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ANALYTICAL SOLUTION OF THE PROBLEM OF SYMMETRIC THERMALLY STRESSED STATE OF THICK PLATES BASED ON THE 3D ELASTICITY THEORY

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An important place among thermoelasticity problems is occupied by the plane elasticity problem obtained from the general three-dimensional problem after using plane stress state hypotheses for thin plates. In the two-dimensional formulation, this problem has become widespread in the study of the effect of temperature loads on the stress state of thin thermosensitive plates. The article proposes a general three-dimensional solution of the static problem of thermoelasticity in a form convenient for practical application. To construct it, a particular solution of the inhomogeneous equation, the thermoelastic displacement potential, was added by us to the general solution of Lamé's equations, the latter solution having been previously found by us in terms of three harmonic functions. It is shown that the use of the proposed solution allows one to satisfy the relation between the static three-dimensional theory of thermoelasticity and boundary conditions, and also to construct a closed system of partial differential equations for the introduced two-dimensional functions without using hypotheses about the plane stress state of a plate. The thermoelastic stress state of a thick or thin plate is divided into two parts. The first part takes into account the thermal effects caused by external heating and internal heat sources, while the second one is determined by a symmetrical force load. The thermoelastic stresses are expressed in terms of deformations and known temperature. A three-dimensional thermoelastic stress-strain state representation is used and the zero boundary conditions on the outer flat surfaces of the plate are precisely satisfied. This allows us to show that the introduced two-dimensional functions will be harmonic. After integrating along the thickness of the plate along the normal to the median surface, normal and shear efforts are expressed in terms of three unknown two-dimensional functions. The three-dimensional stress state of a symmetrically loaded thermosensitive plate was simplified to the two-dimensional state. For this purpose, we used only the hypothesis that the normal stresses perpendicular to the median surface are insignificant in comparison with the longitudinal and transverse ones. Displacements and stresses in the plate are expressed in terms of two two-dimensional harmonic functions and a particular solution, which is determined by a given temperature on the surfaces of the plate. The introduced harmonic functions are determined from the boundary conditions on the side surface of the thick plate. The proposed technique allows the solution of three-dimensional boundary value problems for thick thermosensitive plates to be reduced to a two-dimensional case.

Keywords: *thick thermoelastic plate, thermosensitive material, stress state.*

Introduction

Thin and thick plates, to which power loads are applied and for which temperature fields and internal heat sources are set, are widely used in power engineering, technological and engineering structures [1–3]. In [2, 4], a review of the literature on the equations of the plane theory of thermoelasticity is made using the hypotheses of plane stresses for thin plates.

The aim of the paper is to develop an effective mathematical approach for the reduction of three-dimensional thermoelastic equilibrium of plates under the influence of force and temperature loads to the study of two-dimensional equations of simple form without imposing additional restrictions on stress components and thermoelastic characteristics of the material.

Problem Formulation and Solution Presentation

Consider a three-dimensional static thermoelastic problem for a thick plate of thickness h, whose median surface occupies an area S with a contour L, lying on the plane Oxy of the Cartesian coordinate system: $x_1 = x,\ x_2 = y,\ x_3 = z$. Assume that on both flat surfaces of the plate ($z = h_j$, $h_1 = h/2$, $h_2 = -h/2$) there are no normal and tangential loads, and only given are the temperatures $T^- = T^+$ where the signs "+", "-" respectively describe the functions on the upper $z = h_1$ or lower $z = -h_1$ surfaces. Assume that the function of the temperature $T(x, y, z)$ is known, and on the side surface of the plate, corresponding to the contour *L*, stress boundary conditions are set. Consider a symmetrical compression-tension along the median surface of the thick heat-sensitive plate

$$u_i(x, y, -z) = u_i(x, y, z),\ i = \overline{1,2},\ \ u_3(x, y, -z) = -u_3(x, y, z), \tag{1}$$

where u_i are the displacements in the direction of the respective axes. From conditions (1) it follows that the normal stresses will be symmetric: $\sigma_j(x, y, -z) = \sigma_j(x, y, z),\ j = \overline{1,3}$. Express the thermoelastic stresses for problem (1) in terms of deformations [1, 3]

$$\sigma_k = 2G\left[\varepsilon_k + \frac{\nu}{1-2\nu}e - \frac{1+\nu}{1-2\nu}\alpha T\right],\quad \tau_{kj} = G\gamma_{kj}, k \neq j, \tag{2}$$

where $e = \frac{1-2\nu}{E}\Theta + 3\alpha T$ is the volume deformation, $\Theta = \sigma_1 + \sigma_2 + \sigma_3$.

Substitute relations (2) into the equilibrium equation and write the equation of stationary thermoelasticity for the displacements [1, 3]

$$(1-2\nu)\nabla^2 u_k + \frac{\partial e}{\partial x_k} = 2(1+\nu)\alpha\frac{\partial T}{\partial x_k},\quad k = \overline{1,3}, \tag{3}$$

$$\nabla^2 T = 0,\ \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}, \tag{4}$$

where ν is Poisson's ratio, α is the coefficient of thermal expansion, and the known temperature $T(x, y, z)$ satisfies equation (4).

Represent the particular solution of equation (3), which is called the thermoelastic displacement potential [1, 3], as follows:

$$u_1 = \frac{\partial\psi}{\partial x_1},\quad u_2 = \frac{\partial\psi}{\partial x_2},\quad u_3 = \frac{\partial\psi}{\partial x_3}, \tag{5}$$

where the function ψ satisfies the equation

$$\nabla^2\psi = \frac{1+\nu}{1-\nu}\alpha T. \tag{6}$$

The general solution of equation (6) whose particular solution takes into account the influence of the known temperature will have the form

$$\psi = \beta z\Omega + \Psi, \tag{7}$$

where $\beta = \frac{1+\nu}{2(1-\nu)}\alpha$; Ω, Ψ are three-dimensional harmonic functions, $\Omega = \int_0^z T dz$, $\frac{\partial\Omega}{\partial z} = T$.

Add relations (5), (7) to the solution of Lame's equations [5, 6] and obtain the general solution of equations (3) in the form

$$u_x = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y},\quad u_y = \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x},\quad u_z = \frac{\partial P}{\partial z} - 4(1-\nu)\Phi, \tag{8}$$

where $P = z(\Phi + \beta\Omega) + \Psi$; Φ, Ψ, *Q* are three-dimensional harmonic functions of displacements, Ω, *T* are known harmonic functions. The biharmonic function *P* satisfies the equation

$$\Delta P + \frac{\partial^2}{\partial z^2}P = 2\frac{\partial}{\partial z}(\Phi + \beta\Omega), \tag{9}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is a two-dimensional Laplace operator.

From conditions (2), (8) it follows that the functions P, Ψ, Q, T will be even with respect to the variable z, and the function Φ will be odd. Write the conditions of symmetry

$$\frac{\partial P^+}{\partial z} = -\frac{\partial P^-}{\partial z}, \quad \frac{\partial \Phi^+}{\partial z} = \frac{\partial \Phi^-}{\partial z}, \quad \frac{\partial Q^+}{\partial z} = -\frac{\partial Q^-}{\partial z}, \quad \Phi^- = -\Phi^+. \tag{10}$$

Take into account the representation of displacements (8) and find the deformations, and, according to formula (2), determine the general expression of normal

$$\sigma_j = 2G\left[\frac{\partial^2 P}{\partial x_j^2} - (-1)^j \frac{\partial^2 Q}{\partial x_1 \partial x_2} - 2\nu \frac{\partial \Phi}{\partial x_3} - 2\beta T\right], \quad j = \overline{1,2},$$

$$\sigma_3 = 2G\left[\frac{\partial^2 P}{\partial x_3^2} - 2(2-\nu)\frac{\partial \Phi}{\partial x_3} - 2\beta T\right], \tag{11}$$

and tangential

$$\tau_{12} = G\left[2\frac{\partial^2 P}{\partial x_1 \partial x_2} + \frac{\partial^2 Q}{\partial x_2^2} - \frac{\partial^2 Q}{\partial x_1^2}\right],$$

$$\tau_{j3} = G\left[\frac{\partial}{\partial x_j}\left[2\frac{\partial P}{\partial x_3} - 4(1-\nu)\Phi\right] - (-1)^j \frac{\partial^2 Q}{\partial x_{3-j} \partial x_3}\right], \quad j = \overline{1,2} \tag{12}$$

stresses, where $G = E/2(1+\nu)$, E are shear and Young's moduli. Express the volume deformation and the sum of normal stresses Θ

$$e = -2(1-2\nu)\frac{\partial}{\partial z}\Phi + 2\beta T, \quad \Theta = -2E\left(\frac{\partial}{\partial z}\Phi + \frac{\alpha T}{1-\nu}\right). \tag{13}$$

On the surfaces of the plate, set the three-dimensional boundary conditions

$$\sigma_3(x, y, h_j) = 0, \quad \tau_{3k}(x, y, h_j) = 0, \quad j, k = \overline{1,2}. \tag{14}$$

Substitute equation (11) into relation (14), take into account relations (10), (13) and that the stresses $\sigma_3 << \sigma_1, \sigma_2$, and obtain

$$\frac{\partial P^+}{\partial z} = 2(2-\nu)\Phi^+ + 2\beta\Omega^+, \tag{15}$$

where $\Omega^+ = \int_0^{h_1} T dz$.

Use representation (11) and write conditions (14) about the absence of tangential loads on the surfaces of the plate

$$\frac{\partial}{\partial x_j}\left[\frac{\partial P^+}{\partial x_3} - 2(1-\nu)\Phi^+\right] - \frac{(-1)^j}{2}\frac{\partial^2 Q^+}{\partial x_{3-j}\partial x_3} = 0, \quad j = \overline{1,2}. \tag{16}$$

Take into account relation (15) and simplify equation (16)

$$4\frac{\partial}{\partial x_j}\left(\Phi^+ + \beta\Omega^+\right) = (-1)^j \frac{\partial^2 Q^+}{\partial x_{3-j}\partial x_3}, \quad j = \overline{1,2}. \tag{17}$$

From equations (17), the following conditions of harmonicity for the introduced functions follow:

$$\Delta\left(\Phi^+ + \beta\Omega^+\right) = 0, \quad \Delta\frac{\partial Q^+}{\partial z} = 0. \tag{18}$$

So, the functions $\Phi^+ + \beta\Omega^+$, $\frac{\partial Q^+}{\partial x_3}$ are harmonic, if we know Φ^+, then we know $\frac{\partial Q^+}{\partial x_3}$.

Use relations (10), (11), (15) and express the efforts in the plate. To do this, substitute the representation of stress (11), (12) into the known expressions of normal and tangential efforts [7] and obtain

$$T_1 = 2G\left[\frac{\partial^2 \tilde{P}}{\partial x^2} + \frac{\partial^2 \tilde{Q}}{\partial x \partial y} - 4\nu\Phi^+ - 4\beta\Omega^+\right],\ T_2 = 2G\left[\frac{\partial^2 \tilde{P}}{\partial y^2} - \frac{\partial^2 \tilde{Q}}{\partial x \partial y} - 4\nu\Phi^+ - 4\beta\Omega^+\right],$$

$$S_{12} = S_{21} = 2G\left[\frac{\partial^2 \tilde{P}}{\partial x \partial y} + \frac{1}{2}\left(\frac{\partial^2 \tilde{Q}}{\partial y^2} - \frac{\partial^2 \tilde{Q}}{\partial x^2}\right)\right], \tag{19}$$

where $\tilde{P} = \int_{-h_1}^{h_1} P dz$, $\tilde{Q} = \int_{-h_1}^{h_1} Q dz$, $\tilde{T} = \int_{-h_1}^{h_1} T dz = \int_{-h_1}^{h_1} \frac{\partial \Omega}{\partial z} dz = 2\Omega^+$. After integrating equation (9), taking into account both the harmonicity of functions (18) and condition (15), write the key equations of plate theory for the basic functions $\tilde{P}$, $\tilde{Q}$, Φ^+

$$\Delta\tilde{P} = -4(1-\nu)\Phi^+,\ \ \Delta\tilde{Q} = -2\frac{\partial}{\partial z}Q^+. \tag{20}$$

Equations (20) are consistent with expressions (19) and coincide with the key equations of the plane stress state [5, 7].

Note that if we substitute relationship (19) into the equation of equilibrium of the plate in the efforts [7], then we obtain the fourth-order partial derivative equation

$$\Delta\Delta\tilde{P} = 4\beta(1-\nu)\Delta\Omega^+. \tag{21}$$

Equation (21) also follows from the obtained relations (18), (20).

Representation of Thermoelastic Stresses in Terms of Harmonic Functions

From equation (18), define the representation of the function Φ^+

$$\Phi^+ = -h\frac{\partial^2 \varphi}{\partial y^2} - \beta\Omega^+, \tag{22}$$

where φ is the unknown harmonic function. Use both expression (22) and relationship (17) between the harmonic functions, and obtain the following simple dependence:

$$\frac{\partial Q^+}{\partial z} = 4h\frac{\partial^2 \varphi}{\partial x \partial y}. \tag{23}$$

Consider representations (22), (23) and write the general solution of equations (20)

$$\tilde{P} = 2(1-\nu)h\left[y\frac{\partial \varphi}{\partial y} + \beta\omega_1\right] + hg_1(x, y),\ \tilde{Q} = -4yh\frac{\partial \varphi}{\partial x} + hg_2(x, y), \tag{24}$$

where ω_1 is the particular solution of the equation $h\Delta\omega_1 = 2\Omega^+$, g_j are the harmonic functions that can be represented as

$$g_1 = (1+\nu)h\left[\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial x}\right],\quad g_2 = (1+\nu)h\left[\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial y}\right], \tag{25}$$

ϕ, ψ are harmonic functions.

Substitute functions (22), (24), (25) into relation (19), express the efforts in terms of the introduced functions and see that the function ϕ does not enter into the representation of efforts (20), so it can be ignored.

Therefore, functions (24) will take the form:

$$\tilde{P} = 2(1-\nu)h\left[y\frac{\partial \varphi}{\partial y} + \beta\omega_1\right] - (1+\nu)h\frac{\partial \psi}{\partial x},\ \tilde{Q} = -4hy\frac{\partial \varphi}{\partial x} + (1+\nu)h\frac{\partial \psi}{\partial y},\ \ \Phi^+ = -h\frac{\partial^2 \varphi}{\partial y^2} - \beta\frac{h}{2}\Delta\omega_1, \tag{26}$$

where the functions ω_1, Ω^+ describe the effect of temperature on the stress state of the plate, and the harmonic functions φ, ψ correspond to the plane stress state [5].

Express the efforts that depend only on temperature. To do this, substitute expressions (26) into relation (19). After the transformations, we obtain such simple formulas

$$T_1 = -Eh\alpha \frac{\partial^2 \omega_1}{\partial y^2},\ T_2 = -Eh\alpha \frac{\partial^2 \omega_1}{\partial x^2},\ S_{12} = S_{21} = Eh\alpha \frac{\partial^2 \omega_1}{\partial x \partial y}. \tag{27}$$

Use the representation of stresses in the plane problem [5], relation (27) and write the general representation of stresses in terms of the three basic functions ω_1, φ, ψ

$$\sigma_x = 2E\left\{ y\frac{\partial^3 \varphi}{\partial y^3} + 2\frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^3 \psi}{\partial y^2 \partial x} - \frac{\alpha}{2}\frac{\partial^2 \omega_1}{\partial y^2} \right\},$$

$$\sigma_y = 2E\left\{ y\frac{\partial^3 \varphi}{\partial x^2 \partial y} + \frac{\partial^3 \psi}{\partial x^3} - \frac{\alpha}{2}\frac{\partial^2 \omega_1}{\partial x^2} \right\}, \tag{28}$$

$$\tau_{xy} = -2E\left\{ y\frac{\partial^3 \varphi}{\partial x \partial y^2} + \frac{\partial^2 \varphi}{\partial x \partial y} + \frac{\partial^3 \psi}{\partial x^2 \partial y} - \frac{\alpha}{2}\frac{\partial^2 \omega_1}{\partial x \partial y} \right\}.$$

Determine the harmonic functions φ, ψ from the boundary conditions (14) given on the side surface of the plate, and reduce them to the conditions on the contour L of the region S. Use stresses (28) and write the boundary conditions according to [5, 7]

$$\{\sigma_y \sin^2\theta + \sigma_x \cos^2\theta + \tau_{xy} \sin 2\theta\}|_L = \sigma_n, \qquad [\frac{\sin 2\theta}{2}(\sigma_y - \sigma_x) + \tau_{xy}\cos 2\theta]|_L = \tau_n, \tag{29}$$

where θ is the angle between the normal to the contour L and the axis Ox, $\sigma_n = \frac{1}{h}\int_{-h_1}^{h_1} \sigma_n(x,y,z)dz\,|_L$, $\tau_n = \frac{1}{h}\int_{-h_1}^{h_1} \tau_n(x,y,z)dz\,|_L$. Find the displacements and deformations in the plate after averaging formulas (8).

The obtained stress expressions (28) and boundary conditions (29) make it possible to solve various boundary value problems for thick thermoelastic plates.

Conclusions

Based on the three-dimensional theory of elasticity, a two-dimensional theory of thin and thick thermoelastic plates loaded only on their lateral sides symmetrically and parallel to the median surface is constructed without using hypotheses about zero tangential and normal stresses inside the plate. It was established that the found stresses and displacements are exactly equal to the corresponding average values of the three-dimensional theory of thermoelasticity. It was also established that from the formulas obtained, there emerge representations of stresses of the plane problem of the theory of elasticity.

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Аналітичний розв'язок задачі симетричного термонапруженого стану товстих пластин на основі тривимірної теорії пружності

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Важливе місце серед задач термопружності займає плоска проблема теорії пружності, яка отримана із загальної тривимірної задачі, після використання гіпотез плоского напруженого стану для тонких пластин. У двовимірній постановці ця задача набула широкого поширення під час дослідження впливу температурних навантажень на напружений стан тонких термочутливих пластин. У статті запропоновано загальний тривимірний розв'язок статичної задачі термопружності у формі, зручній для практичного застосування. Для його побудови до раніше знайденого автором загального розв'язку рівнянь Ляме через три гармонічні функції добавлено частковий розв'язок неоднорідного рівняння – термопружний потенціал переміщень. Показано що використання запропонованого розв'язку дозволяє задовольнити співвідношення статичної тривимірної теорії термопружності і крайові умови та побудувати замкнуту систему рівнянь у частинних похідних на введені двовимірні функції без використання гіпотез про плоский напружений стан пластини. Термопружний напружений стан тонкої або товстої пластини розділений на дві частини: перша враховує тепловий вплив, викликаний зовнішнім нагріванням і внутрішніми джерелами тепла; друга визначається симетричними силовими навантаженнями. Термопружні напруження виражені через деформації і відому температуру. Використано подання тривимірного термопружного напружено-деформованого стану і точно задоволено нульові крайові умови на зовнішніх плоских поверхнях пластини. Це дозволило показати, що введені двовимірні функції будуть гармонічними. Після інтегрування по товщині пластини вздовж нормалі до серединної поверхні виражено нормальні і зсувні зусилля через три невідомі двовимірні функції. Тривимірний напружений стан симетрично навантаженої термочутливої пластини спрощено до двовимірного стану. При цьому зведенні використано тільки гіпотезу, що перпендикулярні серединній поверхні нормальні напруження є незначними в порівнянні із повздовжніми та поперечними напруженнями. Переміщення і напруження в пластині виражено через дві двовимірні гармонічні функції і частковий розв'язок, який визначається заданою температурою на поверхнях пластини. Введені гармонічні функції визначаються із крайових умов на бічній поверхні товстої пластини. Запропонована методика дає змогу розв’язок тривимірних крайових задач для товстих термочутливих пластин зводити до двовимірного випадку.

Ключові слова: *товста термопружна пластина, термочутливий матеріал, напружений стан.*

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