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SOLUTION OF THE INVERSE PROBLEM OF IDENTIFYING THE THERMAL CONDUCTIVITY TENSOR IN ANISOTROPIC MATERIALS

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On the basis of A. N. Tikhonov's regularization theory, a technique has been developed for solving inverse heat conduction problems of identifying the thermal conductivity tensor in a two-dimensional domain. Such problems are replaced by problems of identifying the principal heat conductivity coefficients and the orientation angle of the principal axes, with the principal coefficients being approximated by Schoenberg's cubic splines. As a result, the problem is reduced to determining the unknown coefficients in these approximations and the orientation angle of the principal axes. With known boundary and initial conditions, the temperature in the domain will depend only on these coefficients and the orientation angle. If one expresses it by the Taylor formula for two terms of series and substitutes it into the Tikhonov functional, then the determination of the increments of the coefficients and the increment of the orientation angle can be reduced to solving a system of linear equations with respect to these increments. By choosing a certain regularization parameter as well as some functions for the principal thermal conductivity coefficients and the orientation angle as an initial approximation, one can implement an iterative process for determining these coefficients. After obtaining the vectors of the coefficients and the angle of orientation as a result of the converging iterative process, it is possible to determine the root-mean-square discrepancy between the temperature obtained and the temperature measured as a result of the experiment. It remains to choose the regularization parameter in such a way that this discrepancy is within the root-mean-square discrepancy of the measurement error. When checking the efficiency of using the proposed method, a number of two-dimensional test problems for bodies with known thermal conductivity tensors were solved. The influence of random measurement errors on the error in the identification of the thermal conductivity tensor was analyzed.

Keywords: *internal inverse heat conduction problem, thermal conductivity tensor, A. N. Tikhonov's regularization method, stabilizing functional, regularization parameter, identification, approximation, Schoenberg's cubic splines.*

Introduction

At present, inverse problems, that is, such problems in which the causal characteristics of physical processes are determined from measurement results or other investigative manifestations, have confidently occupied their niche in the study of physical processes of various nature, including thermophysical ones. The solution of internal inverse heat conduction problems (IHCP) for the identification of the tensor of thermal conductivity in anisotropic materials is of particular importance at the stage of constructing mathematical models of thermal processes in most heat-shield materials used, for example, in modern aerospace technology (fiberglass, asboplastics, carbon plastics, most graphites, and graphite-containing materials). This is due to the fact that heat transfer in such materials is described by equations containing the thermal conductivity tensor.

In this article, such an IHCP is considered as the problem of identifying a temperature-dependent thermal conductivity tensor with a known heat capacity and boundary conditions at the boundary of the object under study. In [1–5], IHCPs are classified, and methods of their solution are considered. At the same time, in [2–5], the problems of identifying the thermal conductivity coefficient and heat capacity are called coefficient IHCPs, and we, following the classification given in [1], consider all the problems of identifying the thermophysical characteristics inside the object under study as belonging to the class of internal IHCPs by analogy with the external IHCPs of identifying heat fluxes and other thermal characteristics on the surface of the object. In [6], an IHCP solution in an anisotropic half-space is given based on the analytical solution obtained in [7].

The author of [8] uses the conjugate gradient method to determine the tensor of nonlinear components of thermal conductivity in a rectangular plate, and, for example, solves the inverse problem of recovering the tensor of thermal conductivity of a carbon composite material reinforced with unidirectional continuous fibers at an angle of 30° to one of the plate boundaries. Monograph [5] presents approaches to solving direct and inverse problems of heat conduction in anisotropic media. In particular, gradient descent methods are used to identify the nonlinear thermal conductivity tensor.

In this article, the technique, described in [9], for determining the isotropic thermal conductivity coefficient is used to identify the thermal conductivity tensor with respect to the increments of the required parameters, and the search for the minimum of this functional is reduced to solving a system of linear equations with respect to these increments.

Problem Formulation

An internal IHCP of identifying the thermal conductivity tensor for a two-dimensional domain is considered. The thermal process in an anisotropic body is described as follows:

$$C(T)\frac{\partial T}{\partial \tau}=\frac{\partial}{\partial x}\left(\lambda_{xx}(T)\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial x}\left(\lambda_{xy}(T)\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial y}\left(\lambda_{yx}(T)\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\lambda_{yy}(T)\frac{\partial T}{\partial y}\right),\ \ (x,y)\in D, \tag{1}$$

$$\left(R(T)+\alpha T\right)\Big|_{M\in\Gamma}=\alpha T_s, \tag{2}$$

$$T(M,\tau)\Big|_{\tau=0}=T_0, \tag{3}$$

where

$$R(T)=\left(\lambda_{xx}(T)\cdot\frac{\partial T}{\partial x}+\lambda_{xy}(T)\cdot\frac{\partial T}{\partial y}\right)\cdot\cos(\nu,x)+\left(\lambda_{yx}(T)\cdot\frac{\partial T}{\partial x}+\lambda_{yy}(T)\cdot\frac{\partial T}{\partial y}\right)\cdot\cos(\nu,y); \tag{4}$$

D is the space domain occupied by the body; $T=T(M,\tau)$ is the body temperature; M is a point of D; τ is time coordinate; $\lambda_{xx}(T)$, $\lambda_{xy}(T)$, $\lambda_{yx}(T)$ and $\lambda_{yy}(T)$ are the required components of the thermal conductivity tensor; $C(T)$ is the volumetric heat capacity of the body; α is the heat transfer coefficient of the surface Γ; T_s is the specified temperature of the medium; ν is the outer normal to the border of the body; T_0 is initial temperature.

To solve the IHCP in an anisotropic body, time-dependent experimental temperature values are set

$$T(x_k,y_k,\tau_l)=T_{lk}^{ex},\quad l=\overline{1,n_\tau},\quad k=\overline{1,m}, \tag{5}$$

where n_τ is the number of measurements along the time coordinate; m is the number of measurement points; (x_k, y_k) are the points of D in which the temperature T_{lk}^{ex} is measured. Measurement error is a random variable distributed according to the normal law with zero expectation and variance σ^2. The nonlinear components of the thermal conductivity tensor are expressed in terms of the principal coefficients $\lambda_\xi(T)$, $\lambda_\eta(T)$ and the angle of orientation φ of the principal axes $O\xi$ and $O\eta$, as follows [7]:

$$\lambda_{xx}(T)=\lambda_\xi(T)\cdot\cos^2\varphi+\lambda_\eta(T)\cdot\sin^2\varphi, \tag{6}$$

$$\lambda_{yy}(T)=\lambda_\eta(T)\cdot\cos^2\varphi+\lambda_\xi(T)\cdot\sin^2\varphi, \tag{7}$$

$$\lambda_{xy}(T)=\lambda_{yx}(T)=\left(\lambda_\xi(T)-\lambda_\eta(T)\right)\cdot\cos\varphi\cdot\sin\varphi. \tag{8}$$

An internal IHCP of identifying the thermal conductivity tensor (1)–(5) using relations (6)–(8) can be formalized in the form

$$A[\lambda_\xi(T),\lambda_\eta(T),\varphi]=T^{ex},$$

where $\lambda_\xi(T)$ and $\lambda_\eta(T)$ are the principal temperature-dependent coefficients; φ is the angle of orientation of the principal axes $O\xi$ and $O\eta$; T^{ex} is the temperature, which in most cases is known from the experiment (initial data); A is the operator connecting the required dependencies $\lambda_\xi(T)$, $\lambda_\eta(T)$ and φ with the initial data T^{ex}. In this formulation, to identify the thermal conductivity tensor, it is sufficient to determine the principal coefficients $\lambda_\xi(T)$, $\lambda_\eta(T)$, and the orientation angle φ of the principal axes $O\xi$ and $O\eta$.

Such a problem, like any other IHCP, due to causal relationship violation, is ill-posed according to Hadamard, which is the reason for the instability of the solution obtained. To solve such a problem, it is either reduced to a conditionally well-posed one, or left ill-posed, but one of the regularization methods is used [2–5]. Here we use Tikhonov's regularization method [4].

The methodology for solving the problem is discussed below.

Regularization Algorithm for Solving an Internal IHCP of Identifying the Thermal Conductivity Tensor

To solve a nonlinear internal IHCP of identifying the thermal conductivity tensor (1)–(8), Tikhonov's regularization principle is used, which is reduced to minimizing the following functional:

$$J=\int_0^{\tau_0}\int_D\left[T(x,y,\tau)-T^{ex}(x,y,\tau)\right]^2 dxdyd\tau+\beta\cdot\Omega\left[\lambda_\xi(T),\lambda_\eta(T),\varphi\right], \quad (9)$$

where $T(x, y, \tau)$ is the temperature obtained in the process of solving the IHCP; $T^{ex}(x, y, \tau)$ is the experimentally obtained temperature; τ_0 is the moment of the end of the thermal process analysis; β is the regularization parameter; $\Omega[\lambda_\xi(T), \lambda_\eta(T), \varphi]$ is the stabilizing functional.

If the required functions $\lambda_\xi(T)$ and $\lambda_\eta(T)$ are presented in the form

$$\lambda_\xi(T)=\sum_{k=1}^{n_\xi}\xi_k B_{\xi 3}^k(T), \quad (10)$$

$$\lambda_\eta(T)=\sum_{k=1}^{n_\eta}\eta_k B_{\eta 3}^k(T), \quad (11)$$

where $(\xi_1,\xi_2,...,\xi_{n_\xi})=\overrightarrow{\Phi_\xi}$ and $(\eta_1,\eta_2,...,\eta_{n_\eta})=\overrightarrow{\Phi_\eta}$ are the vectors of the required parameters, and $B_{\xi 3}^k(T)$ and $B_{\eta 3}^k(T)$ are Schoenberg's cubic splines, then the identification of the thermal conductivity tensor will be reduced to determining the unknown vectors $\overrightarrow{\Phi_\xi}$ and $\overrightarrow{\Phi_\eta}$, as well as the orientation angle φ of the principal axes $O\xi$ and $O\eta$.

Functional (9) is minimized by the iterative method [9–11]. Since the temperature $T(x, y, \tau)$ depends both on the vectors $\overrightarrow{\Phi_\xi}$, $\overrightarrow{\Phi_\eta}$ and the orientation angle φ of the principal axes $O\xi$ and $O\eta$, it can be written, at the $(p+1)$th iteration, using the Taylor series as follows:

$$T^{p+1}\left(x,y,\tau,\lambda_\xi^{p+1},\lambda_\eta^{p+1},\varphi^{p+1}\right)\approx T^p\left(x,y,\tau,\lambda_\xi^p,\lambda_\eta^p,\varphi^p\right)+\sum_{k=1}^{n_\xi}\frac{\partial T^p}{\partial\xi_k}\Delta\xi_k^{p+1}+\sum_{k=1}^{n_\eta}\frac{\partial T^p}{\partial\eta_k}\Delta\eta_k^{p+1}+\frac{\partial T^p}{\partial\varphi}\Delta\varphi^{p+1}, \quad (12)$$

where $(\Delta\xi_1^{p+1},\Delta\xi_2^{p+1},...,\Delta\xi_{n_\xi}^{p+1})=\Delta\overrightarrow{\Phi_\xi^{p+1}}$ and $(\Delta\eta_1^{p+1},\Delta\eta_2^{p+1},...,\Delta\eta_{n_\eta}^{p+1})=\Delta\overrightarrow{\Phi_\eta^{p+1}}$ are the vectors of the increments $\overrightarrow{\Delta\Phi_\xi^{p+1}}=\overrightarrow{\Phi_\xi^{p+1}}-\overrightarrow{\Phi_\xi^p}$, $\overrightarrow{\Delta\Phi_\eta^{p+1}}=\overrightarrow{\Phi_\eta^{p+1}}-\overrightarrow{\Phi_\eta^p}$, and the increment of the orientation angle $\Delta\varphi^{p+1}=\varphi^{p+1}-\varphi^p$.

At the $(p+1)$th iteration, the stabilizing functional is represented in the form

$$\begin{aligned}\Omega\left[\lambda_\xi^{p+1}(T),\lambda_\eta^{p+1}(T),\varphi^{p+1}\right]=&\int_{T_{\min}}^{T_{\max}}\left(w_{0\xi}\left(\lambda_\xi^{p+1}\right)^2+w_{1\xi}\left(\frac{\partial\lambda_\xi^{p+1}}{\partial T}\right)^2+w_{2\xi}\left(\frac{\partial^2\lambda_\xi^{p+1}}{\partial T^2}\right)^2\right)dT+\\&+\int_{T_{\min}}^{T_{\max}}\left(w_{0\eta}\left(\lambda_\eta^{p+1}\right)^2+w_{1\eta}\left(\frac{\partial\lambda_\eta^{p+1}}{\partial T}\right)^2+w_{2\eta}\left(\frac{\partial^2\lambda_\eta^{p+1}}{\partial T^2}\right)^2\right)dT+w_\varphi\cdot\left(\varphi^{p+1}\right)^2\end{aligned}, \quad (13)$$

where $w_{0\xi},w_{1\xi},w_{2\xi},w_{0\eta},w_{1\eta},w_{2\eta},w_\varphi$ are the weight factors that are selected using the a-priori information about the required dependencies $\lambda_\xi(T)$, $\lambda_\eta(T)$ and φ. In this problem, to determine the required dependencies $\lambda_\xi(T)$, $\lambda_\eta(T)$, the second-order regularization was used, and for φ, the zero-order regularization [5].

If we substitute expressions (9), (10), (11), and (12) into functional (8) and use the necessary condition for the minimum of functional (8), then we can obtain a system of linear equations with respect to $\Delta\xi_1^{p+1},...,\Delta\xi_{n_\xi}^{p+1},\Delta\eta_1^{p+1},...,\Delta\eta_{n_\eta}^{p+1},\Delta\varphi^{p+1}$ at the $(p+1)$th iteration. The elements of the system of linear equations can be obtained in the same way as the elements of the system of linear equations were determined in [10].

This system includes the regularization parameter, which is determined as we did in [9, 10, 12, 13], proceeding from the condition

$$\left(1-\sqrt{\frac{2}{N}}\right)\sigma \le \delta \le \left(1+\sqrt{\frac{2}{N}}\right)\sigma, \tag{14}$$

which was proposed in [4]. Here, N is the total number of thermometric measurements; σ is the root-mean-square error; δ is the standard deviation of the obtained temperature from the measured one.

The regularization parameter is considered to have been chosen correctly if, for the solution obtained according to the iterative scheme proposed above, the two-sided inequality (14) is satisfied.

Numerical Experiment

We considered, for a rectangular anisotropic plate $l_x \times l_y$ heated by a convective heat flux, the internal IHCP of identifying the thermal conductivity tensor $\lambda_{xx}(T)$, $\lambda_{xy}(T)$, $\lambda_{yx}(T)$, $\lambda_{yy}(T)$

$$C(T)\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda_{xx}(T)\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial x}\left(\lambda_{xy}(T)\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial y}\left(\lambda_{yx}(T)\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda_{yy}(T)\frac{\partial T}{\partial y}\right),\ x \in (0, l_x), y \in (0, l_y), \tag{15}$$

$$R(T)\big|_{(x=0)\cup(y=0)} = 0, \tag{16}$$

$$\left(R(T) + \alpha T\right)\big|_{(x=l_x)\cup(y=l_y)} = \alpha T_s,$$

where $R(T)$ is the operator in the form (4). As in [8], 9 points were taken to identify the thermal conductivity tensor (the justification of the minimum number of points for identifying the thermal conductivity tensor in two-dimensional domains is also carried out in [8]). In this test numerical experiment, the points were selected at the corners of the plate, in the middle of each of its outer sides, and in its center.

When carrying out the numerical experiment, taken as the principal temperature-dependent thermal conductivity coefficients were the functions

$$\lambda_\xi(T) = 1 + 0.5 \cdot T - T^2, \tag{17}$$

$$\lambda_\eta(T) = 0.5 - 0.5 \cdot T + 2 \cdot T^2, \tag{18}$$

which are fairly accurately approximated by Schoenberg's cubic splines with a small number of required parameters. Taken as the orientation angle of the principal axes $O\xi$ and $O\eta$ was that of 30°. The obtained numerical solution at the temperature measurement points is superimposed with a random error distributed according to the normal law at $\sigma=0.02$.

After substituting expressions (17) and (18) into (6)–(8) and at $\varphi=30°$, the following expressions are obtained

$$\lambda_{xx}(T) = 0.875 + 0.25 \cdot T - 0.25 \cdot T^2, \tag{19}$$

$$\lambda_{yy}(T) = 0.625 - 0.25 \cdot T + 1.25 \cdot T^2, \tag{20}$$

$$\lambda_{xy}(T) = \lambda_{yx}(T) = 0.4330127019 \cdot \left(0.5 + T - 3 \cdot T^2\right). \tag{21}$$

Figures 1–5 show the dependencies of the principal thermal conductivity coefficients $\lambda_\xi(T)$ and $\lambda_\eta(T)$ obtained using the method described above, as well as the dependencies of the thermal conductivity tensor $\lambda_{xx}(T)$, $\lambda_{xy}(T)$, $\lambda_{yx}(T)$, $\lambda_{yy}(T)$, which are compared with their dependencies in the form (17)–(21) for the following dimensionless quantities: $n_\tau=100$; $m=9$; $\Delta\tau=0.01$; $l_x=1$; $l_y=1$; $\alpha=5$; $T_s=1$; $T_0=0$; $n_\xi=4$; $n_\eta=4$; $w_{0\xi}=0$; $w_{1\xi}=0$; $w_{2\xi}=1$; $w_{0\eta}=0$; $w_{1\eta}=0$; $w_{2\eta}=1$; $w_\varphi=1$; $C(T)=1$. As far as the orientation angle is concerned, it is equal to 29.87º.

Figure 6 shows the dependencies $T(\tau)$ at the plate thermometry point $(x, y)=(0, 0)$, obtained as a result of solving both the direct and inverse problems, the "noisy" temperature at this point, as well as the one obtained using the identified principal thermal conductivity coefficients and the orientation angle of the principal axes.

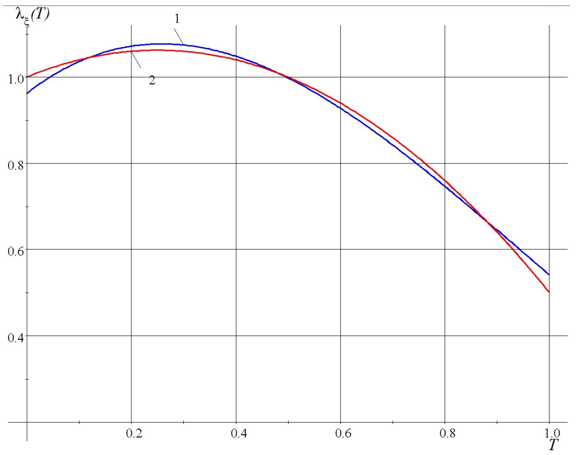

Fig. 1. Dependencies of the principal coefficient $\lambda_\xi(T)$:
1 – in the form (17); 2 – obtained using the iterative method

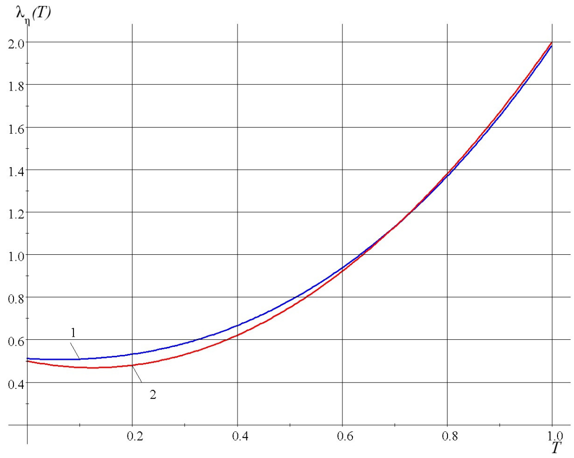

Fig. 2. Dependencies of the principal coefficient $\lambda_\eta(T)$:
1 – in the form (18); 2 – obtained using the iterative method

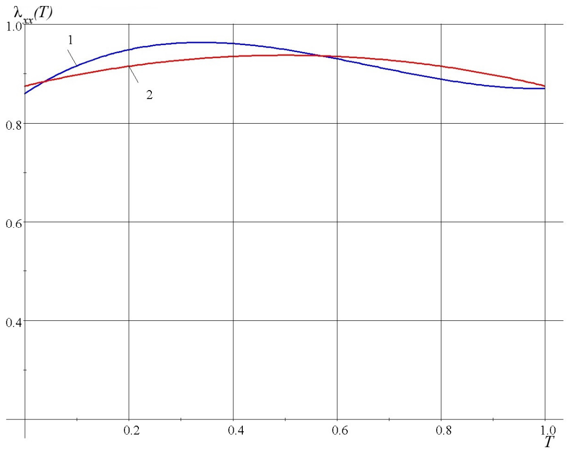

Fig. 3. Dependencies of the coefficient $\lambda_{xx}(T)$:
1 – in the form (19); 2 – using the iterative method

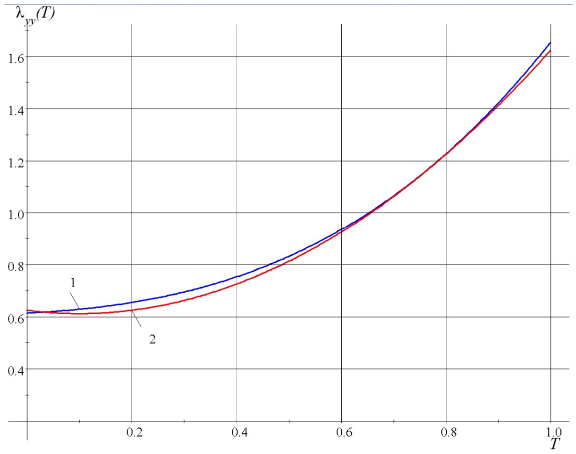

Fig. 4. Dependencies of the coefficient $\lambda_{yy}(T)$:
1 – in the form (20); 2 – using the iterative method

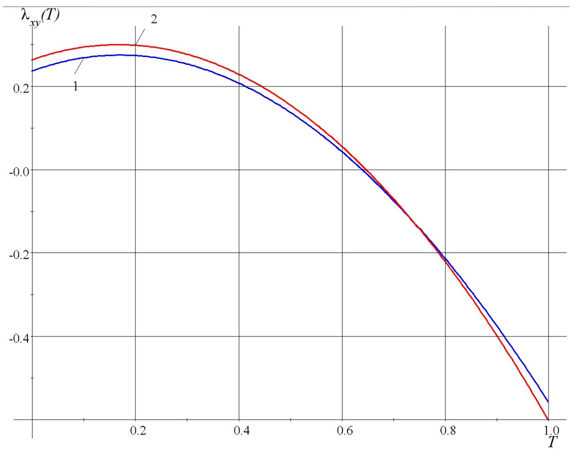

Fig. 5. Dependencies of the coefficient $\lambda_{xy}(T)$ and $\lambda_{yx}(T)$:
1 – in the form (21); 2 – obtained using the iterative method

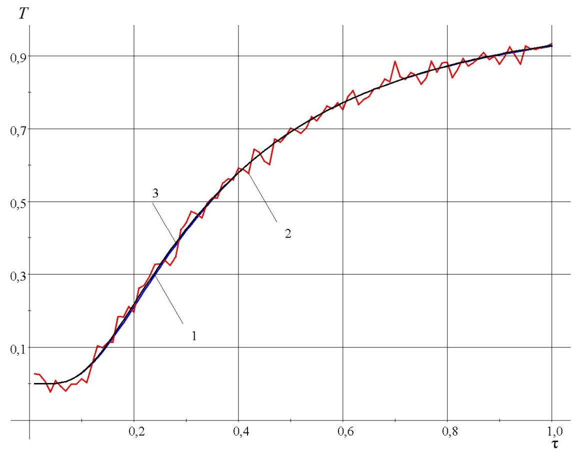

Fig. 6. Dependencies $T(\tau)$ at the plate thermometry point $(x, y)=(0, 0)$:
1 – obtained using the coefficients in the thermal conductivity tensor in the form (18–20);
2 – "noisy" solution of the direct problem;
3 – obtained using the iterative method

The selection of the regularization parameter β began with β=0.01. The iterative process of selecting β after three iterations ended at β=0.0001 when the root-mean-square error reached δ≈0.0199. All the boundary-value problems for determining the temperature field in the object under study were solved using the finite element method and an implicit difference scheme.

Using the proposed technique, we also solved the problem presented in [8], where, in a rectangular plate $l_x \times l_y$, the nonlinear components of the thermal conductivity tensor of a carbon composite material reinforced with unidirectional continuous fibers at an angle of 30° to one of the plate boundaries are recovered. In this case we used equation (15), initial condition (3), the boundary condition

$$T\big|_{(x=0)\cup(y=0)\cup(x=l_x)\cup(y=l_y)} = T_s \tag{22}$$

and experimental data (5) at nine points.

For the numerical experiment, as in [8], the principal components of the thermal conductivity tensor are taken in the following form:

$$\lambda_\xi(T) = -5.625 + 0.25\cdot 10^{-2}\cdot T + 0.46875\cdot 10^{-5}\cdot T^2 \quad \text{W/(m·K)}, \tag{23}$$

$$\lambda_\eta(T) = -4.5 + 0.0125\cdot T - 0.625\cdot 10^{-5}\cdot T^2 \quad \text{W/(m·K)}. \tag{24}$$

The direct problem (15), (16), (22) with the principal components of the thermal conductivity tensor (23), (24) and the orientation angle φ=30° of the principal components of the thermal conductivity tensor to one of the plate boundaries was solved at T_s=1400 K; T_0=600 K; C=2.25×10^6 J/(m^3·K); l_x=0.1 m; l_y=0.06 m. The experimental temperature values were determined at the points $\{(x_i, y_i)\}$, $i=\overline{1,3}$; $j=\overline{1,3}$, where x_i={0.01 m; 0.05 m; 0.09 m}, y_i={0.004 m; 0.012 m; 0.02 m}, at times $\{t_k = k\cdot\Delta t\}$, $k=\overline{1,40}$ at Δt=5 sec. In solving the IHCP, superimposed on the experimental temperature values was a random error distributed according to the normal law at σ=5*K*.

Figures 7–8 show the identified dependencies of the principal thermal conductivity coefficients $\lambda_\xi(T)$ and $\lambda_\eta(T)$, obtained using the method described above, which are compared with their dependencies in the form (23)–(24) for n_ξ=4; n_η=4; $w_{0\xi}$=0; $w_{1\xi}$=0; $w_{2\xi}$=1; $w_{0\eta}$=0; $w_{1\eta}$=0; $w_{2\eta}$=1; w_φ=1. The orientation angle identification converged to a value of 31.27°.

The selection of the regularization parameter β began with β=4×10^4. The iterative process of selecting β after five iterations ended at β=1.5×10^3 when the root-mean-square error reached δ≈5.385. All the boundary-value problems for determining the temperature field in the object under study were solved using the finite element method and an implicit difference scheme.

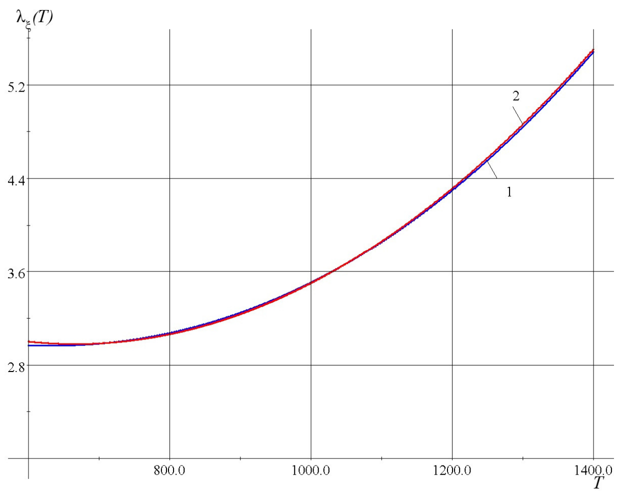

Fig. 7. Dependencies of the principal coefficient $\lambda_\xi(T)$:
1 – in the form (25); 2 – obtained using the iterative method

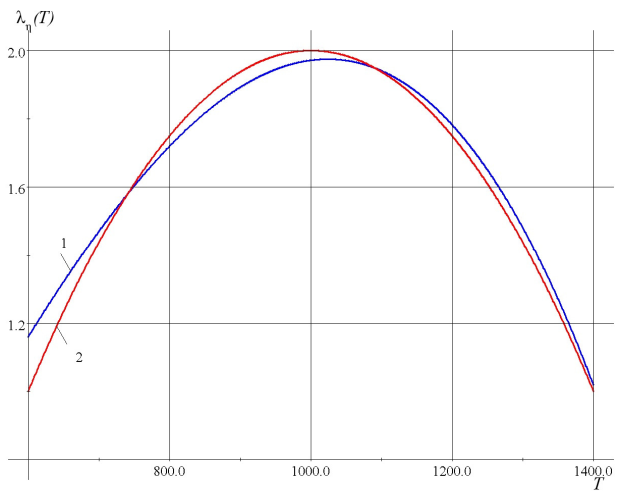

Fig. 8. Dependencies of the principal coefficient $\lambda_\eta(T)$:
1 – in the form (25); 2 – obtained using the iterative method

Conclusions

The presented solution of the nonlinear two-dimensional internal IHCPs of identifying the thermal conductivity tensor indicates that the identification technique presented can be successfully used in the presence of an a priori information about the desired function. If such information is not available, then the proposed approach can also be applied, but the measurement errors should be comparable with those in the solution of the direct problem. The method proposed in this paper for the identification of the nonlinear thermal conductivity tensor gives comparable results with the known methods, and when using the a priori information on the smoothness of tensor components, the efficiency of its application is even higher [8].

The studies presented in the paper were carried out within the framework of budgetary theme III-6-20.

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Розв'язання оберненої задачі з ідентифікації тензора теплопровідності в анізотропних матеріалах

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На основі теорії регуляризації А. М. Тихонова розроблено методику розв'язання обернених задач теплопровідності з ідентифікації тензора теплопровідності двовимірної області. Ці задачі замінюються на задачі з ідентифікації головних коефіцієнтів та кута орієнтації головних осей, а головні коефіцієнти апроксимуються кубічними сплайнами Шьонберга. В результаті задача зводиться до визначення невідомих коефіцієнтів в цих апроксимаціях і кута орієнтації головних осей. За відомих граничних і початкових умов температура в області буде залежати тільки від цих коефіцієнтів і кута орієнтації. Якщо виразити її за формулою Тейлора для двох членів ряду і підставити в функціонал Тихонова, то визначення збільшень коефіцієнтів і збільшення кута орієнтації можна звести до розв'язання системи лінійних рівнянь щодо цих збільшень. Вибравши певний параметр регуляризації і деякі функції для головних коефіцієнтів теплопровідності і кута орієнтації як початкове наближення, можна реалізувати ітераційний процес визначення цих коефіцієнтів. Отримавши вектори коефіцієнтів і кут орієнтації в результаті збігального ітераційного процесу, можна визначити середньоквадратичну нев'язку між одержуваною температурою і температурою, яка вимірюється в результаті проведеного експерименту. Залишається підібрати параметр регуляризації таким чином, щоб ця нев'язка була в межах середньоквадратичної похибки помилки вимірювань. Під час перевірки ефективності використання запропонованого методу розв'язано ряд двомірних тестових задач для тіл з відомими тензорами теплопровідності. Проаналізовано вплив випадкових похибок вимірювань на похибка ідентифікації тензора теплопровідності.

Ключові слова: *внутрішня обернена задача теплопровідності, тензор теплопровідності, метод регуляризації А. М. Тихонова, стабілізуючий функціонал, параметр регуляризації, ідентифікація, апроксимація, кубічні сплайни Шьонберга.*

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