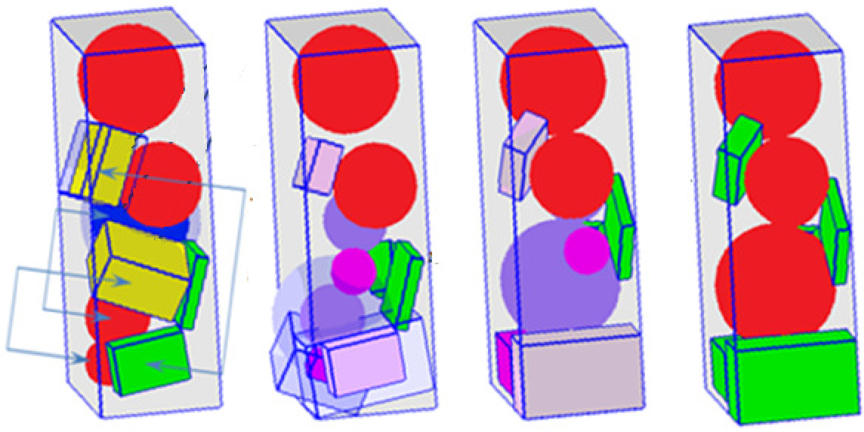
Global optimization for this strategy is based on the idea of searching through local minimums by constructing new promising starting points with using homothetical transformations of objects at the local minimum point obtained. To do this, at the local minimum point, a nonlinear programming problem is solved. As a result of solving such a problem, we get the point where we can identify 2 groups of objects:

1) objects that are surrounded by empty space, and therefore, can be replaced with larger ones;

2) objects around which the space is tightly filled, which makes it impossible to permute their positions with the purpose of reducing the volume of the container.

To determine such appropriate groups of objects, a special auxiliary nonlinear optimization problem is solved, which aims to reduce the volume of the container, provided that the objects placed therein allow homothetical transformations. A peculiarity of the auxiliary problem is the absence of constraints on the maximum value of the homothetic coefficients of objects. As a result, the volume of the container is reduced due to the fact that some objects will be reduced and some, enlarged in sizes. This change in object sizes allows us to define the 2 groups of objects described above. Since with the reduction in the volume of the container, some objects became smaller than it was specified, the next step will be to solve an auxiliary problem that will increase the sizes of objects to their specified values. To solve this auxiliary problem, iterative attempts are made to construct a number of promising starting points. To construct such points, we try, in the sequence given, to permutate the objects from the first group and the objects from the second group. Such a permutation allows us to get into the subregion that is located in the area of gravity of another local minimum. By permutating objects, we reduce their sizes so that they do not overlap with adjacent objects. If it is possible to enlarge the objects to their original sizes, then the point corresponding to this permutation of objects is taken as the promising starting point for finding a new local minimum of the main problem.

The main stages of constructing a promising starting point by the example of the problem of packing non-oriented parallelepipeds and spheres are presented in Fig. 4.



a b c d

***Fig. 4. The main stages of constructing a promising starting point:***
a – the result of solving the auxiliary problem of homothetic transformations of the packed objects in order to reduce the volume of the container and identify two groups of permutable objects;
b – the constructed promising starting point for finding the extremum of the following auxiliary problem;
c – the result of solving the auxiliary problem of maximizing the sum of the homothetic coefficients of the packed objects;
d – the obtained point can be taken as a promising starting point for finding a new local minimum of the main problem

If we cannot find the global extremum of the auxiliary problem of maximizing the sum of the homolthetic coefficients of the packed objects from the series of constructed promising starting points, then the last found local minimum is taken as an approximation to the solution of the problem.

The effectiveness of the proposed strategy is achieved through the implementation of successive changes in the dimension of the solution space during the transition between auxiliary problems. The objective function gradually improves due to the fact that the local extremum point of one auxiliary problem is not the local extremum point of another auxiliary problem.

The details of the implementation of this strategy are presented in [15–17].

*A multi-stage multi-start strategy.* This strategy was used to solve the problem of packing non-convex non-oriented 3D objects. The strategy is focused on finding the optimal placement of non-convex non-oriented objects, which significantly complicates the solution process. Therefore, to reduce large computational and time costs, we decompose the process of solving the problem into several major stages (preparatory and multi-start) and their substages.

Since the strategy is focused on the placement of non-convex non-oriented objects, a clustering method is proposed to construct feasible promising starting points. Local optimization was performed using an interior point method together with a decomposition strategy. To search through local extrema, a multi-start generation of feasible promising initial starting points was used.

During the preparatory phase, a number of nonlinear programming problems are solved that allow data to be obtained to construct the starting points of the main packing problem.

At the multi-start stage, both different starting points and corresponding local minimums are constructed. It should be noted that, depending on the shape of clusters, used are either the optimal packing strategy, or strategy of packing parallelepipeds that allow orthogonal rotations, or spheres, or non-oriented parallelepipeds and spheres. To solve these problems, we use the strategy based either on homothetical transformations and or the construction of starting points.

Thanks to the clustering method for non-convex non-oriented 3D objects, the construction of starting points reduces to solving the problem of packing half of the convex objects of a much simpler spatial shape (parallelepipeds and spheres). This greatly reduces time to construct starting points.

It should be noted that the reduction of computational costs is also facilitated by the fact that the process of finding the local extremum of the problem is divided into two stages: that of solving the linear problem by fixing the angles of rotation and that of solving the nonlinear problem. In addition, since a strategy for finding an approximation to the global extremum is used to place the formed cluster set, the constructed starting point is an approximation to the local extremum of the main problem.

As an approximation to the global minimum of the problem, the best local minimum is selected, obtained as a result of performing the multi-start phase.

Details of the implementation of this strategy are presented in [19].

*Methods for constructing feasible starting points.* To apply local optimization methods, we need to construct the starting points that belong to the feasible region. One of the requirements for methods for the construction of starting points for 3D object packing problems is to generate a variety of points (which will ensure finding different local extrema) and reduce computational costs in order to construct them quickly.

Developed in this work are the following methods: the regular placement method for packing congruent 3D objects; the homothetic transformation method for convex objects whose surfaces are formed by conical, cylindrical, and spherical surfaces [16] (Fig. 5); and the clustering method for convex polyhedral objects [19] (Fig. 6).

*Local optimization methods.* The analysis of the peculiarities of the mathematical models of packing problems revealed that the feasible region is described by a large number of nonlinear inequalities. This fact requires the development of methods that will make it possible to effectively solve the problem of high-dimentional problems.

The main idea behind the proposed local optimization methods is based on the decomposition of the main problem into subproblems with significantly fewer constraints and of smaller dimensions. To do this, the following stages are performed: feasible subregions with starting point are sequentially generated; the subsystem of $\varepsilon$-active constraints is determined; local extrema in the selected subregions are sought with the help of modern second-order NLP solvers; transition to other subregions is organized. A detailed implementation of the developed methods is given in [16].

Fig. 7 depicts local minima for different problems.

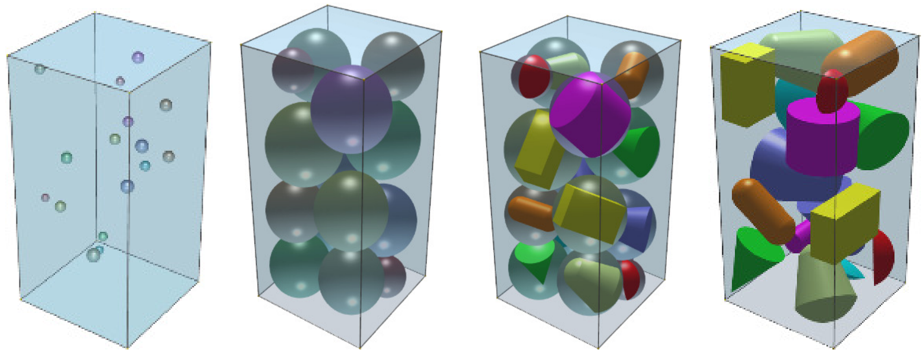


***Fig. 5. An example of constructing a feasible starting point by the homothetic transformation method***

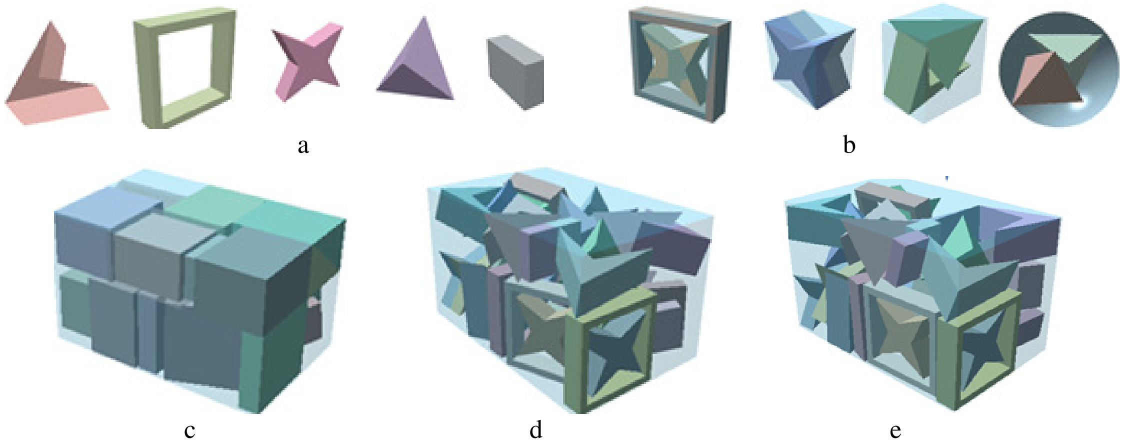


***Fig. 6. Starting point construction by the clustering method:***
a – given forms of polyhedral objects;
b – selected forms of clusters according to the criterion of maximum packing factor;
c – result of packing the cluster subset formed;
d – feasible starting point corresponding to the placement of clusters;
e – local minimum

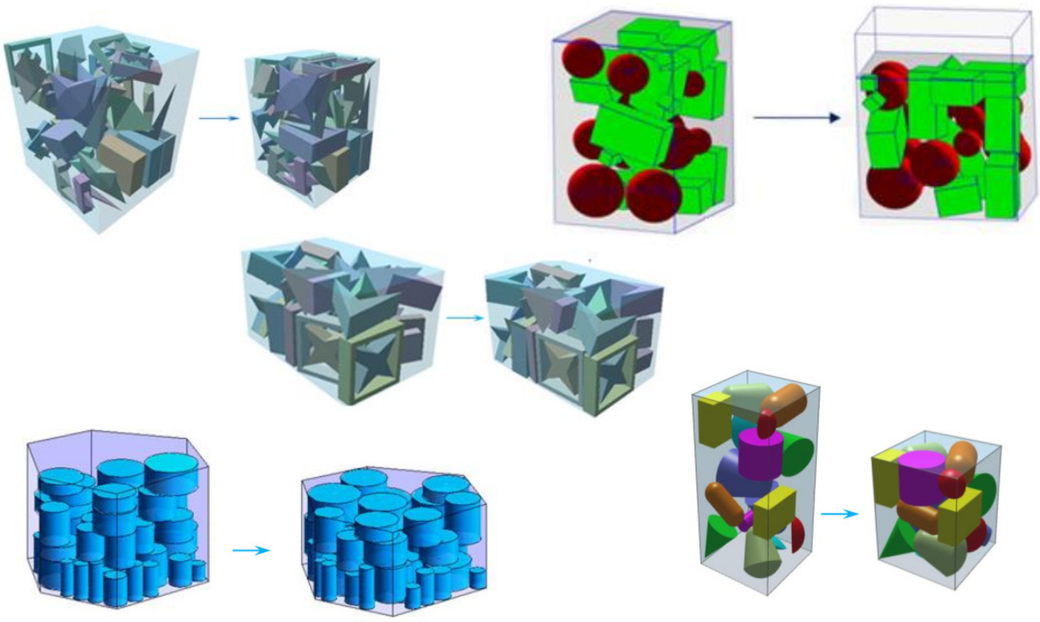


***Fig. 7. Examples of packings corresponding to the points of local extrema of different problems***

**Conclusions**

We propose a common methodology for solving 3D object placement problems. The methodology is a development of the theory of geometric design, and can be used by experts in the field to choose a strategy for solving placement problems.

The methodology focuses on advanced developments in geometric design and the use of powerful software packages to solve both linear and nonlinear programming problems.

The effectiveness of the proposed tools is confirmed by a number of computational experiments, during which the results obtained were compared with those of foreign researchers, and the results obtained were improved both by the values of the objective function and resolution time.

The results obtained are the theoretical and practical basis for engineering calculations during the automation and modeling of the processes of placing objects of different physical nature.

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## Методологія розв’язання задач пошуку оптимального розміщення тривимірних тіл

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*Робота присвячена розв’язанню оптимізаційних задач упаковки тривимірних тіл шляхом побудови точних математичних моделей та розробки підходів, що базуються на застосуванні оптимізаційних методів нелінійного програмування та сучасних розв’язувачів. Розроблено конструктивні засоби математичного та комп'ютерного моделювання відношень орієнтованих та неорієнтованих тривимірних тіл, поверхня яких утворена циліндричними, конічними, сферичними поверхнями та площинами, у вигляді нових класів Ф-функцій та квазі-Ф-функцій. На базі розроблених засобів математичного моделювання побудовано і досліджено базову математичну модель задачі оптимальної упаковки тривимірних тіл, поверхні яких утворені циліндричними, конічними, сферичними поверхнями і площинами, та різні її реалізації, що охоплюють широкий клас наукових і прикладних задач упаковки тривимірних тіл. Розроблено загальну методологію розв’язання задач упаковки тривимірних тіл, що допускають одночасно неперервні повороти та трансляції. Запропоновано стратегії, методи і алгоритми розв’язання оптимізаційних задач упаковки тривимірних тіл з урахуванням технологічних обмежень (мінімально допустимі відстані, зони заборони, можливість неперервних трансляцій та обертань). Виходячи з запропонованих засобів математичного моделювання, математичних моделей, методів і алгоритмів створено програмне забезпечення з використанням технології паралельних обчислень для автоматичного розв’язання оптимізаційних задач упаковки тривимірних тіл. Отримані результати можуть бути застосовані під час розв’язання задач оптимізації компоновочних розв’язків, для комп’ютерного моделювання в матеріалознавстві, у порошковій металургії та нанотехнологіях, під час оптимізації процесу 3D-друку для SLS технології адитивного виробництва, у інформаційно-логістичних системах, що забезпечують оптимізацію перевезення та зберігання вантажів.*

***Ключові слова:*** *упаковка, тривимірні тіла, геометричне проектування, Ф-функції, математичне моделювання, неперервні обертання, нелінійна оптимізація.*

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