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INVERSE PROBLEM OF FRACTURE MECHANICS FOR A PERFORATED STRINGER PLATE

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To determine an optimal contour of holes for a perforated stringer plate weakened by a periodic system of cracks, an inverse problem of fracture mechanics is considered. It is assumed that the material of the plate is elastic or elastic-plastic. The stiffeners (stringers) are symmetrically riveted to the plate. The perforated plate is uniformly stretched at infinity along the stringers. It is assumed that rectilinear cracks are located near the contours of the holes and are perpendicular to the riveted stiffeners. The solution of the formulated inverse problem is based on the principle of equal strength. The optimal shape of the holes satisfies two conditions: the condition for the absence of stress concentration on the hole surface and the condition for the zero stress intensity factors in the vicinity of the crack tips. The unknown contour of holes is looked for in the class of contours close to circular. The action of the stiffeners is replaced by unknown equivalent concentrated forces at the points of their connection with the plate. The sought-for functions (the stresses, displacements, concentrated forces and stress intensity factors) are looked for in the form of expansion in small parameter. The solution to the problem is sought using the apparatus of the theory of analytic functions and the theory of singular integral equations, then the conditional extremum problem is solved. As a result, a closed system of algebraic equations is obtained, which allows to minimize the stress state on the contours of holes and stress intensity factors in the vicinity of the crack tips. The obtained system of algebraic equations allows to determine the form of equal strength contour of holes, the stress-strain state of the perforated stringer plate and also the optimal value of the tangential stress.

***Keywords**: perforated plate, stringers, cracks, optimal contour, equi-strong holes.*

Introduction

To prevent the fracture of the perforated plate, it is very important to know the optimal contour of the holes [1–9]. Hole contour without any areas preferred for brittle failure or plastic deformations (equal strength contour) is optimal [10, 11]. However, optimal contour of the hole must also meet the conditions of immobility of the cracks present in the body. Recently, some problems have been considered for finding the optimal contour of the hole, taking into account the presence of cracks in the body [12–19].

The goal of this paper is to find optimal contour of holes for a perforated plate weakened by linear cracks near the contours of the holes and stiffened with a regular system of stiffness ribs.

Problem statement

We consider an elastic plate weakened by an infinite row of identical holes. The plate was stiffened by a regular system of stringers and is subjected to homogeneous stretching along the stringers by the stress $\sigma_y^\infty = \sigma_0$ (Fig. 1). There are rectilinear cracks near the contours. The plane stress state is realized in the contour. It is accepted that the stress state of the stringers is uniaxial. We assume that the plate and stringers interact in the same plane, and only at the stiffening points the stringers are not subjected to bending and are not weakened by the setting of attachment points.

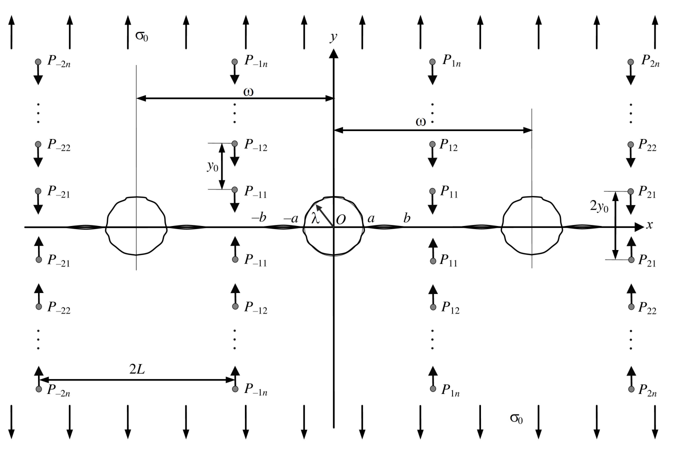

Fig. 1. Design scheme of the inverse problem

The radius of attachment points (coupling area) is small compared to their step and other characteristic sizes. The action of attachment points is modeled with the action of concentrated forces applied at the centers of attachment points. Accordingly, the action of stringers is replaced by unknown equivalent forces P_{mn} applied at the attachment points of stringers and plates.

The problem is in determining the equal strength contour of holes, under which the cracks will not grow, and also the stress-strain state of a perforated built-up plate and the sizes of concentrated forces P_{mn}.

The boundary conditions of the problem are of the form:

– on the unknown contours L_m (m=0, 1, 2, …) of the holes

$$\sigma_n = 0\,;\quad \tau_{nt} = 0\,;\quad \sigma_t = \sigma_* = \text{const}\,; \tag{1}$$

– on the crack faces

$$\sigma_y = 0\,;\quad \tau_{xy} = 0\,;\quad a + m\omega \le |x| \le b + m\omega\,.$$

Here t and n are a tangent and normal to the hole contour. The value σ_* for elastic plate should be determined, whereas for elastoplastic plate we accept the plasticity condition [20]

$$f(\sigma_n, \sigma_t, \tau_{nt}) = 0\,, \tag{2}$$

where f is the given function. It is assumed that plastic area appears for the first time on the contour hole and coves the entire contour at once, does not penetrate deep into it. It is known [10, 11] that such a body is most durable in the sense of uniform distribution of stresses over all points of the hole contour.

It is required to find such a form of holes under which the crack will not grow, and tangential initial stress acting on the hole contours will be constant. Note that it follows from condition (2) that the stress σ_t is constant everywhere on the tensile boundary of hole contour and equals the material strength. According to Irwin-Orowan theory of quasi-brittle failure, the stress intensity factors characterize the stress state in the vicinity of crack tips. Therefore, we require that condition (1) will be satisfied on the hole contours, and the condition

$$K_{\mathrm{I}}^{a+m\omega} = 0\,;\quad K_{\mathrm{I}}^{b+m\omega} = 0\,, \tag{3}$$

will be satisfied in the vicinity of the crack tips. Here $K_{\mathrm{I}}^{a+m\omega}$, $K_{\mathrm{I}}^{b+m\omega}$ are the stress intensity factors in the vicinity of crack tips. Since the cracks were located symmetrically, $K_{\mathrm{I}}^{a+m\omega} = K_{\mathrm{I}}^{-a-m\omega}0$, $K_{\mathrm{I}}^{b+m\omega} = K_{\mathrm{I}}^{-b-m\omega}$.

The solution of the boundary value problem

We will look for the unknown contour L_m (m=0, 1, 2, …) of holes as close to the circular one as possible.

We represent it in the form $r = \rho(\theta) = \lambda + \varepsilon H(\theta)$, where $\varepsilon = R_{\max}/\lambda$ is a small parameter, $R_{\max}$ is the greatest height of the contour profile L_m of the hole from the circle r=λ. The function $H(\theta)$ will be found in the process of solving the problem. Without loss of generality of the considered problem, it is accepted that the sought-for function $H(\theta)$ is symmetric about the coordinate axes and can be represented in the form of Fourier series $H(\theta) = \sum_{k=1}^{\infty} d_{2k} \cos 2k\theta$.

We will look for the sought-for functions (the stresses, displacements, concentrated forces P_{mn} and stress intensity factors K_{I}) in the form of expansion in small parameter

$$\sigma_n = \sigma_n^{(0)} + \varepsilon\sigma_n^{(1)} + \ldots\,;\quad \sigma_t = \sigma_t^{(0)} + \varepsilon\sigma_t^{(1)} + \ldots\,;\quad \tau_{nt} = \tau_{nt}^{(0)} + \varepsilon\tau_{nt}^{(1)} + \ldots\,;$$

$$u = u^{(0)} + \varepsilon u^{(1)} + \ldots\,;\quad v = v^{(0)} + \varepsilon v^{(1)} + \ldots\,;$$

$$P_{mn} = P_{mn}^{(0)} + \varepsilon P_{mn}^{(1)} + \ldots\,;\quad K_{\mathrm{I}} = K_{\mathrm{I}}^{(0)} + \varepsilon K_{\mathrm{I}}^{(1)} + \ldots\,,$$

in which we neglect for simplicity the terms containing ε of degree greater than one.

Each of approximations satisfies the system of differential equations of the plane problem of elasticity theory.

We obtain the values of stress tensor components for r=ρ(θ) by expanding in series the expressions for stresses in the vicinity of r=λ.

Solution of the problem in zero approximation

Taking into account the known formulas [21] for the stress components σ_n and τ_{nt}, the boundary conditions of the problem take the following form:

Allowing for known formulas [21] for stress components σ_n and τ_{nt} the boundary conditions, of the problem take the following form:

– in the zero approximation

on the contour $r=\lambda$

$$\sigma_r^{(0)} = 0\,; \quad \tau_{r\theta}^{(0)} = 0\,, \tag{4}$$

on the crack faces

$$\sigma_y^{(0)} = 0\,; \quad \tau_{xy}^{(0)} = 0\,;\; a + m\omega \le |x| \le b + m\omega\,, \tag{5}$$

– in the first approximation

on the contour $r=\lambda$

$$\sigma_r^{(1)} = N\,; \quad \tau_{r\theta}^{(1)} = T\,, \tag{6}$$

on the crack faces

$$\sigma_y^{(1)} = 0\,; \quad \tau_{xy}^{(1)} = 0\,;\; a + m\omega \le |x| \le b + m\omega\,. \tag{7}$$

Here $N = -H(\theta)\dfrac{\partial \sigma_r^{(0)}}{\partial r} + \dfrac{2}{\lambda}\tau_{r\theta}^{(0)}\dfrac{dH(\theta)}{d\theta}$; $T = \dfrac{1}{\lambda}(\sigma_\theta^{(0)} - \sigma_r^{(0)})\dfrac{dH(\theta)}{d\theta} - H(\theta)\dfrac{\partial \tau_{r\theta}^{(0)}}{\partial r}$.

Since the boundary conditions and geometry of the domain D occupied by the plate material possess the symmetry with respect to coordinate axes, the stresses are periodic functions of period ω. Based on the Kolosov-Muskhelishvili formulas [21] and boundary conditions on the hole contours and crack faces, the problem (4)–(6) in the zero approximation is reduced to determining two analytic functions $\Phi^{(0)}(z)$ and $\Psi^{(0)}(z)$ from the conditions

$$\Phi^{(0)}(\tau) + \overline{\Phi^{(0)}(\tau)} - e^{2i\theta}\left[\bar{\tau}\Phi'^{(0)}(\tau) + \Psi^{(0)}(\tau)\right] = 0 \quad \text{for } \tau = \lambda e^{i\theta} + m\omega\,; \tag{8}$$

$$\Phi^{(0)}(x) + \overline{\Phi^{(0)}(x)} + x\overline{\Phi'^{(0)}(x)} + \overline{\Psi^{(0)}(x)} = 0\,; \quad a + m\omega \le |x| \le b + m\omega\,. \tag{9}$$

We will look for the solution of problem (8)–(9) in the form

$$\Phi^{(0)}(z) = \Phi_0^{(0)}(z) + \Phi_1^{(0)}(z) + \Phi_2^{(0)}(z)\,; \quad \Psi^{(0)}(z) = \Psi_0^{(0)}(z) + \Psi_1^{(0)}(z) + \Psi_2^{(0)}(z)\,. \tag{10}$$

The potentials $\Phi_0^{(0)}(z)$ and $\Psi_0^{(0)}(z)$ determine stress and strain fields in the solid plate under the action of tensile stresses σ_0 and the system of concentrated forces $P_{mn}^{(0)}$ and are of the form

$$\Phi_0^{(0)}(z) = -\frac{1}{4}\sigma_0 - \frac{i}{2\pi h(1+\kappa)}\sum_{mn}{}' P_{mn}^{(0)}\left[\frac{1}{z - mL + iny_0} - \frac{1}{z - mL - iny_0}\right];$$

$$\Psi_0^{(0)}(z) = \frac{1}{2}\sigma_0 - \frac{i\kappa}{2\pi h(1+\kappa)}\sum_{mn}{}' P_{mn}^{(0)}\left[\frac{1}{z - mL + iny_0} - \frac{1}{z - mL - iny_0}\right] +$$

$$+ \frac{i}{2\pi h(1+\kappa)}\sum_{mn}{}' P_{mn}^{(0)}\left[\frac{mL - iny_0}{(z - mL + iny_0)^2} - \frac{mL + iny_0}{(z - mL - iny_0)^2}\right], \tag{11}$$

where $\kappa=(3-\nu)/(1+\nu)$; ν is the Poisson ratio of the plate material; the prime at the sum sign shows that when summing, the index $m=n=0$ is excluded.

The functions $\Phi_1^{(0)}(z)$ and $\Psi_1^{(0)}(z)$ corresponding to the unknown normal displacements along the cracks are sought in the explicit form

$$\Phi_1^{(0)}(z) = \frac{1}{2\omega}\int_{L'} g^{(0)}(t)\,\mathrm{ctg}\frac{\pi}{\omega}(t - z)\,dt\,; \quad \Psi_1^{(0)}(z) = -\frac{\pi z}{2\omega^2}\int_{L'} g^{(0)}(t)\sin^{-2}\frac{\pi}{\omega}(t - z)\,dt\,, \tag{12}$$

here $L'=[-a, -b]+[a, b]$; the function $g^{(0)}(x)$ characterizes the derivative of the crack faces opening.

To find the complex potentials $\Phi_2^{(0)}(z)$ and $\Psi_2^{(0)}(z)$, we represent conditions (8)–(9) in the form

$$\Phi_2^{(0)}(\tau)+\overline{\Phi_2^{(0)}(\tau)}+\left[\overline{\tau}\Phi_2'^{(0)}(\tau)+\Psi_2^{(0)}(\tau)\right]e^{2i\theta}=f_1(\theta)+if_2(\theta)+\phi_1(\theta)+i\phi_2(\theta)\,; \qquad (13)$$

$$f_1(\theta)+if_2(\theta)=-\Phi_0^{(0)}(\tau)-\overline{\Phi_0^{(0)}(\tau)}+\left[\overline{\tau}\Phi_0'^{(0)}(\tau)+\Psi_0^{(0)}(\tau)\right]e^{2i\theta}\,; \qquad (14)$$

$$\phi_1(\theta)+i\phi_2(\theta)=-\Phi_1^{(0)}(\tau)-\overline{\Phi_1^{(0)}(\tau)}+\left[\overline{\tau}\Phi_1'^{(0)}(\tau)+\Psi_1^{(0)}(\tau)\right]e^{2i\theta}\,. \qquad (15)$$

We will look for the potentials $\Phi_2^{(0)}(z)$ and $\Psi_2^{(0)}(z)$ in the following form

$$\Phi_2^{(0)}(z)=\alpha_0^{(0)}+\sum_{k=0}^{\infty}\alpha_{2k+2}^{(0)}\frac{\lambda^{2k+2}\rho^{(2k)}(z)}{(2k+1)!}\,;\quad \Psi_2^{(0)}(z)=\sum_{k=0}^{\infty}\beta_{2k+2}^{(0)}\frac{\lambda^{2k+2}\rho^{(2k)}(z)}{(2k+1)!}+\sum_{k=0}^{\infty}\alpha_{2k+2}^{(0)}\frac{\lambda^{2k+2}S^{(2k)}(z)}{(2k+1)!}\,. \qquad (16)$$

Here $\rho(z)=\left(\frac{\pi}{\omega}\right)^2\sin^{-2}\left(\frac{\pi}{\omega}z\right)-\frac{1}{3}\left(\frac{\pi}{\omega}\right)^2$; $S(z)=\sum_{m,n}{}'\left(\frac{P_m}{(z-P_m)^2}-\frac{2z}{P_m^2}-\frac{1}{P_m}\right)$.

Relations (10)–(12) and (16) determine the class of symmetric problems with periodic stress distribution. Taking into account the symmetry with regard to coordinate axes, we have

$$\operatorname{Im}\alpha_{2k+2}^{(0)}=0\,;\quad \operatorname{Im}\beta_{2k+2}^{(0)}=0\,;\quad k=0,\,1,\,2.$$

From the condition of constancy of the principal vector of forces acting on the arc connecting two congruent points in the domain D we have

$$\alpha_0^{(0)}=\pi^2\beta_2^{(0)}\lambda^2/24\,.$$

The unknown coefficients $\alpha_{2k+2}^{(0)}$ and $\beta_{2k+2}^{(0)}$ must be determined from the boundary condition (13). We will consider that $f_1(\theta)+if_2(\theta)$ and $\phi_1(\theta)+i\phi_2(\theta)$ on the contour $|\tau|=\lambda$ expand in Fourier series. By symmetry, these series are of the form

$$f_1(\theta)+if_2(\theta)=\sum_{k=-\infty}^{\infty}A_{2k}e^{2ik\theta}\,;\quad \operatorname{Im}A_{2k}=0\,;\quad A_{2k}=\frac{1}{2\pi}\int_0^{2\pi}\left(f_1(\theta)+if_2(\theta)\right)e^{-2ik\theta}d\theta\,; \qquad (17)$$

$$\phi_1(\theta)+i\phi_2(\theta)=\sum_{k=-\infty}^{\infty}B_{2k}e^{2ik\theta}\,;\quad \operatorname{Im}B_{2k}=0\,;\quad B_{2k}=\frac{1}{2\pi}\int_0^{2\pi}\left(\phi_1(\theta)+i\phi_2(\theta)\right)e^{-2ik\theta}d\theta\,. \qquad (18)$$

Having substituted the relation (14) in (17), after calculating the integrals by means of residue theory, we find

$$A_0=-\frac{1}{2}\sigma_0+\frac{1}{\pi h(1+\kappa)}\sum_{m,n}{}'P_{mn}^{(0)}\frac{2ny_0}{C\overline{C}}\,;$$

$$A_2=\frac{1}{2}\sigma_0-\frac{1}{\pi h(1+\kappa)}\sum_{m,n}{}'P_{mn}^{(0)}\left(\frac{\lambda^2\sin 3\phi_3}{\rho_1^3}+\frac{\kappa\sin\phi_3}{\rho_1}-\frac{\sin 3\phi_3}{\rho_1}\right);$$

$$A_{2k}=\frac{1}{\pi h(1+\kappa)}\left[\sum_{m,n}{}'P_{mn}^{(0)}\left(\frac{\lambda^2\sin(2k+1)\phi_3}{\rho_1^{2k+1}}+\frac{(-2)(-3)\ldots(-2k)\lambda^{2k}\sin(2k+1)\phi_3}{(2k-1)!\rho_1^{2k+1}}-\frac{\sin 3\phi_3}{\rho_1}\right)-\right.$$
$$\left.-\frac{\kappa\lambda^{2k-2}\sin(2k-1)\phi_3}{\rho_1^{2k-1}}+\frac{(-2)(-3)\ldots(1-2k)\lambda^{2k-2}\sin(2k+1)\phi_3}{(2k-1)!\rho_1^{2k+1}}\right];\ k=2,\,3,\,\ldots;$$

$$A_2=\frac{1}{\pi h(1+\kappa)}\sum_{m,n}{}'P_{mn}^{(0)}\frac{\lambda^{2k}\sin(2k+1)\phi_3}{\rho_1^{2k+1}}\,;\ k=1,\,2,\,\ldots,$$

where $C=mL+iny_0$; $\rho_1=\sqrt{C\overline{C}}$; $\phi_3=\operatorname{arctg}\frac{ny_0}{mL}$.

And also, when substituting (15) in (18) and calculating the integral by means of the residue method, we obtain

$$B_{2k} = -\frac{1}{2\omega}\int_{L'} g^{(0)}(t) f_{2k}(t)dt\,,$$

where

$$f_0(t) = 2\gamma(t)\,;\ f_2(t) = -\frac{\lambda^2}{2}\gamma^{(2)}(t)\,;\ \gamma(t) = \mathrm{ctg}\frac{\pi}{\omega}t\,;$$

$$f_{2k}(t) = -\frac{\lambda^{2k}(2k-1)}{(2k)!}\gamma^{(2k)}(t) + \frac{\lambda^{2k-2}}{(2k-3)!}\gamma^{(2k-2)}(t)\,;\ k=2,\,3,\,\ldots;$$

$$f_{-2k}(t) = -\frac{\lambda^{2k}}{(2k)!}\gamma^{(2k)}(t)\,;\ k=1,\,2,\,\ldots.$$

Since the periodicity conditions are fulfilled, when solving the problem it suffices to consider one strip of periods, for example, the main one (with a hole contour L_0, $\tau=\lambda e^{i\theta}$).

The system of equations (13) from which the unknown coefficients $\alpha_{2k}^{(0)}$ and $\beta_{2k}^{(0)}$ are determined, degenerates to one functional equation. To construct equations regarding the coefficients $\alpha_{2k}^{(0)}$ and $\beta_{2k}^{(0)}$, as well as the functions $\Phi_2^{(0)}(z)$ and $\Psi_2^{(0)}(z)$, we expand these functions in Laurent series in the vicinity of the point z=0. Having substituted $\tau=\lambda e^{i\theta}$ into the left-hand of the boundary condition (13) on the contour instead of the functions $\Phi_2^{(0)}(z)$, $\overline{\Phi_2^{(0)}}$, $\Phi_2'^{(0)}$, $\Psi_2^{(0)}(z)$ and their expansion – in Laurents series in the vicinity of the point z=0, as well as the Fourier series (15) and (18) to the right side of (13) instead of the function $f_1(\theta) + if_2(\theta)$ and $\phi_1(\theta) + i\phi_2(\theta)$, we compare the coefficients at the identical powers $e^{i\theta}$. As a result, we obtain two infinite systems of algebraic equations regarding the coefficients $\alpha_{2k}^{(0)}$ and $\beta_{2k}^{(0)}$.

After a number of transformations, we obtain an infinite system of algebraic equations with respect to $\alpha_{2k+2}^{(0)}$

$$\alpha_{2j+2}^{(0)} = \sum_{k=0}^{\infty} A_{j,k}^{*}\alpha_{2k+2}^{(0)} + b_j^{(0)}\,. \tag{19}$$

The constants $\beta_{2k+2}^{(0)}$ are determined from the relations

$$\beta_{2k+2}^{(0)} = \frac{1}{K_1}\left(-M_0 + 2\sum_{k=0}^{\infty}\frac{g_{k+1}\lambda^{2k+2}}{2^{2k+2}}\alpha_{2k+2}^{(0)}\right);$$

$$\beta_{2j+4}^{(0)} = (2j+3)\alpha_{2j+2}^{(0)} + \sum_{k=0}^{\infty}\frac{(2j+2k+3)!\,g_{k+1+2}\lambda^{2j+2k+4}}{(2j+2)!(2k+1)!2^{2j+2k+4}}\alpha_{2k+2}^{(0)} - M_{-2j-2}\,. \tag{20}$$

Here $K_1 = 1 - \frac{\pi^2}{12}\lambda^2$.

Requiring that the functions (10) satisfy the boundary condition (9), after a number of transformations we obtain a singular integral equation for the function $g^{(0)}(x)$

$$\frac{1}{\omega}\int_{L'} g^{(0)}(t)\mathrm{ctg}\frac{\pi}{\omega}(t-x)dx + H_0(x) = 0\,, \tag{21}$$

where

$$H_0(x) = \Phi_*(x) + \overline{\Phi_*(x)} + x\Phi_*'(x) + \Psi_*(x)\,;$$

$$\Phi_*(x) = \Phi_0^{(0)}(x) + \Phi_2^{(0)}(x)\,;\quad \Psi_*(x) = \Psi_0^{(0)}(x) + \Psi_2^{(0)}(x)\,.$$

Singular integral equation (21) and algebraic systems (19), (20) contain unknown quantities of concentrated forces $P_{mn}^{(0)}$. To determine these quantities, we use the Hooke's law and the method of "gluing" of two asymptotes of the sought-for solution.

According to the Hooke's law, the magnitude of the concentrated force $P_{mn}^{(0)}$ acting on each attachment point as viewed from the stringer equals

$$P_{mn}^{(0)} = \frac{E_s A_s}{2 y_0 n} \Delta \nu_{m,n}^{(0)} \quad (m,\, n=1,\, 2,\, \ldots),$$

where E_s is the Young's modulus of the material; A_s is the cross-section area of the stringer; $2y_0n$ is the distance between the attachment points; $\Delta\nu_{m,n}^{(0)}$ is the relative displacement of the considered attachment points equal to the elongation of the appropriate section of the stringer.

To find concentrated forces $P_{mn}^{(0)}$ we require the fulfillment of the condition of compatibility of displacements, i.e. we accept that the relative elastic displacement of the points $z=mL+i(y_0n-a_0)$ and $z=mL-i(y_0n-a_0)$ equals the relative displacement $\Delta\nu_{m,n}^{(0)}$ of the attachment points. Here a_0 is the radius of the attachment points (the coupling area).

By means of complex potentials (10)–(12), (16) and the Kolosov-Muskhelishvili formulas for displacements, we find the relative displacement $\Delta\nu_{m,n}^{(0)}$. Knowing the relative displacements $\Delta\nu_{m,n}^{(0)}$ we determine the sought-for magnitudes of concentrated forces from the system

$$P_{pr}^{(0)} = \frac{E_s A_s}{2 y_0 r} \Delta \nu_{p,r}^{(0)} \quad (p,\, r=1,\, 2,\, \ldots) \tag{22}$$

Algebraic systems (19), (20), (22) and singular integral equation (21) are connected and should be solved jointly.

Using the expansion

$$\frac{\pi}{\omega} ctg \frac{\pi}{\omega} z = \frac{1}{z} - \sum_{j=0}^{\infty} g_{j+1} \frac{z^{2j+1}}{\omega^{2j+2}}$$

we can reduce the equation (21) to the form

$$\frac{1}{\pi} \int_{L_1} \frac{g^{(0)}(t)}{t-x} dt + \frac{1}{\pi} \int_{L_1} g^{(0)}(t) K(t-x) dt + H_0(x) = 0\,, \tag{23}$$

where $K(t) = -\sum_{j=0}^{\infty} g_{j+1} \frac{t^{2j+1}}{\omega^{2j+2}}$.

Taking into account that the function $g^{(0)}(x)$ is odd, using the change of variables, we reduce the equation (23) to the standard form

$$\frac{1}{\pi} \int_{-1}^{1} \frac{g_*^{(0)}(\tau)}{\tau - \eta} d\tau + \frac{1}{\pi} \int_{-1}^{1} g_*^{(0)}(\tau) B(\eta, \tau) d\tau + H_{0*}(\eta) = 0\,. \tag{24}$$

We represent the solution of singular integral equation (24) in form

$$g_*(\eta) = \frac{g_0(\eta)}{\sqrt{1-\eta^2}}\,.$$

The function $g_0(\eta)$ is Holder continuous in the area $[-1, 1]$ and is replaced by the Lagrange polynomial structured by Chebyshev nodes [22, 23]. Using [22, 23] square formulas, we reduce the integral equation (24) to the system of M linear algebraic equations regarding approximate values $g_k^{(0)}$ of the sought-for function at nodal points. Using the algebraization procedure [22, 23] the singular integral equation (21) under additional condition

$$\int_{a+m\omega}^{b+m\omega} g_0^{(0)}(t) dt = 0\,; \quad \int_{-a-m\omega}^{-b-m\omega} g_0^{(0)}(t) dt = 0 \quad (m=0,\, 1,\, 2,\, \ldots),$$

that provides uniqueness of displacements when bypassing the contours of cracks, is reduced to the system of M linear algebraic equations for determining M unknowns $g^{(0)}(\tau_m)$ (m=1, 2, …, M). After some transformations, the integral equation is replaced by the system of algebraic equations

$$\sum_{k=1}^{M} a_{m,k}\, g_k^{(0)} + \frac{1}{2} H_{0*}(\eta_m) = 0 \quad (m=1, 2, \ldots, M-1); \tag{25}$$

$$\sum_{k=1}^{M} g_k^{(0)}(\eta_m) = 0 ,$$

where $a_{m,k} = \frac{1}{2M}\left(\frac{1}{\sin\theta_m} ctg \frac{\theta_m + (-1)^{|m-k|}\theta_k}{2} + B(\tau_m, \eta_k) \right)$; $\theta_m = \frac{2m-1}{2M}\pi$; $g_k^{(0)} = g^{(0)}(\tau_k)$; $\eta_m = \cos\theta_m$; $\tau_k = \eta_k$.

For the stress intensity factor in the vicinity of crack tips for $x=a+m\omega$ in zero approximation we have

$$K_{\mathrm{I}}^{(0)} = \sqrt{\pi(b-a)} \sum_{m=1}^{M} (-1)^{m+M} g^{(0)}(t_m) \operatorname{tg} \frac{2m-1}{4M}\pi ,$$

in the vicinity of crack tips for $x=b+m\omega$ we have

$$K_{\mathrm{I}}^{(0)} = \sqrt{\pi(b-a)} \sum_{m=1}^{M} (-1)^{m} g^{(0)}(t_m) \operatorname{ctg} \frac{2m-1}{4M}\pi .$$

The solution of the problem in the first approximation

After finding the solution in zero approximation we solve the problem in the first approximation. The boundary conditions (6)–(7) of the problem for the first approximation are written as

$$\Phi^{(1)}(\tau) + \overline{\Phi^{(1)}(\tau)} - e^{2i\theta}\left[\overline{\tau}\Phi'^{(1)}(\tau) + \Psi^{(1)}(\tau)\right] = N - iT \quad \text{for } \tau = \lambda e^{i\theta} + m\omega \tag{26}$$

$$\Phi^{(1)}(x) + \overline{\Phi^{(1)}(x)} + x\overline{\Phi'^{(1)}(x)} + \overline{\Psi^{(1)}(x)} = 0 ;\ a + m\omega \le |x| \le b + m\omega . \tag{27}$$

We look for the solution to the boundary value problem (26) similar to the zero approximation in the form

$$\Phi^{(1)}(z) = \Phi_0^{(1)}(z) + \Phi_1^{(1)}(z) + \Phi_2^{(1)}(z) ; \quad \Psi^{(1)}(z) = \Psi_0^{(1)}(z) + \Psi_1^{(1)}(z) + \Psi_2^{(1)}(z) , \tag{28}$$

where the potentials $\Phi_0^{(1)}(z)$ and $\Psi_0^{(1)}(z)$ describe stress and strain field under the action of the system of concentrated forces $P_{mn}^{(1)}$ and are determined by the formulas similar to (11), where $\sigma_0 = 0$, $P_{mn}^{(0)}$ and should be replaced by $P_{mn}^{(1)}$.

We look for the potentials $\Phi_1^{(1)}(z)$ and $\Psi_1^{(1)}(z)$ in the form similar to (12), and this time the function $g^{(0)}(x)$ should be replaced by $g^{(1)}(x)$. We find $\Phi_2^{(1)}(z)$ and $\Psi_2^{(1)}(z)$ from the boundary condition (28), using the N. I. Muskhelishvili method again

$$\Phi_2^{(1)}(z) = \Phi^*(z) + \sum_{k=0}^{\infty} a_{2k} z^{-2k} ; \quad \Psi_2^{(1)}(z) = \Psi^*(z) + \sum_{k=0}^{\infty} b_{2k} z^{-2k} .$$

Here $\Phi^*(z)$, $\Psi^*(z)$ are determined from the formulas similarly to (16), where $\alpha_k^{(0)}$ and $\beta_k^{(0)}$ replaced by $\alpha_k^{(1)}$ and $\beta_k^{(1)}$, respectively. The coefficients, a_{2k} and b_{2k} are found from the formulas

$$a_{2n} = C_{2n}R^{2n} \ (n=1, 2, \ldots); \quad a_0=0;$$

$$b_{2n} = (2n-1)R^{2n} a_{2n-2} - R^{2n} a_{-2n+2} \ (n\ge 2); \tag{29}$$

$$b_0=0; \quad b_2 = -C_0 R^2 ; \quad N - iT = \sum_{k=-\infty}^{\infty} C_{2k} e^{-2ki\theta} .$$

The coefficients $\alpha_k^{(1)}$ and $\beta_k^{(1)}$ are determined by the algebraic system similarly to (19)–(20). For the concentrated $P_{mn}^{(1)}$ we have

$$P_{mn}^{(1)} = \frac{E_s A_s}{2 y_0 n} \Delta\nu_{m,n}^{(1)}, \tag{30}$$

where the mutual displacement $\Delta\nu_{m,n}^{(1)}$ is determined similarly to the zero approximation.

Requiring the function (28) to satisfy the boundary conditions (27) on the crack faces in the first approximation, after some transformation we obtain a singular integral equation with respect to $g^{(1)}(x)$.

As in the zero approximation, using the algebraization procedure [20, 21], the singular integral equation

$$\frac{1}{\pi}\int_{L_1} \frac{g^{(1)}(t)}{t-x}dt + \frac{1}{\pi}\int_{L_1} g^{(1)}(t) K(t-x) dt + H_1(x) = 0$$

under the additional condition

$$\int_{a+m\omega}^{b+m\omega} g_0^{(1)}(t)dt = 0; \quad \int_{-a-m\omega}^{-b-m\omega} g_0^{(1)}(t)dt = 0 \ \ (m=0, 1, 2, \ldots),$$

providing uniqueness of displacement, bypassing the crack conditions in the first approximation, is reduced to the system of M linear algebraic equation for determining M unknowns $g^{(1)}(\tau_m)$ (m=1, 2, …, M)

$$\sum_{k=1}^{M} a_{m,k} g_k^{(1)} + \frac{1}{2} H_{1*}(\eta_m) = 0 \quad (m=1, 2, \ldots, M-1), \tag{31}$$

$$\sum_{k=1}^{M} g_k^{(1)}(\eta_m) = 0,$$

where $g_k^{(1)} = g^{(1)}(\tau_k)$.

In the first approximation, for stress intensity factor in the vicinity of crack tips for $x=a+m\omega$ we have

$$K_{\rm I}^{(1)} = \sqrt{\pi(b-a)} \sum_{m=1}^{M} (-1)^{m+M} g^{(1)}(t_m) \,\mathrm{tg}\, \frac{2m-1}{4M}\pi,$$

in the vicinity of crack tips $x=b+m\omega$ we have

$$K_{\rm I}^{(1)} = \sqrt{\pi(b-a)} \sum_{m=1}^{M} (-1)^{m} g^{(1)}(t_m) \,\mathrm{ctg}\, \frac{2m-1}{4M}\pi.$$

The obtained systems of equations of the first approximation are not still closed, since the right side of these systems contains the coefficients d_{2k} of expansion of the function $H(\theta)$ in Fourier series.

Solution of optimization problem

To build missing equations, we use the boundary condition (1) under additional constraints (3). By means of the obtained solution we find σ_t in the surface layer of the contour L_0 ($r = \rho(\theta)$) within first order sizes with respect of small parameter ε

$$\sigma_t = \sigma_t^{(0)}(\theta)\Big|_{r=\lambda} + \varepsilon\left[H(\theta)\frac{\partial\sigma_t^{(0)}(\theta)}{\partial\lambda} + \sigma_t^{(1)}(\theta)\right]\Bigg|_{r=\lambda}.$$

The stresses $\sigma_t^{(1)}(\theta)$ depend on the coefficients d_{2k} of the Fourier series of the sought-for function $H(\theta)$. To build missing equations that allow to find the coefficients d_{2k}, we require that the stress distribution on the contour of holes close to uniform one will be satisfied.

Reducing the stress concentration on the hole contours is carried out by minimizing the criterion

$$U = \sum_{i=1}^{M} \left[\sigma_t(\theta_i) - \sigma_*\right]^2 \to \min,$$

here σ_* is an unknown value of the normal tangential stress on surface hole.

The stated optimization problem is in finding the values of unknown coefficients d_{2k} best providing the values of the function $\sigma_t(\theta_i)$ according to condition (1) under additional constraints (3).

The function U and stress intensity factors depend on the coefficients d_{2k} and thus, we come to the problem for the conditional extremum of the function $U(\sigma_*, d_{2k})$, when the coefficients d_{2k} are associated with the additional condition

$$K_{\mathrm{I}}^{(0)a+m\omega} + \varepsilon K_{\mathrm{I}}^{(1)a+m\omega} = 0\,; \quad K_{\mathrm{I}}^{(0)b+m\omega} + \varepsilon K_{\mathrm{I}}^{(1)b+m\omega} = 0\,. \qquad (32)$$

It is necessary to find minimum value of the function $U(\sigma_*, d_{2k})$, and the k+1 arguments of this function are not independent but are subjected to two additional conditions (32).

To solve the problem for conditional extremum we use the method of Lagrange undetermined multipliers. Let's consider the auxiliary function

$$U_0 = U + \lambda_1 K_{\mathrm{I}}^{a+m\omega} + \lambda_2 K_{\mathrm{I}}^{b+m\omega}\,,$$

with two undetermined multiplies λ_1, λ_2.

The necessary k+1 conditions for extremum are of the form

$$\frac{\partial U_0}{\partial d_{2k}} = 0 \ \ (k=1, 2, \ldots, n); \quad \frac{\partial U_0}{\partial \sigma_*} = 0\,. \qquad (33)$$

The obtained n+1 equations with two additional equations (32) make up the system of equations with n+1+2 unknowns σ_*, d_{2k} (k=1, 2, …, n), λ_1, λ_2. Adding this system of equations to the obtained algebraic system (19), (20), (22), (25), (29)–(31) and to the system for the coefficients $\alpha_k^{(1)}$, $\beta_k^{(1)}$, we obtain the closed algebraic system for determining all unknowns, including σ_* and the coefficients d_{2k}.

Analysis of the results

The system of equations (33) together with previously obtained algebraic systems of an elasticity theory problem in zero and first approximations allows to determine the form of equal strength contour of holes, the stress-strain state of the perforated stringer plate and also the optimal value of the tangential stress σ_*.

When performing the calculation, the obtained systems were solved by the Gauss method with the choice of the main element. The interval $[0, 2\pi]$ of the change of the variable θ was divided into M equal parts, where $M>2m+1$, m is the number of parameters left for particle calculations. Since the function $\sigma_t(\theta_i, d_{2k})$ is linear with respect to unknown parameters, then the compiling and solving the system (33) of equations is greatly simplified.

Calculations were carried out for the following values of free parameters a_0/L=0.01; y_0/L=0.25. For simplification A_s/y_0h=1 were accepted. The stringers were made of the composite Al-steel, the plate from the alloy B95, E=7.1×10^4 MPa; E_s=11.5×10^4 MPa. It was accepted that the amount of stringers at the attachment points equals 14, M=72. The results of calculations of the sought-for function $H(\theta)$ are in the Table.

Table. The values of the Fourier coefficients for equal strength contour

λ	0.2	0.3	0.4	0.5	0.6	0.7
d_0	0.1003	0.1102	0.1214	0.1298	0.1405	0.1561
d_2	−0.0128	−0.0457	−0.0789	−0.1007	−0.1220	−0.1406
d_4	0.0093	0.0118	0.0154	0.0168	0.0189	0.0205
d_6	0.0008	0.0052	0.0096	0.0109	0.0124	0.0143

When performing calculations, the obtained systems were solved by the Gauss method with the choice of the main element. Each of infinity systems (19), (20), (22), (29) and (30) was reduced to a large number of equations depending on the distance between holes.

Calculations showed that for the range $0<\lambda<0.8$ it suffices to reduce the systems (19), (20), (22), (29) and (30) to five equations. For $\lambda>0.8$ the systems (19), (20), (22), (29) and (30) were reduced to 30 equations. It should be noted that the value of the parameter $\lambda>0.8$ falls out of the working range of change λ. Quite quick convergence of the solution of the system of equations (19), (20), (22), (29) and (30) in the range $0<\lambda<0.8$ is explained by the fact that the coefficients of the system (19), (20), (22), (29) and (30) contain high degrees of the parameter λ.

Conclusions

The solution of the problem of determining the optimal shape of holes for a perforated stringer plate weakened by a periodic system of linear cracks, was found.

The represented mathematical model allows to determine optimal contour of holes, meeting the requirements of immobility of cracks and equal strength condition. Thus, the obtained solution of the stated inverse problem makes it possible to increase the strength of the plate and prevent its fracture.

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Обернена задача механіки руйнування для перфорованої стрингер-плити

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Для визначення оптимального контуру отворів у перфорованій стрингер-плиті, що ослаблена періодичною системою тріщин, розглядається обернена задача механіки руйнування. Вважається, що матеріал плити є пружним або пружно-пластичним. Ребра жорсткості (стрингери) симетрично закріплені на плиті. Перфорована плита рівномірно розтягується на нескінченність вздовж стрингерів. Вважається, що прямолінійні тріщини розташовані поблизу контурів отворів та перпендикулярно до прикріплених ребер жорсткості. Розв'язок сформульованої оберненої задачі базується на принципі рівної міцності. Оптимальна форма отворів задовольняє дві умови: умову відсутності концентрації напружень на поверхні отвору та умову нульових коефіцієнтів інтенсивності напружень поблизу вершин тріщини. Невідомий контур отворів шукається в класі контурів, близьких до кругових. Дія ребер жорсткості замінюється невідомими еквівалентними зосередженими силами в точках їх з'єднання з плитою. Шукані функції (напруження, переміщення, зосереджені сили та коефіцієнти інтенсивності напружень) розглядаються у вигляді розкладу за малим параметром. Розв'язок задачі здійснюється з використанням апарату теорії аналітичних функцій та теорії сингулярних інтегральних рівнянь, після чого розв'язується задача умовного екстремуму. В результаті отримано замкнену систему алгебраїчних рівнянь, яка дає змогу мінімізувати напружений стан на контурах отворів та коефіцієнти інтенсивності напружень поблизу вершин тріщини. Отримана система алгебраїчних рівнянь дозволяє визначити форму контуру рівної міцності отворів, напружено-деформований стан перфорованої стрингер-плити, а також оптимальне значення тангенціального напруження.

***Ключові слова**: перфорована плита, стрингери, тріщини, оптимальний контур, рівноміцні отвори.*

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