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NUMERICAL ANALYSIS OF THE STRESS STATE OF NEAR-CIRCULAR HOLLOW CYLINDERS MADE OF FUNCTIONALLY GRADED MATERIALS

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Hollow cylinders of circular cross-section, made of functionally graded materials, are used in many branches of economy as structural elements and parts of machines and units. During manufacturing or in the process of operation of such cylinders, the shape of their cross-sections may differ from the circular one to some extent. A solution of the equilibrium problem of hollow cylinders of non-uniform thickness, which are close to circular ones, in a 3D formulation under certain boundary conditions at the ends is considered in this paper. The cross-sections of the considered cylinders are described using Pascal's limacon equation. A two-component continuously non-homogeneous material, which elastic properties, characterizing Young's modulus and Poisson's ratio, can be determined using concentration of the composition materials along the thickness, was chosen as the cylinder material. The aim of the paper is numerical analysis of the stress state of cylinders of such class depending on the law of variation of elastic properties of their material. The solution of the problem is based on reduction of the original three-dimensional boundary value problem for the system of partial differential equations with variable coefficients to a one-dimensional boundary value problem for a system of ordinary differential equations with constant coefficients of higher order. At the same time, the analytical method of separating variables in two coordinate directions with approximation of functions by discrete Fourier series is used. The one-dimensional boundary value problem is solved by the stable numerical method of discrete orthogonalization. The analysis of the stress state of cylinders depending on the dent size that appear in the neighborhood of the reference surface diameter and the law of variation of the material elastic properties was performed. It is shown that the nonlinearity of the law of the elastic properties distribution along the thickness leads to an increase/decrease of maximum values of normal displacements and longitudinal stresses by 1.3 times compared to the linear law. At the same time, an increase in the dent size leads to an increase of both the displacements and normal stresses by 2-3 times in the zone of the dent maximum dimension compared to the diametrically opposite zone. The results obtained in the paper can be used in strength calculations of structural elements and parts of machines of a similar type.

Keywords: *discrete Fourier series, stress state, hollow cylinders, 3D elasticity theory, Pascal's limacon equation, functionally graded materials.*

Introduction

The use of hollow cylinders as elements of structures and parts of machines requires ensuring strength and reliability during their operation. We note that in order to make optimal design and construction decisions during the development of new designs of machines, equipment, and buildings at the early stages of design, detailed information on the influence on the stress state of both the design parameters, the properties of the material from which they will be made, and the corresponding operating conditions is needed [1, 2]. In case of absence of experimental data, such information can be obtained only with the help of theoretical models. Therefore, the development of promising types of constructions depends on the discovery of effective means of mathematical modeling as soon as possible.

We note that, based on the specifics of operation and the given requirements, various methods and approaches are currently used to solve classes of problems about the stress strain state of plates and shells to ensure the strength and reliability of the considered structures and their elements. For example, a technique that allows the solution of three-dimensional boundary value problems for thick thermosensitive plates to be reduced to a two-dimensional case is proposed in paper [3]. In [4], within the framework of the spatial theory of elasticity, two options of the semi-analytical method of finite elements are proposed in relation to the study of the frequencies of free oscillations and the forms of the distribution of displacements in them. Paper [5] is devoted to the development of semi-analytical methods. In paper [6], an improved finite element method using a multiparameter wavelet finite element was used to determine the stress strain state of non-thin cylindrical shells.

At the same time, structural elements in the form of hollow cylinders very often work under difficult conditions, in which the outer and inner surfaces are in environments with different properties. At the same time, for the construction of such structures, it is advisable to use functionally graded materials (FGM), which have continuously variable mechanical properties along a certain direction. By the way, at the beginning of their use, continuous non-homogeneous (functionally graded) materials were proposed as an alternative to homogeneous coatings and layers in parts of aerospace vehicles, which are subject to the influence of extremely high temperatures [7, 8]. Studies have shown that materials with variable elastic properties along the thickness have a higher degree of resistance to wear and cracking. In addition, machine parts made of functionally graded materials (FGM) have a longer service life compared to conventional all-ceramic parts [9].

Most often, ceramics and metal are chosen for the manufacture of FGM. The problems of the static and dynamic state of plates and shells made of FGM are considered in papers [10–13]. To study the static behavior and conduct the stress state analysis of thick-walled circular cylinders, the general theory of shear deformation using the generalized differential quadrature method and the Newton Raphson iterative method was used in [14]. The problem of the stressed state of thick hollow cylinders made of FGM, in which the properties change in the radial and axial directions under the action of thermal and mechanical loads, was considered in [15]. At the same time, the meshless local Petrov-Galerkin method with time integration of linear differential equations by the Crank-Nicolson and Newmark method is applied.

One of the approaches to solving problems about the stress state of circular hollow cylinders, which are under the action of a surface load under certain boundary conditions at the ends, consists in presenting the solution in the form of a double Fourier series (in the direction of the generating and guiding cylinder), which makes it possible to reduce the original three-dimensional boundary value problem for a system of partial differential equations with constant coefficients to a one-dimensional one for a system of ordinary differential equations, which is solved by one of the numerical methods (by the thickness of the cylinder) [16]. In the case of a non-circular cross-section, the initial three-dimensional boundary value problem is described by a system of partial differential equations with variable coefficients, which does not allow to separate the variables in the direction of the guide. To overcome this problem, a following approach is proposed, according to which, after separating the variables in the direction of the generator to the system of differential equations being solved, additional functions, which are subsequently approximated by discrete Fourier series, which makes it possible to formally present the solving system of partial differential equations with variable coefficients as a system with constant coefficients and to separate the variables in the direction of the guide, are introduced. The integration of the obtained system of ordinary differential equations is carried out on the basis of a stable numerical method of discrete orthogonalization, with the definition of the integration of the amplitude values of the complementary functions at each step using the standard procedure for calculating the coefficients of the discrete Fourier series. The advantage of this approach is that it allows to accurately satisfy the boundary conditions on the side surfaces of the cylinder, take into account the change in the mechanical parameters of the material along the thickness of the cylinder, and obtain a solution of the problem that is close to the precise.

Aim of paper is to study the influence of changes in the law of elastic properties of material on the stress state of near-circular hollow cylinders, which are under the action of surface distributed load, based on an unconventional approach based on the use of analytical methods of separating variables in two coordinate directions with the parallel application of approximation of functions by discrete Fourier series and the numerical method of discrete orthogonalization [17].

Problem statement and solution method

Elastic bodies in the form of hollow cylinders bounded by two curved surfaces, between which the distance, which determines the thickness of the body, is constant, are considered. The material of an elastic body is isotropic and non-uniform in thickness, as a result, the body obeys generalized Hooke's law. Taking into account the fact that the displacement compared to the thickness, and the relative displacements compared to the unit, are insignificant, the equations of the linear spatial theory of elasticity are taken as the main ones [18]. At the same time, elastic bodies are solid bodies that do not undergo breaks.

In this paper, hollow cylinders are assigned to the curvilinear coordinate system s, t, γ, where s=*const*, t=*const* – lines of main curvatures on some coordinate cylindrical surface, and the coordinate γ is laid along the normal to this surface. The arc coordinate t is offset from a certain fixed generator, and we consider the parameter s of the generator to be equal to the distance to one of the boundary contours of the coordinate surface. The middle surface of the cylinder equidistant from its bounding surfaces is chosen as the coordinate reference surface.

The square of the length of the linear element in the selected coordinate system has the form

$$dS^2 = ds^2 + H_2^2 dt^2 + d\gamma^2 ,$$

where $H_1=H_3=1$, $H_2=1+\gamma/R_t$, here R_t – is the radius of curvature of the guide reference surface.

The problem is solved under the following conditions: the direction of the coordinate surface is an arbitrary continuous piecewise smooth closed surface; mechanical properties can be variable along the thickness, remaining constant along the design, the applied load can be set arbitrarily.

When solving problems of the theory of elasticity for hollow cylinders, it is necessary to satisfy not only the basic equations of the theory of elasticity, but also the corresponding boundary conditions on the bounding surfaces.

The cylinders are assumed to be under the action of an external normal load $q=q_0\cdot\sin(\pi s/L)$ (q_0=*const*, L is the length of the cylinder), then the boundary conditions on the side surfaces have the form

$$\sigma_\gamma = 0 , \ \tau_{s\gamma} = 0 , \ \tau_{t\gamma} = 0 \ \text{ at } \gamma = -h/2;$$

$$\sigma_\gamma = q_0 \cdot \sin(\pi s / L) , \ \tau_{s\gamma} = 0 , \ \tau_{t\gamma} = 0 \ \text{ at } \gamma = h/2, \tag{1}$$

where h is the cylinder thickness.

Cylinders closed along the guide are considered. In this case the conditions of periodicity of all factors of the stress state are met (P is the period)

$$\sigma_\gamma(s,t,\gamma) = \sigma_\gamma(s,t+P,\gamma) , \ \tau_{s\gamma}(s,t,\gamma) = \tau_{s\gamma}(s,t+P,\gamma) , \ u_\gamma(s,t,\gamma) = u_\gamma(s,t+P,\gamma) . \tag{2}$$

At the ends of the cylinders there is a diaphragm that is absolutely rigid in its plane and flexible when leaving it, i.e.

$$\sigma_s(t,\gamma) = 0 , \ u_t(t,\gamma) = u_\gamma(t,\gamma) = 0 \ \text{ at } s=0, s=L. \tag{3}$$

The functions in which the boundary conditions on the lateral surfaces are formulated, i.e. the three stress components σ_γ, $\tau_{s\gamma}$, $\tau_{t\gamma}$, to which three components of displacements u_γ, u_s, u_t are added, are taken as solving functions. After certain transformations from the basic equations (three equilibrium equations, six expressions of deformations due to displacement and six relations of generalized Hooke's law), it is possible to obtain a solving system of partial differential equations of the sixth order with variable coefficients, which, along with the boundary conditions (1)–(3), describes a three-dimensional boundary value problem in the form

$$\frac{\partial \sigma_\gamma}{\partial \gamma} = -\frac{1}{H_2 R_t}\sigma_\gamma - \frac{\partial \tau_{s\gamma}}{\partial s} - \frac{1}{H_2}\frac{\partial \tau_{t\gamma}}{\partial t} + \frac{1}{H_2 R_t}\left[\frac{E\nu}{1-\nu^2}\frac{\partial u_s}{\partial s} + \frac{\nu}{1-\nu}\sigma_\gamma + \frac{E}{1-\nu^2}\frac{1}{H_2}\left(\frac{\partial u_t}{\partial t} + \frac{1}{R_t}u_\gamma\right)\right];$$

$$\frac{\partial \tau_{s\gamma}}{\partial \gamma} = -\frac{1}{H_2 R_t}\tau_{s\gamma} - \frac{\partial}{\partial s}\left[\frac{E\nu}{1-\nu^2}\frac{\partial u_s}{\partial s} + \frac{E\nu}{1-\nu^2}\frac{1}{H_2}\left(\frac{\partial u_t}{\partial t} + \frac{1}{R_t}u_\gamma\right) + \frac{\nu}{1-\nu}\sigma_\gamma\right] - \frac{1}{H_2}\frac{\partial}{\partial t}\left[\frac{E}{2(1+\nu)}\left(\frac{1}{H_2}\frac{\partial u_s}{\partial t} + \frac{\partial u_t}{\partial s}\right)\right];$$

$$\frac{\partial \tau_{t\gamma}}{\partial \gamma} = -\frac{2}{H_2 R_t}\tau_{t\gamma} - \frac{1}{H_2}\frac{\partial}{\partial t}\left[\frac{E\nu}{1-\nu^2}\frac{\partial u_s}{\partial s} + \frac{E}{1-\nu^2}\frac{1}{H_2}\left(\frac{\partial u_t}{\partial t} + \frac{1}{R_t}u_\gamma\right) + \frac{\nu}{1-\nu}\sigma_\gamma\right] - \frac{\partial}{\partial s}\left[\frac{E}{2(1+\nu)}\left(\frac{1}{H_2}\frac{\partial u_s}{\partial t} + \frac{\partial u_t}{\partial s}\right)\right];$$

$$\frac{\partial u_\gamma}{\partial \gamma} = -\frac{\nu}{1-\nu}\frac{\partial u_s}{\partial s} - \frac{\nu}{1-\nu}\frac{1}{H_2}\left(\frac{\partial u_t}{\partial t} + \frac{1}{R_t}u_\gamma\right) + \frac{1-\nu-2\nu^2}{E(1-\nu)}\sigma_\gamma ; \tag{4}$$

$$\frac{\partial u_s}{\partial \gamma} = -\frac{2(1+\nu)}{E}\tau_{s\gamma} - \frac{\partial u_\gamma}{\partial s};$$

$$\frac{\partial u_t}{\partial \gamma} = -\frac{2(1+\nu)}{E}\tau_{s\gamma} - \frac{1}{H_2}\frac{\partial u_\gamma}{\partial t} + \frac{1}{H_2}\frac{1}{R_t}u_t ;$$

$$(0 \le s \le L;\quad t_0 \le t \le t_0 + P;\quad -h/2 \le \gamma \le h/2).$$

The presence of boundary conditions (3) allows to separate the variables in the direction of the generator, giving the solving functions, taking into account the applied load in the form of expansions in Fourier series along the coordinate s

$$X(t,s,\gamma) = \sum_{n=1}^{N}\left[X_n(t,\gamma)\sin\lambda_n s\right];\quad Y(t,s,\gamma) = \sum_{n=0}^{N}\left[Y_n(t,\gamma)\cos\lambda_n s\right], \tag{5}$$

where $X = \{\sigma_\gamma, \tau_{t\gamma}, u_\gamma, u_t, q\}$; $Y = \{\tau_{s\gamma}, u_s\}$; $\lambda_n = \dfrac{\pi n}{L}$.

Substituting the series (5) and the boundary conditions (1) into the solution system (4), and separating the variables, we obtain the solution system of partial differential equations with variable coefficients for each member of the series (5), which describes the two-dimensional boundary value problem.

Since hollow cylinders with a close but not circular cross-sectional shape are considered, the system of differential equations to be solved has coefficients that depend on the two coordinates t, γ and do not allow to separate the variables in the direction of the guide. To overcome this obstacle, the products of the solvable functions with the specified coefficients are formally written in the form of so-called complementary functions, thereby transforming the formally solvable system of partial differential equations with variable coefficients into a system with constant coefficients. By omitting the index n in the notation of the solving functions and denoting the complementary functions through

$$\varphi_1^j(t,\gamma) = \frac{1}{H_2}\frac{1}{R_t}\left\{\sigma_\gamma; \tau_{s\gamma}; u_\gamma; u_s; \frac{1}{H_2}\frac{1}{R_t}u_\gamma\right\},\ (j = \overline{1,5});$$

$$\varphi_2^j(t,\gamma) = \frac{1}{H_2}\frac{1}{R_t}\{\tau_{t\gamma}; u_t\},\ (j = \overline{1,2});$$

$$\varphi_3^j(t,\gamma) = \frac{1}{H_2}\left\{\frac{\partial \sigma_\gamma}{\partial t}; \frac{\partial u_\gamma}{\partial t}; \frac{\partial u_s}{\partial t}\right\},\ (j = \overline{1,3}); \tag{6}$$

$$\varphi_4^j(t,\gamma) = \frac{1}{H_2}\left\{\frac{\partial \tau_{t\gamma}}{\partial t}; \frac{\partial u_t}{\partial t}; \frac{1}{R_t}\frac{\partial u_t}{\partial t}\right\},\ (j = \overline{1,3});$$

$$\varphi_5^j(t,\gamma) = \frac{1}{H_2}\frac{\partial \varphi_1^3}{\partial t};\ \varphi_6^j(t,\gamma) = \frac{1}{H_2}\frac{\partial \varphi_3^3}{\partial t};\ \varphi_7^j(t,\gamma) = \frac{1}{H_2}\frac{\partial \varphi_4^2}{\partial t}$$

we obtain a solving system of partial differential equations with constant coefficients in the form

$$\frac{\partial \sigma_\gamma}{\partial \gamma} = \lambda_n \tau_{s\gamma} + \left(\frac{\nu}{1-\nu} - 1\right)\varphi_1^1 - \varphi_4^1 - \frac{E\nu}{1-\nu^2}\lambda_n \varphi_1^4 + \frac{E}{1-\nu^2}\left(\varphi_4^3 - \varphi_1^5\right);$$

$$\frac{\partial \tau_{s\gamma}}{\partial \gamma} = \frac{\nu}{1-\nu}\lambda_n \sigma_\gamma + \frac{E}{1-\nu^2}\lambda_n^2 u_s - \varphi_1^2 - \left(\frac{E\nu}{1-\nu^2} + \frac{E}{1-\nu^2}\right)\lambda_n \varphi_4^2 - \frac{E\nu}{1-\nu^2}\lambda_n \varphi_1^3 - \frac{E}{2(1+\nu)}\varphi_6;$$

$$\frac{\partial \tau_{t\gamma}}{\partial \gamma} = \frac{E}{2(1+\nu)}\lambda_n^2 u_t - 2\varphi_2^1 + \left(\frac{E\nu}{1-\nu^2} + \frac{E}{2(1+\nu)}\right)\lambda_n \varphi_3^3 - \frac{E}{1-\nu^2}(\varphi_7 + \varphi_5) - \frac{\nu}{1-\nu}\varphi_3^1; \tag{7}$$

$$\frac{\partial u_\gamma}{\partial \gamma} = \frac{1-\nu-2\nu^2}{(1-\nu)E}\sigma_\gamma + \frac{\nu}{1-\nu}(\lambda_n u_s - \varphi_4^2 - \varphi_1^3);$$

$$\frac{\partial u_s}{\partial \gamma} = \frac{2(1+\nu)}{E}\tau_{s\gamma} - \lambda_n u_\gamma ;$$

$$\frac{\partial u_t}{\partial \gamma} = \frac{2(1+\nu)}{E}\tau_{t\gamma} - \varphi_3^2 + \varphi_2^3 .$$

As mentioned above, the resulting system of differential equations (7) is formally a system with constant coefficients. To separate the variables in the direction of the guide, we will present the solving, complementary functions and load components in the form of expansions in Fourier series along the coordinate t

$$\tilde{X}(t,\gamma) = \sum_{k=0}^{K} \tilde{X}_k(\gamma)\cos\lambda_k t ; \quad \tilde{Y}(t,\gamma) = \sum_{k=1}^{K} \tilde{Y}_k(\gamma)\sin\lambda_k t ; \quad \lambda_k = \frac{2\pi k}{P}, \tag{8}$$

where $\tilde{X} = \{\sigma_\gamma, \tau_{t\gamma}, u_\gamma, u_t, q, \varphi_1^j, \varphi_4^j, \varphi_6\}$; $\tilde{Y} = \{\tau_{s\gamma}, u_s, \varphi_2^j, \varphi_3^j, \varphi_5, \varphi_7\}$.

After substituting into the system being solved (7) the series (8) and the corresponding boundary conditions, as well as separation of variables, we get to the solution system of ordinary differential equations with constant coefficients relative to the amplitude values of the series (8), which describes a one-dimensional boundary value problem

$$\frac{\partial \sigma_{\gamma,k}}{\partial \gamma} = \lambda_n \tau_{s\gamma,k} + \left(\frac{\nu}{1-\nu} - 1\right)\varphi_{1,k}^1 - \varphi_{4,k}^1 - \frac{E\nu}{1-\nu^2}\lambda_n \varphi_{1,k}^4 + \frac{E}{1-\nu^2}\left(\varphi_{4,k}^3 - \varphi_{1,k}^5\right);$$

$$\frac{\partial \tau_{s\gamma,k}}{\partial \gamma} = \frac{\nu}{1-\nu}\lambda_n \sigma_{\gamma,k} + \frac{E}{1-\nu^2}\lambda_n^2 u_{s,k} - \varphi_{1,k}^2 - \left(\frac{E\nu}{1-\nu^2} + \frac{E}{1-\nu^2}\right)\lambda_n \varphi_{4,k}^2 - \frac{E\nu}{1-\nu^2}\lambda_n \varphi_{1,k}^3 - \frac{E}{2(1+\nu)}\varphi_{6,k} ;$$

$$\frac{\partial \tau_{t\gamma,k}}{\partial \gamma} = \frac{E}{2(1+\nu)}\lambda_n^2 u_{t,k} - 2\varphi_{2,k}^1 + \left(\frac{E\nu}{1-\nu^2} + \frac{E}{2(1+\nu)}\right)\lambda_n \varphi_{3,k}^3 - \frac{E}{1-\nu^2}(\varphi_{7,k} + \varphi_{5,k}) - \frac{\nu}{1-\nu}\varphi_{3,k}^1 ; \tag{9}$$

$$\frac{\partial u_{\gamma,k}}{\partial \gamma} = \frac{1-\nu-2\nu^2}{(1-\nu)E}\sigma_{\gamma,k} + \frac{\nu}{1-\nu}(\lambda_n u_{s,k} - \varphi_{4,k}^2 - \varphi_{1,k}^3) ;$$

$$\frac{\partial u_{s,k}}{\partial \gamma} = \frac{2(1+\nu)}{E}\tau_{s\gamma,k} - \lambda_n u_{\gamma,k} ;$$

$$\frac{\partial u_{t,k}}{\partial \gamma} = \frac{2(1+\nu)}{E}\tau_{t\gamma,k} - \varphi_{3,k}^2 + \varphi_{2,k}^3$$

with boundary conditions

$$\sigma_{\gamma,k} = 0 ;\ \tau_{s\gamma,k} = 0 ;\ \tau_{t\gamma,k} = 0 \ \text{at}\ \gamma = -h/2;$$

$$\sigma_{\gamma,k} = q_0 ;\ \tau_{s\gamma,k} = 0 ;\ \tau_{t\gamma,k} = 0 \ \text{at}\ \gamma = h/2. \tag{10}$$

If the amplitude values of the complementary functions are known in the system of equations (9), then the problem is reduced to a two-point linear boundary value problem with constant coefficients, which is solved by the stable numerical method of discrete orthogonalization simultaneously for all harmonics of series (8). To establish the amplitude values of complementary functions, the method of their approximation by discrete Fourier series is used. For this purpose, at each step of integration, for a fixed value of the coordinate γ, in a number of points of the guide, taking into account the expressions (6), the tabular values of the complementary functions are calculated based on the current value of the solving functions. Using the standard procedure for determining the Fourier coefficients for the functions given by the table of their values, these coefficients are calculated, substituted into the solution system (9) and the next step of integration is made. At the beginning of the integration, the amplitude values of the complementary functions are determined based on the boundary conditions (10).

The reliability of the obtained results is ensured by the use of strict equations of the spatial theory of elasticity, the use of analytical methods of separation of variables and stable numerical method of discrete orthogonalization. The solution error may occur when applying the numerical method, as well as when calculating the amplitude values of the complementary functions specified by the table. It is possible to influence the accuracy of the obtained results by varying the number of integration and orthogonalization points when applying the numerical method and the number of points for determining tabular values of complementary functions and the corresponding number of terms contained in discrete Fourier series. Some examples of assessing the accuracy and reliability of the results for an isotropic non-thin shell when applying the considered approach are given in the paper [19].

Numerical results and their analysis

On the basis of the considered methodology, the problem of the stress state of hollow near-circular cylinders with a disturbed cross-sectional surface around one of the diameters was solved (Fig. 1). The reference surface is described by Pascal's limacon equation, which in the polar coordinate system has the form

$$\rho = a\cos\psi + l\,, \qquad (11)$$

where ρ is the polar radius; ψ is the polar angle in cross-section; $0 \le \psi \le 2\pi$; a – is the radius of the original circle; l is the distance at which the point is displaced along the radius vector.

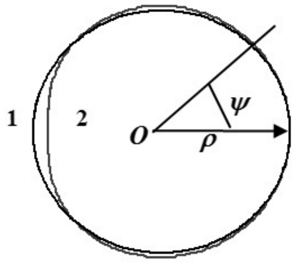


Fig. 1. Cross-section of the reference surface:
1 – circular, 2 – with a dent

Since the initial information about the shape of the reference surface (11) is given through the parameter ψ, which is different from t, the transition coefficient must be taken into account when calculating the corresponding derivatives

$$\frac{dt}{d\psi} = \left[\rho^2 + (\rho')^2\right]^{\frac{1}{2}} = \omega(\psi)\,.$$

Then, when defining certain additional functions, we get the following dependence for each function $V = V(\gamma, t(\psi))$

$$\frac{\partial V}{\partial t} = \frac{\partial V}{\partial \psi}\frac{1}{\omega(\psi)}\,,$$

and the radius of curvature of the reference surface in the cross-section has the form

$$R_\psi = \frac{\left[\rho^2 + (\rho')^2\right]^{\frac{3}{2}}}{\rho^2 + 2(\rho')^2 - \rho\rho''}\,.$$

For the geometric parameters of the cylinders, the following are selected: length $L = 60 \cdot l_0$, thickness $h = 6 \cdot l_0$, Pascal's limacon parameters $a = 5 \cdot l_0$; $20 \cdot l_0$; $l = 40 \cdot l_0$.

Cylinders are made of FGM, for which the relation between the elastic modulus E and Poisson's ratio ν with the corresponding parameters of the materials included in the composition are determined by the formulas [10]

$$E = (E_2 - E_1)V + E_1\,;\ \nu = (\nu_2 - \nu_1)V + \nu_1\,,$$

where E_1, E_2, ν_1, ν_2 are mechanical parameters of the first and second materials; V is the concentration of the second material depending on the thickness coordinate $-h/2 \le \gamma \le h/2$. During calculations, the power law of change of elastic properties $V = (\gamma/h + 0.5)^m$ is adopted. Mechanical properties of composition materials are: aluminum: $E = 70$ GPa; $\nu = 0.3$; SiC: $E = 427$ GPa; $\nu = 0.17$. A value is selected for the exponent parameter $m = 0.5$; 1; 2. For $m = 1$, the nature of the distribution of the elastic modulus and Poisson's ratio along the thickness of the shell is linear, which is violated for other values of the parameter m.

The results of solving the problem are shown in the average cross-section of the length of the cylinder ($L = 30 \cdot l_0$) for the distribution of fields of normal displacements u_γ in the Table and Fig. 2, as well as for the distribution of normal stress fields σ_s and σ_ψ in the form of graphs in Figs. 3–5. Due to symmetry, the results are given along the guide in the section $0 \le \psi \le \pi$.

The table shows the values of normal displacements u_γ for cylinders with two types of dents ($a = 5$, $a = 20$) depending on the power of change in the law of the elastic properties of the FGM in two sections of the guide ($\psi = 0$, $\psi = \pi$) on the inner ($\gamma = -3$), outer ($\gamma = 3$) and reference surfaces ($\gamma = 0$). Fig. 2 shows the graphs of the distribution of normal displacements along the guide cylinder for three values of the power of the law of variation of elastic properties. Solid lines correspond to displacements in cylinders for a=5, dashed – for a=20.

Table. Distribution of normal displacements u_γ

m	γ	u_γ/q_0			
		a=5		a=20	
		ψ=0	ψ=π	ψ=0	ψ=π
0.5	-3	9.532	9.735	11.800	26.738
	0	9.420	9.630	11.700	26.900
	3	9.334	9.538	11.574	26.669
1	-3	11.974	12.238	14.769	34.294
	0	11.800	12.100	14.600	34.500
	3	11.689	11.955	14.447	34.296
2	-3	15.888	16.235	19.578	45.245
	0	15.700	16.000	19.300	45.600
	3	15.468	15.819	19.109	45.344

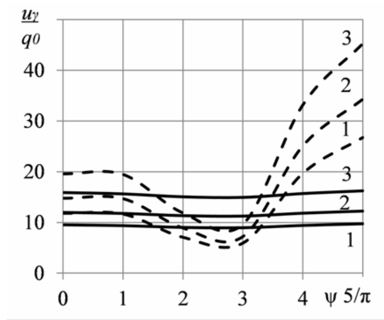


Fig. 2. Distribution of normal displacements u_γ along the cylinder guide:
1 – m=0,5; 2 – m=1; 3 – m=2

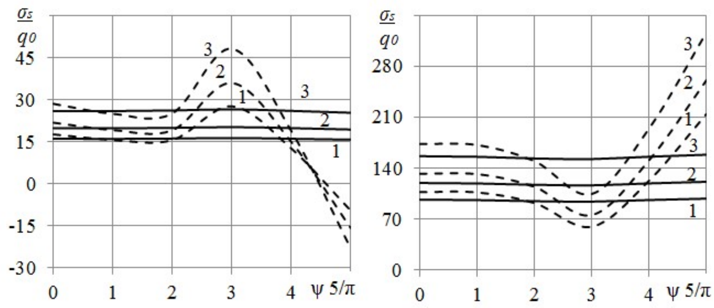
From the Table and Fig. 2, it can be seen that the normal displacements reach their maximum values in the zone of the maximum size of the dent (ψ=π). At the same time, in the case of a small dent (a=5), the displacement values differ insignificantly along the guide for all values of the parameter m. For cylinders with dent parameter a=20, displacement values in the zone of its maximum value (ψ=π) increase by approximately 2.3 times, compared to similar values in the cross-section (ψ=0). An increase in the dent leads to an increase in displacements by approximately 1.25 times in the section ψ=0 and by 2.8 times in the section ψ=π.

The non-linearity of the law of the distribution of elastic properties of the material causes a decrease in the value of displacements for m=0.5 and an increase in their value for m=2 by 1.3 times, compared to similar values in the case of a linear law for m=1.

Fig. 3 shows the distribution graphs of normal longitudinal stress fields σ_s along the guide cylinder on the inner (Fig. 3, a) and outer surfaces of the cylinder (Fig. 3, b). Line designations are the same as in Fig. 2.

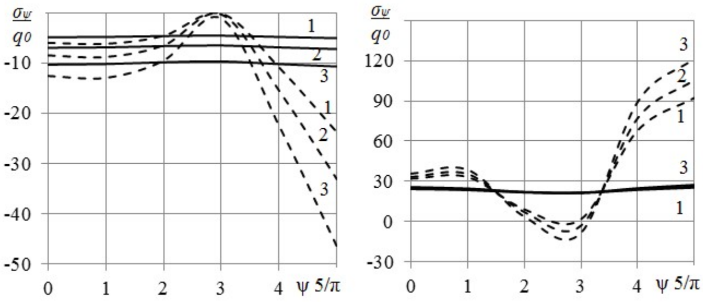
From the given graphs, it can be seen how the increase of the dent affects the stress distribution σ_s for three values of the power of the law of variation of elastic properties of FGM. Thus, for a minor dent, the distribution of longitudinal stresses is linear. When the size of the dent increases, the linearity is violated and the stresses σ_s reach their maximum amplitude values on the inner surface in the cross-section ψ=π/2, increasing by approximately 1.8 times, and on the outer surface in the cross-section ψ=π, increasing by 2.2 times. A change in the law of distribution of the elastic properties of the FGM has the same effect on the magnitude of stresses as it does for displacements.

Fig. 4, similarly to Fig. 3, shows the graphs of the distribution of normal circular stresses σ_ψ.



a b

Fig. 3. Distribution of longitudinal stresses σ_s along the cylinder guide:
1 – m=0.5; 2 – m=1; 3 – m=2;
a – on the inner surface, b – on the outer surface



a b

Fig. 4. Distribution of circular stresses σ_ψ along the guide cylinder:
1 – m=0.5; 2 – m=1; 3 – m=2;
a – on the inner surface, b – on the outer surface

From the graphs shown in Fig. 4, it can be seen that, in contrast to the longitudinal stresses, the circular stresses acquire their maximum amplitude value in the cross-section of the maximum dent size ($\psi=\pi$) on both the inner and outer surfaces for a=20. At the same time, when changing the power of the law of elastic properties of the material, the value of the maximum stresses increases by 15% for m=2 and decreases by 12% for m=0.5, compared to the linear law for m=1. For the dent parameter a=5, the distribution law of circular stresses is almost linear and at the same time, the law of variation of elastic properties in the section $\psi=\pi$ does not affect the magnitude of stresses σ_ψ.

Fig. 5 shows the graphs of the distribution of normal stresses σ_s and σ_ψ along the thickness of the cylinders, depending on the size of the dent and the power of the distribution law of the elastic properties of the FGM.

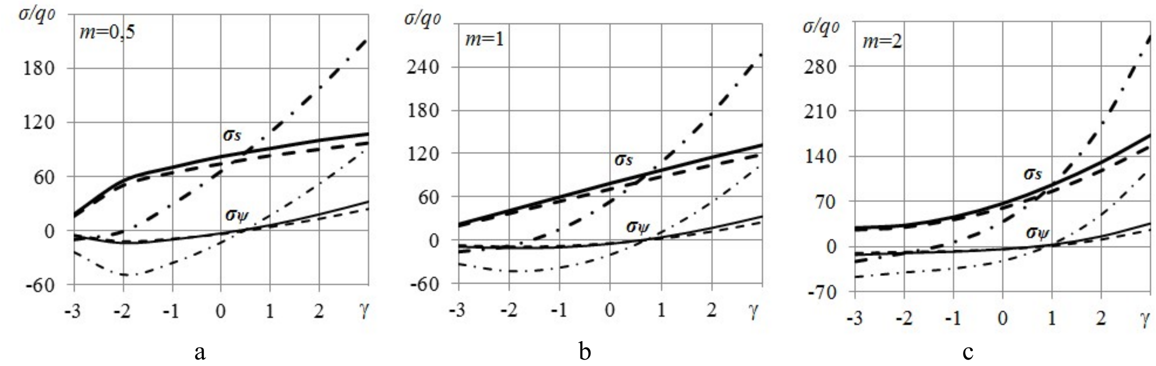


Fig. 5. Distribution of longitudinal σ_s and circular σ_ψ stresses along the cylinder thickness:
a – m=0.5; b – m=1; c – m=2

The solid lines correspond to the stress distribution curves in the section ψ=0 for the dent parameter a=20; dash-dotted lines – in the section $\psi=\pi$ for a=20; dashed lines – for a=5 in the cross-section $\psi=\pi$ (since for cylinders with a slight dent (a=5) the value of normal stresses differs little along the guide, Fig. 5 shows the curves in only one cross-section of the guide). Thick lines correspond to longitudinal stress distribution curves σ_s, thin lines correspond to circular ones σ_ψ.

From the graphs shown in Fig. 5, it can be seen that for all values of the power of the law of variation of the elastic properties of the material, the longitudinal stresses that acquire their maximum values in the zone of the maximum dent size in the section $\psi=\pi$ are predominant. At the same time, for the linear law of change in the power of elastic properties (m=1) for a small dent (a=5) and the parameter a=20 in the section ψ=0, there is a linear law of longitudinal stress distribution. In addition, in the section ψ=0, the values of both circumferential and longitudinal stresses are close to each other for both forms of cylinders and all values of the power law of the elastic parameters of the FGM. The change in the power of the law of the elastic properties of FGM leads to an increase in the maximum circular stresses by approximately 15% for m=2 and a decrease in their value by approximately 13% for m=0.5 compared to the corresponding values for m=1.

Conclusions

1. In the spatial formulation, the problem of the stress state of hollow near-circular cylinders, which are under the action of a distributed load, under certain boundary conditions at the ends made of FGM, is solved. When solving the problem, an approach based on a numerical analytical technique was used, which allows reducing the original three-dimensional boundary value problem for a system of partial differential equations with variable coefficients to a one-dimensional boundary value problem for a system of ordinary differential equations with constant coefficients, which is solved by stable numerical method.

2. Pascal's limacon equation was used to describe the cross-section of the reference surface of the studied cylinders. Two forms with a dent around one of the cylinder diameters are considered. An analysis of the properties of the stress state of the mentioned cylinders was carried out depending on the size of the dent and the power law indicator of the change in the elastic properties of the FGM along the thickness.

3. It was established that the nonlinearity in the distribution of the elastic properties of the material leads to a decrease in the value of the maximum normal displacements and longitudinal stresses for m=0.5

and an increase for m=2, compared to the corresponding values for m=1 by 1.3 times. Accordingly, the maximum values of circular stresses decrease by 13% for m=0.5 and increase by 15% for m=2, compared to the linear law of changes in the elastic properties of the material for m=1.

4. For a small dent (a=5), the values of normal stresses almost do not differ in magnitude along the direction of the cylinder. The presence of a more significant dent (a=20) leads to a violation of the linear law of the distribution of normal stresses along the guide, while their values increase by almost 2.8 times in the section $\psi=\pi$, compared to the section ψ=0.

The results obtained in the paper can be used when calculating the strength of structural elements and machine parts of a similar type.

References

1. Filatov, H. V. (2020). Optimal design of smooth shells both with and without taking into account initial imperfections. *Journal of Mechanical Engineering – Problemy Mashynobuduvannia*, vol. 23, no. 1, pp. 58–63. https://doi.org/10.15407/pmach2020.01.058.
2. Filatov, H. V. (2021) Optimal design of single-layered reinforced cylindrical shells. *Journal of Mechanical Engineering – Problemy Mashynobuduvannia*, vol. 24, no. 1, pp. 58–64. https://doi.org/10.15407/pmach2021.01.058.
3. Revenko, V. P. (2023). Solutions of three-dimensional problems of the theory of elasticity for orthotropic solids. *Journal of Mathematical Sciences*, vol. 273, pp. 92–100. https://doi.org/10.1007/s10958-023-06486-y.
4. Marchuk, A. V., Reneiskaya, S. V., & Leshchuk, O. N. (2020). Three-dimensional analysis of the free vibrations of layered composite plates based on the semianalytic finite-element method. *International Applied Mechanics*, vol. 56, iss. 4, pp. 481–497. https://doi.org/10.1007/s10778-020-01031-9.
5. Nguyen-Sy, T., Vu, M. N., Nguyen, T. K., Tran-Le, A.-D., Thai, M.-Q., & Nguyen, T.-T. (2021). Poroelastic response of a functionally graded hollow cylinder under an asymmetric loading condition. *Archive of Applied Mechanics*, vol. 91, iss. 1, pp. 3171–3189. https://doi.org/10.1007/s00419-021-01958-6.
6. Zhang, X., He, Y., Li, Z., Yan, R., & Chen, X. (2020). Static and dynamic analysis of cylindrical shell by different kinds of B-spline wavelet finite elements on the interval. *Engineering with Computers*, vol. 36, iss. 4, pp. 1903–1914. https://doi.org/10.1007/s00366-019-00804-2.
7. Gholami, R., Darvizeh, A., Ansari, R., & Pourashraf, T. (2018). Analytical treatment of the size-dependent nonlinear postbuckling of functionally graded circular cylindrical micro-/nano-shells. *Iranian Journal of Science and Technology, Transactions of Mechanical Engineering*, vol. 42, iss. 2, pp. 85–97. https://doi.org/10.1007/s40997-017-0080-6.
8. Najibi, A., Alizadeh, P., & Ghazifard, P. (2021). Transient thermal stress analysis for a short thick hollow FGM cylinder with nonlinear temperature-dependent material properties. *Journal of Thermal Analysis and Calorimetry*, vol. 146, iss. 5, pp. 1971–1982. https://doi.org/10.1007/s10973-020-10442-2.
9. Kushnir, R. M. & Zhydyk, U. V. (2019) Temperature stresses in a functionally graded cylindrical shell. *Materials Science*, vol. 54, iss. 5, pp. 666–677. https://doi.org/10.1007/s11003-019-00231-0.
10. Grigorenko, A. Ya., Efimova, T. L., & Korotkikh, Yu. A. (2015). Free axisymmetric vibrations of cylindrical shells made of functionally graded materials. *International Applied Mechanics*, vol. 51, iss. 6, pp. 654–663. https://doi.org/10.1007/s10778-015-0722-6.
11. Kurpa, L. V., Shmatko, T. V., & Linnik, A. B. (2023). Analysis of stability and vibrations of porous power and sigmoid functionally graded sandwich plates by the R-functions method. *Journal of Mechanical Engineering – Problemy Mashynobuduvannia*, vol. 26, no. 4, pp. 38–49. https://doi.org/10.15407/pmach2023.04.038.
12. Fateeva, Yu. & Gristchak, V. Z. (2017). An approximate analytical approach for nonlinear thermodynamic problem of FGM shallow spherical shells with time dependent parameters. Proceeding of the 7th International Conference on Mechanics and Materials in Design (Albufeira, Portugal, June 11–15, 2017), pp. 1109–1110.
13. Shi, P., Xie, J., & Hao, S. (2021). Static response of functionally graded piezoelectric-piezomagnetic hollow cylinder/spherical shells with axial/spherical symmetry. *Journal of Mechanical Science and Technology*, vol. 35, iss. 4, pp. 1583–1596. https://doi.org/10.1007/s12206-021-0322-9.
14. Khorshidi, M. A. & Soltani, D. (2020). Analysis of functionally graded thick-walled cylinders with high order shear deformation theories under non-uniform pressure. *SN Applied Sciences*, vol. 2, article no. 1362. https://doi.org/10.1007/s42452-020-3179-0.
15. Salehi, A. & Ahmadi, I. (2022). Transient thermal and mechanical stress analysis of 2D-functionally graded finite cylinder: A Truly Meshless formulation. *Iranian Journal of Science and Technology, Transactions of Mechanical Engineering*, vol. 46, iss. 3, pp. 573–598. https://doi.org/10.1007/s40997-021-00432-6.
16. Grigorenko, Ya. M., Vasylenko, A. T., & Pankratova, N. D. (1985). *Statika anizotropnykh tolstostennykh obolochek* [Statics of anisotropic thick-walled shells]. Kyiv: Vyshcha shkola, 190 p. (in Russian).

17. Grigorenko, Ya. M. & Rozhok, L. S. (2014). Applying discrete Fourier series to solve problems of the stress state of hollow noncircular cylinders. *International Applied Mechanics*, vol. 50, iss. 2, pp. 105–127. https://doi.org/10.1007/s10778-014-0616-z.
18. Timoshenko, S. P. & Goodier, J. N. (1951). Theory of elasticity. New York: McGraw-Hill Book Company, 519 p.
19. Grigorenko, Ya .M. & Rozhok, L. S. (2023). On the equilibrium of nonthin cylindrical shells with a dent. *Journal of Mathematical Sciences*, vol. 272, no. 2, pp. 80–92. https://doi.org/10.1007/s10958-023-06401-5.

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Чисельний аналіз напруженого стану порожнистих циліндрів, близьких до кругових, виготовлених із функціонально-градієнтних матеріалів

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Порожнисті циліндри кругового поперечного перерізу, виготовлені з функціонально-градієнтних матеріалів, застосовуються в багатьох галузях господарювання як елементи конструкцій та деталі машин і агрегатів. Під час виготовлення або у процесі експлуатації таких циліндрів форма їх поперечного перерізу може певною мірою відрізнятися від кругової. Стаття присвячена розв'язанню задачі про рівновагу неоднорідних по товщині порожнистих циліндрів, близьких до кругових, у просторовій постановці за певних граничних умов на торцях. Поперечний переріз розглядуваних циліндрів описано за допомогою рівняння равлика Паскаля. Для матеріалу обрано двокомпонентний неперервно-неоднорідний матеріал, пружні властивості якого, що характеризують модуль Юнга та коефіцієнт Пуассона, можуть бути визначені за допомогою концентрації матеріалів композиції вздовж товщини. Метою роботи є проведення чисельного аналізу напруженого стану циліндрів даного класу залежно від закону зміни пружних властивостей матеріалу. Розв'язок задачі базується на редукції вихідної тривимірної крайової задачі для системи рівнянь у частинних похідних зі змінними коефіцієнтами до одномірної крайової задачі для системи звичайних диференціальних рівнянь зі сталими коефіцієнтами більш високого порядку. При цьому застосовується аналітичний метод відокремлення змінних у двох координатних напрямках, із паралельним використанням апроксимації функцій дискретними рядами Фур'є. Одномірна крайова задача розв'язується за допомогою стійкого чисельного методу дискретної ортогоналізації. Проведено аналіз напруженого стану розглядуваних циліндрів залежно від величини вм'ятини, що має місце в околі одного з діаметрів поверхні відліку, і закону зміни пружних характеристик матеріалу. Показано, що нелінійність закону розподілу пружних характеристик вздовж товщини призводить до збільшення/зменшення максимальних значень нормальних переміщень і поздовжніх напружень в 1,3 рази, порівняно з лінійним законом. При цьому збільшення величини вм'ятини призводить до зростання значень як переміщень, так і нормальних напружень у 2–3 рази в зоні максимальної величини вм'ятини, порівняно з діаметрально протилежною зоною. Отримані в роботі результати можуть бути використані при розрахунках на міцність елементів конструкцій та деталей машин подібного типу.

***Ключові слова**: дискретні ряди Фур'є, напружений стан, порожнисті циліндри, просторова теорія пружності, рівняння равлика Паскаля, функціонально-градієнтні матеріали.*

Література

1. Filatov H. V. Optimal design of smooth shells both with and without taking into account initial imperfections. *Journal of Mechanical Engineering – Problemy Mashynobuduvannia*. 2020. Vol. 23. No. 1. P. 58–63. https://doi.org/10.15407/pmach2020.01.058.
2. Filatov H. V. Optimal design of single-layered reinforced cylindrical shells. *Journal of Mechanical Engineering – Problemy Mashynobuduvannia*. 2021. Vol. 24. No. 1. P. 58–64. https://doi.org/10.15407/pmach2021.01.058.
3. Ревенко В. П. Розв'язки тривимірних задач теорії пружності для ортотропних тіл. *Математичні методи та фізико-механічні поля*. 2020. Т. 63. № 3. С. 78–84.
4. Марчук А. В., Ренейская С. В., Лещук О. Н. Трехмерный анализ свободных колебаний слоистых композитных плит на основе полуаналитического метода конечных элементов. *Прикладная механика*. 2020. Т. 56. № 4. С. 97–116.

5. Nguyen-Sy T., Vu M. N., Nguyen T. K., Tran-Le A.-D., Thai M.-Q., Nguyen T.-T. Poroelastic response of a functionally graded hollow cylinder under an asymmetric loading condition. *Archive of Applied Mechanics*. 2021. Vol. 91. Iss. 1. P. 3171–3189. https://doi.org/10.1007/s00419-021-01958-6.
6. Zhang X., He Y., Li Z., Yan R., Chen X. Static and dynamic analysis of cylindrical shell by different kinds of B-spline wavelet finite elements on the interval. *Engineering with Computers*. 2020. Vol. 36. Iss. 4. P. 1903–1914. https://doi.org/10.1007/s00366-019-00804-2.
7. Gholami R., Darvizeh A., Ansari R., Pourashraf T. Analytical treatment of the size-dependent nonlinear postbuckling of functionally graded circular cylindrical micro-/nano-shells. *Iranian Journal of Science and Technology, Transactions of Mechanical Engineering*. 2018. Vol. 42. Iss. 2. P. 85–97. https://doi.org/10.1007/s40997-017-0080-6.
8. Najibi A., Alizadeh P., Ghazifard P. Transient thermal stress analysis for a short thick hollow FGM cylinder with nonlinear temperature-dependent material properties. *Journal of Thermal Analysis and Calorimetry*. 2021. Vol. 146. Iss. 5. P. 1971–1982. https://doi.org/10.1007/s10973-020-10442-2.
9. Кушнір Р. М., Жидик У. В. Температурні напруження у функціонально-градієнтній циліндричній оболонці. *Фізико-хімічна механіка матеріалів*. 2018. Т. 54. № 5. С. 56–65.
10. Григоренко А. Я., Ефимова Т. Л., Коротких Ю. А. Осесимметричные свободные колебания цилиндрических оболочек из непрерывно неоднородных материалов. *Прикладная механика.* 2015. Т. 51. № 6. С. 61–71.
11. Kurpa L. V., Shmatko T. V., Linnik A. B. Analysis of stability and vibrations of porous power and sigmoid functionally graded sandwich plates by the R-functions method. *Journal of Mechanical Engineering – Problemy Mashynobuduvannia*. 2023. Vol. 26. No. 4. P. 38–49. https://doi.org/10.15407/pmach2023.04.038.
12. Fateeva Yu., Gristchak V. Z. An approximate analytical approach for nonlinear thermodynamic problem of FGM shallow spherical shells with time dependent parameters. Proceeding of the 7th International Conference on Mechanics and Materials in Design (Albufeira, Portugal, June 11–15, 2017), 2017. P. 1109–1110.
13. Shi P., Xie J., Hao S. Static response of functionally graded piezoelectric-piezomagnetic hollow cylinder/spherical shells with axial/spherical symmetry. *Journal of Mechanical Science and Technology*. 2021. Vol. 35. Iss. 4. P. 1583–1596. https://doi.org/10.1007/s12206-021-0322-9.
14. Khorshidi M. A., Soltani D. Analysis of functionally graded thick-walled cylinders with high order shear deformation theories under non-uniform pressure. *SN Applied Sciences*. 2020. Vol. 2. Article no. 1362. https://doi.org/10.1007/s42452-020-3179-0.
15. Salehi A., Ahmadi I. Transient thermal and mechanical stress analysis of 2D-functionally graded finite cylinder: A Truly Meshless formulation. *Iranian Journal of Science and Technology, Transactions of Mechanical Engineering*. 2022. Vol. 46. Iss. 3. P. 573–598. https://doi.org/10.1007/s40997-021-00432-6.
16. Григоренко Я. М., Василенко А. Т., Панкратова Н. Д. (1985). Статика анизотропных толстостенных оболочек. Киев: Вища школа, 1985. 190 с.
17. Григоренко Я. М., Рожок Л. С. Применение дискретных рядов Фурье к решению задач о напряженном состоянии полых цилиндров с некруговым поперечным сечением. *Прикладная механика*. 2014. Т. 50. № 2. С. 3–26.
18. Тимошенко С. П., Гудьер Дж. Теория упругости. М.: Наука, 1975. 576 с.
19. Григоренко Я. М., Рожок Л. С. Про рівновагу нетонких циліндричних оболонок із вм'ятиною. *Математичні методи та фізико-механічні поля*. 2020. Т. 63. № 2. С. 72–82.