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METHOD OF SOLVING GEOMETRICALLY NONLINEAR BENDING PROBLEMS OF THIN SHALLOW SHELLS OF COMPLEX SHAPE

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A new numerical analytical method for solving geometrically nonlinear bending problems of thin shallow shells and plates of complex shape is given in the paper. The problem statement is performed within the framework of the classic geometrically nonlinear formulation. The parameter continuation method was used to linearize the nonlinear bending problem of shallow shells and plates. An increasing parameter t related to the external load, which characterizes the shell loading process, is introduced. For the variational formulation of the linearized problem, a functional in the Lagrange form, defined on the kinematically possible movement speeds, is constructed. To find the main unknowns of the problem of nonlinear bending of the shell (displacement, deformation, stress), the Cauchy problem was formulated by the parameter t for the system of ordinary differential equations, which was solved by the fourth order Runge-Kutta-Merson method with automatic step selection. The initial conditions are found from the solution of the problem of geometric linear deformation. The right-hand sides of the differential equations at fixed values of the parameter t, corresponding to the Runge-Kutta-Merson scheme, were found from the solution of the variational problem for the functional in the Lagrange form. Variational problems were solved using the Ritz method combined with the R-function method, which allows to accurately take into account the geometric information about the boundary value problem and provide an approximate solution in the form of a formula - a solution structure that exactly satisfies all (general structure) or part (partial structure) of boundary conditions. The test problem for the nonlinear bending of a square clamped plate under the action of a uniformly distributed load of different intensity is solved. The results for deflections and stresses obtained using the developed method are compared with the analytical solution and the solution obtained by the finite element method. The problem of bending of a clamped plate of complex shape is solved. The effect of the geometric shape on the stress-strain state is studied.

Keywords: *flexible shallow shell, complex shape, R-function method, parameter continuation method.*

Introduction

Thin shallow shells and plates are widely used as structural elements in aerospace engineering, mechanical engineering, energy, chemical industry and other industries. Theories and methods of calculating the stress-strain state of shells, in particular, in a geometrically nonlinear setting, are thoroughly covered in the literature. Despite this, they continue developing. A fairly complete review of methods for solving linear and nonlinear problems of the theory of shells is done, for example, in papers [1–3]. In studies, plates and shells of a canonical geometric shape in plan are considered most often. According to their results, it can be concluded that it is possible to obtain the solution of the boundary value problem in an analytical form under certain conditions of loading and fixing, but if the plate or shell have a complex geometric shape, it turns out to be impossible. In this case, it is necessary to use universal methods that allow to find an approximate solution in areas of complex shape, for example, the finite elements method [4–6], the R-function method [7, 8], the "immersion" method [9], etc. In addition, the analysis of the available literature showed that the number of papers devoted to the study of geometrically nonlinear deformation of plates and shallow shells of a complex shape in the plan is quite limited. Which seems quite logical because the search for effective methods of linearization and solving nonlinear problems of the theory of shells and plates of complex shape is currently ongoing.

The purpose of the paper is the development of a numerical analytical method for solving problems of geometrically nonlinear deformation of thin shallow shells and plates of complex shape in the plan, based on the R-function method.

Problem statement. Solution method

An isotropic thin shallow shell of thickness h and arbitrary shape Ω in the plan is considered in a rectangular Cartesian coordinate system $O\,x_1\,x_2\,z$. The axis Oz is perpendicular to the shell plane. The temperature is constant. For thin shallow shells, it is assumed that the internal geometry of the coordinate surface z=0 is no different from the Euclidean geometry on the plane. Shell lifting arrow is f<0.2a, where a is the smallest characteristic size of the shell. At the same time, the coefficients of the first quadratic form are taken as $A_i\approx 1$, (i=1, 2), and the main curvatures of the coordinate surface are constant: k_i=const [3, 10]. The shell is under the action of a transverse load of intensity equal to $q_z^* = q_z^*(x_1, x_2)$.

If the deflection arrow can be compared with the shell thickness ($w_{\max}\geq 0.25h$), then to state the problem, it is necessary to use the nonlinear theory of shells, which takes into account large deflections.

The displacements of the shell points along the axes Ox_1, Ox_2, Oz is defined by expressions [3, 10]:

$$v_1(x_1, x_2, z) = u_1 - zw_{,1}\,,\ \ v_2(x_1, x_2, z) = u_2 - zw_{,2}\,,\ \ v_3(x_1, x_2, z) = w\,, \tag{1}$$

where $u_1(x_1, x_2)$, $u_2(x_1, x_2)$, $w(x_1, x_2)$ are the displacements of coordinate surface points of the shell along the axes Ox_1, Ox_2, Oz, respectively.

With the usual simplifications for thin shallow shells and taking into account nonlinear members, which are significant in the case of large deflections and small deformations, the latter ones are related to displacements by the following nonlinear relations [3, 10]:

$$\varepsilon_{11} = u_{1,1} - zw_{,11} + k_1 w + 0{,}5w_{,1}^2\,,\ \ \varepsilon_{22} = u_{2,2} - zw_{,22} + k_2 w + 0{,}5w_{,2}^2\,,$$

$$\gamma_{12} = 2\varepsilon_{12} = u_{1,2} + u_{2,1} - 2zw_{,12} + w_{,1}\,w_{,2}\,,\ \ \varepsilon_{i3} = 0,\ (i = 1,2,3), \tag{2}$$

Stresses and strains are related by Hooke's law

$$\sigma_{11} = \frac{E}{1-\nu^2}(\varepsilon_{11} + \nu\varepsilon_{22}),\ \ \sigma_{22} = \frac{E}{1-\nu^2}(\varepsilon_{22} + \nu\varepsilon_{11}),\ \ \sigma_{12} = G\gamma_{12}\,. \tag{3}$$

Here E, ν are Young's modulus and Poisson's ratio of the shell material, $G = \dfrac{E}{2(1+\nu)}$ is the shear modulus.

One of the most common methods of analyzing the nonlinear deformation of mechanical systems is considered to be tracking of their deformations development process with changes of any characteristic parameter. Equations describing the nonlinear behavior of deformed systems contain a parameter or can be given in a form that includes a parameter, for example, related to load, time, or any other parameter that quantitatively characterizes the deformation of the system and explicitly or implicitly contains in the equations.

To linearize and state the problem of geometrically nonlinear deformation of shallow shells and plates, the parameter continuation method [11], which in our case is naturally associated with the external load, will be used. The increasing parameter $t\in[t_0, t_*]$ characterizing the shell loading process is considered. In this case, t_0 is the value of the parameter at which the deflections are small, and therefore, the problem of deformation is geometrically linear, t_* corresponds to the given level of shell loading $q_z(t_*) = q_z^*$. For the external transverse load, the linear law

$$q_z(t) = q_{z0} + tq_{z1}\,, \tag{4}$$

where $t\in[0, t_*]$.

It should be noted that since the elastic problem is being solved, the final result does not depend on the load path and other load laws can be taken.

After differentiating relation (2) by the parameter t, formulas relating the derivatives of deformations and displacements are obtained

$$\dot\varepsilon_{11} = \dot u_{1,1} - z\dot w_{,11} + k_1\dot w + w_{,1}\,\dot w_{,1}\,,\ \ \dot\varepsilon_{22} = \dot u_{2,2} - z\dot w_{,22} + k_2\dot w + w_{,2}\,\dot w_{,2}\,,$$

$$\dot\gamma_{12} = 2\dot\varepsilon_{12} = \dot u_{1,2} + \dot u_{2,1} - 2z\dot w_{,12} + w_{,1}\,\dot w_{,2} + w_{,2}\,\dot w_{,1}\,. \tag{5}$$

In this case, the dot above the symbols denotes the full derivative of the argument t. Further, the derivatives of t will be referred to as rates.

If the rotation angles $w_{,i}$ are considered as given functions, then the relations (5) are linear.

After differentiating Hooke's law (3) by t and taking into account (5), for the stress rates we write

$$\dot{\sigma}_{11} = \frac{E}{1-\nu^2}\left(\dot{u}_{1,1} + \nu\dot{u}_{2,2} - z\left(\dot{w}_{,11} + \nu\dot{w}_{,22}\right) + \left(k_1 + \nu k_2\right)\dot{w} + w_{,1}\dot{w}_{,1} + \nu w_{,2}\dot{w}_{,2}\right),$$

$$\dot{\sigma}_{22} = \frac{E}{1-\nu^2}\left(\dot{u}_{2,2} + \nu\dot{u}_{1,1} - z\left(\dot{w}_{,22} + \nu\dot{w}_{,11}\right) + \left(k_2 + \nu k_1\right)\dot{w} + w_{,2}\dot{w}_{,2} + \nu w_{,1}\dot{w}_{,1}\right),$$

$$\dot{\sigma}_{12} = G\left(\dot{u}_{1,2} + \dot{u}_{2,1} - 2z\dot{w}_{,12} + w_{,1}\dot{w}_{,2} + w_{,2}\dot{w}_{,1}\right), \qquad (6)$$

For the variational problem statement, the principle of virtual work for quasi-static problems [12] is used. The corresponding functional in the Lagrange form, written with respect to the rates of displacements for a three-dimensional body, has the form [12]

$$L(\dot{v}_i) = 0{,}5\iiint_V \left(\dot{\sigma}_{kl}\dot{\varepsilon}_{kl} + \sigma_{ij}\dot{v}_{k,i}\dot{v}_{k,j}\right) - \iint_{S_p} \dot{P}_i\dot{v}_i dS, \ (i, j, k, l{=}1, 2, 3), \qquad (7)$$

where $\dot{v}_i$ are kinematically possible displacement rates.

Considering that $\dot{v}_{i,j} \sim \dot{w}_{,i}^2 << 1$, $(i, j{=}1, 2)$, and neglecting terms of higher order of smallness in expression (7), for a flexible thin shell we will have

$$L = 0{,}5\iint_\Omega \int_{(h)} \left(\dot{\sigma}_{kl}\dot{\varepsilon}_{kl} + \sigma_{11}\dot{w}_{,1}^2 + \sigma_{22}\dot{w}_{,2}^2 + 2\sigma_{12}\dot{w}_{,1}\dot{w}_{,2}\right)dx_1dx_2dz - -\iint_\Omega \dot{q}_z\dot{w}dx_1dx_2, \ (k, l{=}1, 2). \qquad (8)$$

By substituting (5), (6) into (8) and integrating over z, we obtain a functional in the Lagrange form for the bending problem of a flexible shallow shell

$$L = L_l + L_n, \qquad (9)$$

where $L_l(\dot{u}_1, \dot{u}_2, \dot{w})$, $L_n(\dot{u}_1, \dot{u}_2, \dot{w})$ are "linear" and "nonlinear" parts of the functional, which are determined by the following formulas:

$$L_l = 0{,}5\iint_\Omega \Big[A_1\left(\dot{u}_{1,1}^2 + \dot{u}_{2,2}^2 + \dot{w}^2\left(k_1^2 + k_2^2\right) + 2k_1\dot{w}\dot{u}_{1,1} + 2k_2\dot{w}\dot{u}_{2,2}\right) + 2A_2\left(\dot{u}_{1,1}\dot{u}_{2,2} + \dot{w}\left(k_1\dot{u}_{2,2} + k_2\dot{u}_{1,1}\right) + k_1k_2\dot{w}^2\right) + A_3\left(\dot{u}_{1,2} + \dot{u}_{2,1}\right)^2 -$$
$$-2B_1\left(\dot{u}_{1,1}\dot{w}_{,11} + \dot{u}_{2,2}\dot{w}_{,22} + \dot{w}\left(k_1\dot{w}_{,11} + k_2\dot{w}_{,22}\right)\right) - 2B_2\left(\dot{u}_{1,1}\dot{w}_{,22} + \dot{u}_{2,2}\dot{w}_{,11} + \dot{w}\left(k_1\dot{w}_{,22} + k_2\dot{w}_{,11}\right)\right) - 2B_3\dot{w}_{,12}\left(\dot{u}_{1,2} + \dot{u}_{2,1}\right) +$$
$$+D_1\left(\dot{w}_{,11}^2 + \dot{w}_{,22}^2\right) + 2D_2\dot{w}_{,11}\dot{w}_{,22} + D_3\dot{w}_{,12}^2\Big]dx_1dx_2 - \iint_\Omega \dot{q}_z\dot{w}dx_1dx_2 \qquad (10)$$

$$L_n = 0{,}5\iint_\Omega \Big[A_1\left(w_{,1}^2\dot{w}_{,1}^2 + w_{,2}^2\dot{w}_{,2}^2 + 2w_{,1}\dot{u}_{1,1}\dot{w}_{,1} + 2w_{,2}\dot{u}_{2,2}\dot{w}_{,2} + 2\dot{w}\left(k_1w_{,1}\dot{w}_{,1} + k_2w_{,2}\dot{w}_{,2}\right)\right) +$$
$$+2A_2\left(w_{,1}\dot{u}_{2,2}\dot{w}_{,1} + w_{,2}\dot{u}_{1,1}\dot{w}_{,2} + \dot{w}\left(k_1w_{,2}\dot{w}_{,2} + k_2w_{,1}\dot{w}_{,1}\right) + w_{,1}w_{,2}\dot{w}_{,1}\dot{w}_{,2}\right) +$$
$$+A_3\left(w_{,1}^2\dot{w}_{,2}^2 + w_{,2}^2\dot{w}_{,1}^2 + 2\left(w_{,1}\dot{w}_{,2} + w_{,2}\dot{w}_{,1}\right)\left(\dot{u}_{1,2} + \dot{u}_{2,1}\right) + 2w_{,1}w_{,2}\dot{w}_{,1}\dot{w}_{,2}\right) -$$
$$-2B_1\left(w_{,1}\dot{w}_{,1}\dot{w}_{,11} + w_{,2}\dot{w}_{,2}\dot{w}_{,22}\right) - 2B_2\left(w_{,1}\dot{w}_{,1}\dot{w}_{,22} + w_{,2}\dot{w}_{,2}\dot{w}_{,11}\right) - 2B_3\dot{w}_{,12}\left(w_{,1}\dot{w}_{,2} + w_{,2}\dot{w}_{,1}\right) +$$
$$+f_{11}\dot{w}_{,1}^2 + f_{22}\dot{w}_{,2}^2 + 2f_{12}\dot{w}_{,1}\dot{w}_{,2}\Big]dx_1dx_2, \qquad (11)$$

where Ω is the domain in which an approximate solution to the problem is sought; $A_1 = \int_{(h)} \frac{E}{1-\nu^2}dz$, $A_2 = \nu A_1$,

$A_3 = \int_{(h)} Gdz$, $B_1 = \int_{(h)} \frac{Ez}{1-\nu^2}dz$, $B_2 = \nu B_1$, $B_3 = 2\int_{(h)} Gzdz$, $D_1 = \int_{(h)} \frac{Ez^2}{1-\nu^2}dz$, $D_2 = \nu D_1$, $D_3 = 4\int_{(h)} Gz^2dz$,

$f_{11} = \int_{(h)} \sigma_{11}dz$, $f_{22} = \int_{(h)} \sigma_{22}dz$, $f_{12} = \int_{(h)} \sigma_{12}dz$.

Here, rotations $w_{,1}$, $w_{,2}$ and stresses σ_{11}, σ_{22}, σ_{12} are considered to be preset for each fixed value of the parameter t and do not vary.

The main unknown functions at each point of the shell for the values t>0 can be found from the solution of the Cauchy problem in terms of the parameter t for the system of ordinary differential equations, which will be written in the following generalized form

$$\frac{dG_k}{dt} = F_k(G_l), \quad \left(k = \overline{1,11}\right), \tag{12}$$

where $\mathbf{G}=\{G_k\}=\{u_i, w, w_{,i}, \varepsilon_{ij}, \sigma_{ij}\}$, $(i, j=1, 2)$. The right-hand sides of equations (12) have the form: $F_1 = \dot{u}_1$, $F_2 = \dot{u}_2$, $F_3 = \dot{w}$; $F_4 = \dot{w}_{,1}$, $F_5 = \dot{w}_{,2}$; F_k (k=6, 7, 8) are determined by formulas (5); F_k (k=9, 10, 11) by formulas (6).

The initial conditions for equations (12) are found from the solution of the problem of linear deformation at $q_z(0)=q_{z0}$. To solve it, the functional in the form (10) can be used, replacing the rates of the functions with the functions themselves.

The most famous methods for solving initial problems include the Euler method, the Adams-Bashforth method, and the Runge-Kutta methods of varying degrees of accuracy. In this paper, we will solve the initial problem for the system of equations (12) using the fourth order Runge-Kutta-Merson (RKM) method with automatic step selection [13].

The right-hand sides of the equations, with fixed values $t\neq 0$ corresponding to the RKM scheme, are found from the solution of the variational problem for the functional (9). Variational problems will be solved using the Ritz method combined with the R-function method [7], which allows to accurately take into account geometric information about the boundary value problem and provide an approximate solution in the form of a formula – a solution structure that exactly satisfies all (general structure) or part (partial structure) of the boundary conditions.

Numerical results

The bending of a clamped square plate [14] subjected to uniformly distributed transverse loads is considered as a test example. The plate dimensions are: side length $2a$=7.62 m, thickness h=0.0762 m. Elastic characteristics of the material: $E=2.1\times10^5$ MPa, ν=0.316.

The clamped boundary conditions have the form

$$\dot{w} = 0,\ \dot{w}_{,n} = 0,\ \dot{u}_1 = 0,\ \dot{u}_2 = 0\ ,$$

and the corresponding solution structure

$$\dot{w} = \omega^2\Phi_1,\ \dot{u}_1 = \omega\Phi_2,\ \dot{u}_2 = \omega\Phi_3\ .$$

In this case, Φ_1, Φ_2, Φ_3 are the undefined components of the solution structure; the function $\omega=\omega(x_1, x_2)$ is constructed using the R-function theory [7] and satisfies the conditions: $\omega=0$, $\omega_{,n}=-1$, on the boundary $\partial\Omega$, $\omega>0$ – inside Ω (**n** – is the external normal to the contour $\partial\Omega$).

In the case of a square plate, the function ω has the form

$$\omega = \omega_1 \wedge_0 \omega_2\ ,$$

where $\omega_1 = \frac{1}{2a}\left(a^2 - x_2^2\right)$, $\omega_2 = \frac{1}{2a}\left(a^2 - x_1^2\right)$, and symbol $\wedge_0$ denotes the R-conjunction [7]: $f_1 \wedge_0 f_2 = f_1 + f_2 - \sqrt{f_1^2 + f_2^2}$.

During the numerical implementation, the undefined components of the solution structure were given in the form of finite series $\Phi_i(x_1, x_2, t) = \sum_n C_n^{(i)}(t) f_n^{(i)}(x_1, x_2)$, ($i$=1, 2, 3), where $C_n^{(i)}(t)$ are the undefined coefficients that were found at each step by the Ritz method; t is a fixed value of the load parameter; $\left\{f_n^{(i)}\right\}$ – systems of linearly independent functions. Power polynomials of the form $x_1^k x_2^l$ were used as $\left\{f_n^{(i)}\right\}$.

Table 1. Non-dimensional deflections $\overline{w}$ in the center of the plate

$\overline{q}$	Analytical	FEM	RFM
17.8	0.237	0.2392	0.2356
38.3	0.471	0.4738	0.4676
63.4	0.695	0.6965	0.6903
95.0	0.912	0.9087	0.9042
134.9	1.121	1.1130	1.1110
184.0	1.323	1.3080	1.3090
245.0	1.521	1.5010	1.5040
318.0	1.714	1.6880	1.6930
402.0	1.902	1.8860	1.8720

Table 2. Non-dimensional stresses $\overline{\sigma}_{11}$ in the center of the plat

$\overline{q}$	Analytical	FEM	RFM
17.8	2.6	2.414	2.530
38.3	5.2	5.022	5.212
63.4	8.0	7.649	7.893
95.0	11.1	10.254	10.535
134.9	13.3	12.850	13.155
184.0	15.9	15.420	15.754
245.0	19.2	18.060	18.440
318.0	21.9	20.741	21.197
402.0	25.1	23.423	24.000

Tables 1, 2 show the results of the calculation of non-dimensional deflections $\bar{w} = \frac{w}{h}$ and normal stresses $\bar{\sigma}_{11} = \frac{4\sigma_{11}a^2}{Eh^2}$ in the center on the lower surface of the plate, at z=0.5h, depending on the value of the non-dimensional load $\bar{q} = \frac{16q_z^* a^4}{Eh^4}$. In this case, the analytical solution of Levy [15], who solved the problem using a double Fourier series (Analytical), the results obtained by the finite element method (FEM) [14] and the R-function method (RFM) are given. In formula (4) it was assumed that: $q_{z0}=q_{z1}=10^{-2}$ MPa. The initial step and the specified calculation error in the RKM method were, respectively, equal to: $\Delta t=10^{-3}$, $\varepsilon=10^{-3}$. The analytical results and the results of FEM were obtained on the basis of the first-order shear deformation theory (FSDT).

The results listed in the Tables 1, 2 show that the method proposed in the paper provides a close match with the results obtained by other methods.

Next, the bending of clamped plate with circular corner cutouts under the action of a uniformly distributed load is considered (Fig. 1). Geometric dimensions are: $2a=2b=7.62$ m, $r=1.5$ m, $h=0.0762$ m. Elastic constants are the same as in the test example.

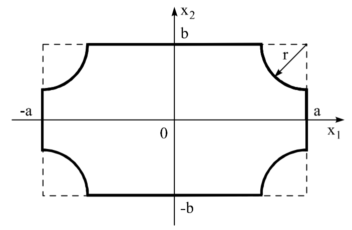

Fig. 1. The plate with circular cutouts

The equation of the boundary in Fig. 1 can be written as follows:

$$\omega = (\omega_1 \wedge_0 \omega_2) \wedge_0 ((\omega_3 \wedge_0 \omega_4) \wedge_0 (\omega_5 \wedge_0 \omega_6)) = 0,$$

where $\omega_1 = \frac{1}{2b}\left(b^2 - x_2^2\right)$, $\omega_1 = \frac{1}{2a}\left(a^2 - x_1^2\right)$, $\omega_3 = \frac{1}{2r}\left((x_1 - a)^2 + (x_2 - b)^2 - r^2\right)$,

$\omega_4 = \frac{1}{2r}\left((x_1 + a)^2 + (x_2 - b)^2 - r^2\right)$, $\omega_5 = \frac{1}{2r}\left((x_1 + a)^2 + (x_2 + b)^2 - r^2\right)$, $\omega_6 = \frac{1}{2r}\left((x_1 - a)^2 + (x_2 + b)^2 - r^2\right)$.

Figs. 2, 3 depict the non-dimensional central deflections $\bar{w}$ and normal stresses $\bar{\sigma}_{11}$ as a functions of the load parameter $\bar{q}$. The solid lines show the results for a complex-shaped plate, and the dashed lines show the results for a square plate. As expected, the corner cutouts make the plate more rigid, reducing the level of deflections and stresses.

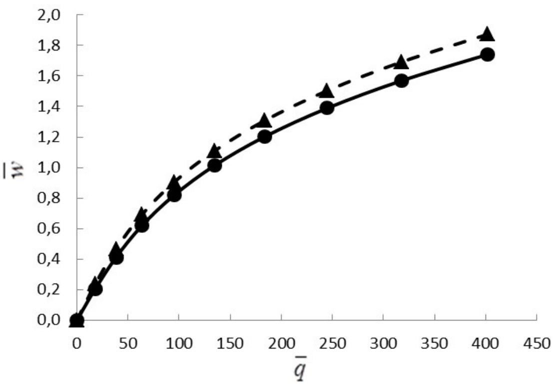

Fig. 2. Non-dimensional deflections in the center of the plate

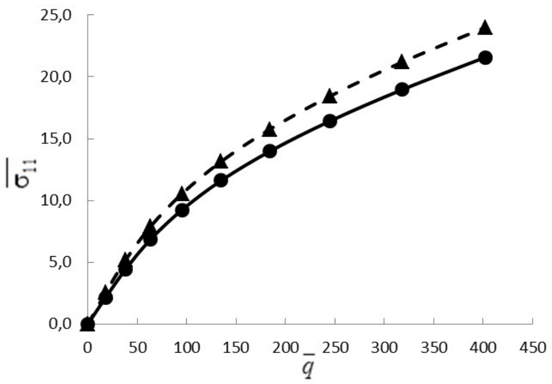

Fig. 3. Non-dimensional stresses in the center of the plate

Conclusions

A new numerical-analytical method for solving geometrically nonlinear problems of bending of thin shallow shells and plates of complex shape in the plan, which is based on the R-function method and the parameter continuation method is developed in the paper. First, the test problem was solved, a coincidence with the analytical solution and the solution by the finite element method was obtained, and secondly, the problem of bending of a clamped plate of complex shape was solved. The influence of the geometric shape on the stress-strain state has been studied.

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Метод розв'язання геометрично нелінійних задач вигину тонких пологих оболонок складної форми

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У статті представлено новий чисельно-аналітичний метод розв'язання геометрично нелінійних задач вигину тонких пологих оболонок і пластин складної форми. Постановку задачі виконано у рамках класичної геометрично нелінійної постановки. Для лінеаризації нелінійної задачі вигину пологих оболонок і пластин використовувався метод продовження за параметром. Введено зростаючий параметр t, пов'язаний із зовнішнім навантаженням, який характеризує процес навантаження оболонки. Для варіаційної постановки лінеаризованої задачі побудовано функціонал у формі Лагранжа, заданий на кінематично можливих швидкостях переміщень. Для знаходження основних невідомих задачі нелінійного вигину оболонки (переміщення, деформації, напруження) сформульовано задачу Коші за параметром t для системи звичайних диференціальних рівнянь, що *розв'язувалася методом Рунґе-Кутти-Мерсона з автоматичним вибором кроку. Початкові умови знаходяться із розв'язку задачі геометрично лінійного деформування. Праві частини диференціальних рівнянь при фіксованих значеннях параметра t, що відповідають схемі Рунґе-Кутти-Мерсона, знаходилися із розв'язку варіаційної задачі для функціонала у формі Лагранжа. Варіаційні задачі розв'язувалися методом Рітца в поєднанні з методом R-функцій, що дозволяє точно врахувати геометричну інформацію про крайову задачу і подати наближений розв'язок у вигляді формули – структури розв'язку, яка точно задовольняє всім (загальна структура) або частині (часткова структура) граничних умов. Розв'язано тестову задачу для нелінійного вигину квадратної жорстко закріпленої пластини під дією рівномірно розподіленого навантаження різної інтенсивності. Результати для прогинів і напружень, отримані за допомогою розробленого методу, порівняні з аналітичним розв'язком і розв'язком, отриманим методом скінченних елементів. Розв'язано задачу вигину жорстко закріпленої пластини складної форми. Досліджено вплив геометричної форми на напружено-деформований стан.*

Ключові слова: *гнучка полога оболонка, складна форма, метод R-функцій, метод продовження за параметром.*

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