UDC 519.168
DOI: 10.15587/2312-8372.2017.108909

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SOLVING CERTAIN PROBLEMS OF SCHEDULING THEORY BY THE METHODS OF QUADRATIC OPTIMIZATION

Досліджено конвеєрну задачу теорії розкладів для (flow shop) зі стандартними обмеженнями. Шляхом деяких перетворень ці обмеження зведені до квадратичних форм змінних задачі. Показано, що будь-яка задача теорії розкладів для конвеєрної системи машин зводиться до задачі мінімуму лінійної функції з набором лінійних і квадратичних обмежень, тобто до задачі опуклої оптимізації. Розглянуто модельний приклад та методом точної квадратичної регуляризації отримано оптимальний розклад.

Ключові слова: *теорія розкладів, опукла оптимізація, метод точної квадратичної регуляризації.*

1. Introduction

The ordering of processes and algorithms is necessary in any field of activity, where the number of such processes is sufficiently large [1]. A separate class of problems of this kind is associated with the ordering of the execution of a certain number of processes (tasks) on a given number of devices, the so-called «scheduling problems» [2]. The problems of information processing, technological processes of industrial enterprises, transport problem – all of them (and not only) are reduced to the problems of scheduling theory. In terms of content, many problems in scheduling theory are optimized. They consist in selecting (finding) among the set of admissible schedules (schedules allowed by the conditions of the problem) of those solutions on which the «optimal» value of the objective function is achieved. It is assumed that at the beginning of the planning period, a list of performed tasks is known. Each task is a set of interdependent works, which are performed on separate devices. For each job, you specify the processing time on each machine, the order of maintenance, and the execution time. The complexity of scheduling problems is determined by the number of tasks (n) and the number of processing units (m). A separate class of tasks «flow shop» fixes the order of using machines and assumes a consistent multi-stage execution of each task on each machine in the established order. The solution of the problem is the sequence of the tasks, at which the total processing time is minimized. Despite the simplicity of setting the tasks of scheduling theory and the increasing number of works devoted to their solution, an algorithm that will find an optimal or approximate solution for «reasonable» time is not found. The proposed approach to solving scheduling problems by the methods of convex optimization allows one to obtain an optimal or approximate solution for the polynomial time.

2. The object of research and its technological audit

The object of research are the problems of scheduling theory, which include certain resources (processing devices), a set of tasks, precedence constraints, and methods for estimating schedules. The so-called «flow shop problem» of scheduling theory is considered, which includes all the components listed above, while the order of processing of each job on the devices is the same and is determined. The required schedule should ensure the minimum time T for all jobs. The graph of the flow shop problem is shown in Fig. 1.

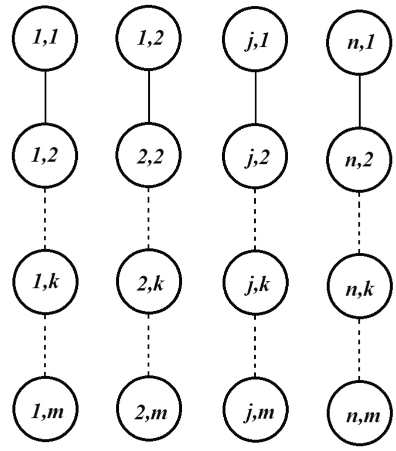

Fig. 1. The flow shop problem.
Each job in the same sequence goes through all the devices

An optimal solution to the formulated problem can be obtained by considering all possible priorities for the fulfillment of the initial tasks. The number of possible variants of execution of tasks is equal to the number of permutations from n elements, that is, $n!$ options. Therefore, for $n \sim 10–20$, it is already impossible to obtain an optimal solution, since the amount of computation and the time of their execution will exceed all reasonable limits.

3. The aim and objectives of research

The aim of this research is development of a method for solving pipeline scheduling problems based on convex optimization methods. This method should ensure the

ISSN 2226-3780

calculation of the optimal schedule with a given accuracy within an acceptable time.

To achieve this aim it is necessary:

1. To reformulate the limitations of the flow shop problem in the form of a set of analytic functions of the variables of this problem.

2. To rewrite the constraints in the form of convex functions and formulate the problem of convex optimization.

3. To choose the optimal method for solving multi-extremal convex optimization problems.

4. Research of existing solutions of the problem

Classification of various problems in scheduling theory is considered in [3]. As is known, the problems of scheduling theory in the general formulation with the number of processing devices greater than $m>2$ do not have a solution algorithm and are NP-complete [1, 2]. An attempt was made in [4] to use the Johnson algorithm ($m=2$) to construct an optimal schedule with a large number of processing devices. But, as the authors note, this approach does not lead to an optimal solution, but allows to find some approximation to it. Evaluation of the difference between the obtained solution and the optimal one is practically impracticable. The traditional approach to solving such problems uses the method of sequential opening of modules or the method of permutation of tasks [5], which leads to a large expenditure of time for calculating options. The method of branches and bound [1], as well as its various modifications [6], improves the methods of permutation of tasks used in solving the pipeline problem of scheduling theory. But they include additional procedures – the procedure for estimating schedules, the procedure for branching, the procedure for dropping out and the procedure for stopping. The main disadvantage of the branch and bound method is the need to define estimates at each vertex of the branch tree, and for a large number of tasks, the number of vertices becomes significant [7]. This does not allow to review all the vertices of the branch tree. It was noted in [8] that modern models and methods of scheduling theory can be divided into two groups: approximate (heuristic) and permutable with various variants of optimization of permutations. As a result, using the methods of the first group, the accuracy of solutions is lost, and the time of solving the problem by the methods of the second group can go beyond reasonable limits. Therefore, the use of general methods of global optimization [9, 10], allowing a reasonable time to obtain a solution to the problem of scheduling theory, is promising. This paper is devoted to the mathematical formulation of the problems of scheduling theory in a form convenient for using the methods of global optimization.

5. Methods of research

Let m – the number of processing devices; n – the number of jobs performed on these devices, each job including processing on each of the m devices. Denote by:

$$t_{\min}=\min(t_{ij}),$$

where t_{ij} – the execution time of the i-th job on the j-th device, the time for the start of the task execution on the selected device will be denoted as x_{ij} (task schedule variables). Let's introduce the auxiliary function of the employment of devices [8] in the form:

$$g_{ij}\left(t-x_{ij}\right)= =\frac{1}{2}\left(\frac{\left|t-x_{ij}\right|}{t_{ij}}+\frac{\left|t-x_{ij}-t_{ij}-t_{\min}\right|}{t_{\min}}-\left|t-x_{ij}-t_{ij}\right|\times\left(\frac{1}{t_{ij}}+\frac{1}{t_{\min}}\right)\right), \quad (1)$$

the graph of which is shown in Fig. 2.

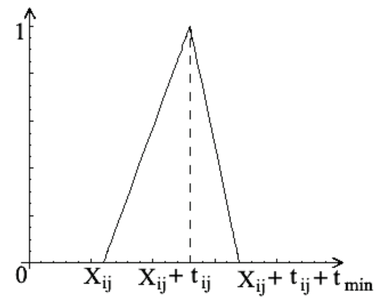

Fig. 2. The graph of the function $g_{ij}(t-x_{ij})$

Condition: on one device at a given time, only one job is processed is equivalent to the inequality:

$$\sum_{i=1}^{n} g_{ij}\left(t-x_{ij}\right)\le 1,\ \forall t\ge 0,\ j=1,\ldots,m. \quad (2)$$

Similarly, the second condition: each job can't be simultaneously executed on two devices, takes the form:

$$\sum_{j=1}^{m} g_{ij}\left(t-x_{ij}\right)\le 1,\ \forall t\ge 0,\ i=1,\ldots,n. \quad (3)$$

The total execution time of all tasks on the devices satisfies an additional set of linear constraints:

$$T\ge x_{im}+t_{im},\ i=1,\ldots n, \quad (4)$$

and is the target function of the flow shop problem. Minimization of the T value is achieved by changing the variables (x_{kj}). Let's pass at the following minimization problem:

$$\min\left(T\sum_{j=1}^{m} g_{ij}\left(t-x_{ij}\right)\le 1,\ \sum_{i=1}^{n} g_{ij}\left(t-x_{ij}\right)\le 1,\ T\ge x_{im}+t_{im}\right). \quad (5)$$

Let's estimate the restrictions on the variables (x_{kj}), which follow from the relations (2), (3) of the problem (5).

Let's consider the system of inequalities (2), which must be satisfied at any time t, including for the moments of time corresponding to the maximum of each of the functions g_{ij} entering into the sum, i. e., the instants of time $t=x_{1j}+t_{1j}$, $t=x_{2j}+t_{2j}$, $t=x_{3j}+t_{3j}$, and so on. As a result, let's obtain the following system of inequalities:

$$\sum_{i=1}^{n} g_{ij}\left(x_{kj}+t_{kj}-x_{ij}\right)\le 1,\ j=1,\ldots m,\ k=1,\ldots.n. \quad (6)$$

In the case $i=k$, in each of the inequalities (6) there is a summand of the form $g_{ij}\left(t_{ij}\right)$, which by identity (1) is identically equal to one, and the system of inequalities (6) can be rewritten as follows:

$$\sum_{i=1}^{n} g_{ij}\left(x_{kj}+t_{kj}-x_{ij}\right)\le 0,\ j=1,...m,\ k=1,...n,\ k\ne i. \quad (7)$$

Since the functions g_{ij} are non-negative by definition, each of the terms in the sum of the summands must be zero. Let's pass to a set of restrictions on the difference $x_{kj}-x_{ij}$, and $k\ne i$. These differences occur twice in equations with rearranged indices i and k, where the signs of the differences are opposite, therefore, two conditions must be fulfilled simultaneously:

$$g_{ij}\left(x_{kj}-x_{ij}+t_{kj}\right)=0 \text{ and } g_{kj}\left(x_{ij}-x_{kj}+t_{ij}\right)=0,$$
$$j=1,...m,\ k=1,...n,\ k\ne i. \quad (8)$$

A zero of a function $g_{ij}(z)$ is achieved for values of the argument $z<0$ or $z>t_{ij}+t_{\min}$, which leads to inequalities:

$$x_{kj}-x_{ij}+t_{kj}\le 0,\quad x_{kj}-x_{ij}+t_{kj}\ge t_{ij}+t_{\min},$$
$$x_{ij}-x_{kj}+t_{ij}\le 0,\quad x_{ij}-x_{kj}+t_{ij}\ge t_{kj}+t_{\min}$$

or

$$x_{kj}-x_{ij}\le -t_{kj},\quad x_{kj}-x_{ij}\ge t_{ij}-t_{kj}+t_{\min},$$
$$x_{kj}-x_{ij}\ge t_{ij},\quad x_{kj}-x_{ij}\le -t_{kj}+t_{ij}-t_{\min}. \quad (9)$$

In the expression (9), the fulfillment of the first inequality automatically leads to the fulfillment of the fourth inequality, and the fulfillment of the third inequality automatically leads to the fulfillment of the second. Thus, two inequalities are valid from the set of inequalities (9):

$$x_{kj}-x_{ij}\le -t_{kj},\quad x_{kj}-x_{ij}\ge t_{ij}, \quad (10)$$

and the execution of any of them nullifies each of the functions in expression (8).

The first of the inequalities (10) indicates that the execution of the i-th job on the j-th device can't begin until this processing ends on the k-th job. The second expression indicates that the execution of the k-th job on the j-th device can't begin until the i-th job processing on this device has finished. Depending on the sequence of tasks, one of these inequalities will be implemented, but both can't be performed at the same time. The combination of these conditions is possible in the expression:

$$(x_{kj}-x_{ij}+t_{kj})(x_{kj}-x_{ij}-t_{ij})\ge 0. \quad (11)$$

Expression (11) defines two parallel hyperplanes in the space of variables (x_{kj}). The region of space between hyperplanes does not satisfy condition (11) and there can't be admissible solutions of problem (5). All possible solutions of problem (5) (including the optimal one) will be located outside of hyperplanes. Thus, the complete system of inequalities (9), which places restrictions on variables x_{kj}, x_{ij} (only one job is processed on a single device at a given time), can be represented as:

$$(x_{kj}-x_{ij}+t_{kj})(x_{kj}-x_{ij}-t_{ij})\ge 0,\ j=1,...m,$$
$$i=1,...n-1,\ k=i+1,...n. \quad (12)$$

For the case of three restrictions of the form (12) in Fig. 3 shows the mutual arrangement of pairs of hyperplanes and the range of possible solutions of problem (5).

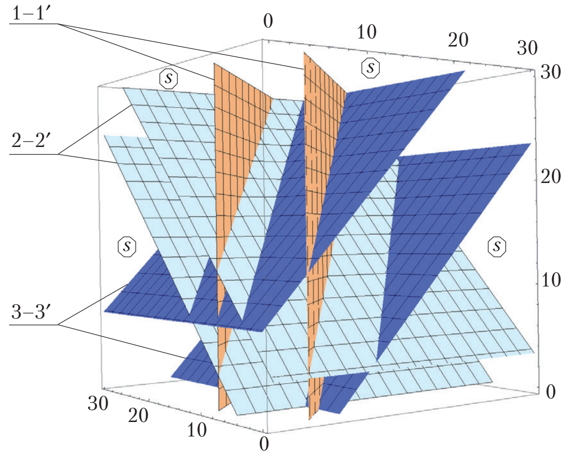

Fig. 3. The mutual disposition of pairs of hyperplanes and the domain of possible solutions of problem (5): 1–1′, 2–2′, 3–3′ – pairs of hyperplanes corresponding to constraints (11)

The admissible solutions in Fig. 1 are possible in regions outside all pairs of hyperplanes. These areas are designated as S. The system of inequalities (3) is treated similarly, with the corresponding replacement of the indices, which leads to the following restrictions:

$$(x_{ij}-x_{iq}+t_{ij})(x_{ij}-x_{iq}-t_{iq})>0,$$
$$i=1,...n,\ q=1,...m,\ j=q+1,...m. \quad (13)$$

In the case of a flow shop problem, the sequence of tasks on the devices is defined and the inequalities (13) must be replaced by the conditions:

$$x_{ik+1}\ge x_{ik}+t_{ik},\ i=1,....n,\ k=1,....m-1. \quad (14)$$

Comparing the results with the graph of the conveyor problem in Fig. 1, it is possible to state that for each line of the graph that determines the sequence of operations in the job, a linear restriction of the form (14) is written. If the sequence of operations on devices is indifferent for the selected job, then in this case it is necessary to use the constraint of the form (13). For tasks performed on one device, but not connected by lines of the graph, constraints of the form (12) must be written.

6. Research results

The procedure proposed in this paper allows to write down the constraints of the problem of scheduling theory in the form of a set of smooth, convex, twice differentiable functions whose number does not exceed:

$$N_{gen}=\frac{1}{2}n\cdot m(m+n-2). \quad (15)$$

Constraints of the form (14) are always less than constraints of the form (13); therefore, for the pipeline problem their total number will be less than in (15):

$$N_{gen}=n\cdot(m-1)+\frac{1}{2}m\cdot n(n-1)=\frac{1}{2}n\cdot m(n+1)-n. \quad (16)$$

Thus, any pipeline problem in the scheduling theory reduces to the problem of the minimum of the linear

ISSN 2226-3780

function (min(T)) (4) with a set of linear and quadratic constraints (12), (14), that is, to the so-called convex optimization problem. The development of the theory of convex optimization in recent years has been very intensive and quite reliable and effective methods of finding the optimal solution of the problem with a given accuracy have been developed [10]. These methods are represented by several «competing» directions, which have their strengths and weaknesses, the description of which is detailed in [11]. The greatest progress in global optimization was obtained using the method of exact quadratic regularization of EQR [12]. This method is applicable for a wide class of multi-extremal problems and allows them to be divided into two classes of complexity. The first class reduces to minimizing the norm of a vector on a convex set, and the second to maximizing the norm of a vector on a convex set. The problem of finding the minimum of a linear function (min(T)) (4) with a set of linear and quadratic constraints (12), (14) belongs to the second class of complexity [9].

The use of the method of exact quadratic regularization transforms the problem (min(T)) with constraints (12)–(14) to the following form:

$$\max\left\{\|x\|^2 \middle| T+s+(r-1)\right| \|x\|^2 \le d\Big\},$$
$$(x_{kj}-x_{ij}+t_{kj})(x_{ij}-x_{kj}+t_{ij})+r\|x\|^2 \le d,$$
$$x_{ik+1} \ge x_{ik}+t_{ik}, \tag{17}$$

where s – a fixed parameter satisfying the condition:

$$T^*+s \ge \|x^*\|^2, \tag{18}$$

and T^* and x^* – a solution of problem (17), the value of the parameter $r>0$ is chosen such that the admissible domain of solutions of problem (17) is convex. Indeed, the parameter r enters into all the restrictions of the problem (17), the Hessians of these functions with a suitable choice of the parameter r are positive definite matrices and the restriction functions are convex. The components of the vector x are the required variables of the schedule task. In problem (17), it is necessary to find the minimum value of the parameter $d>0$, for which simultaneously with the condition (18) the following condition is fulfilled:

$$r\|x\|^2 = d. \tag{19}$$

The parameter s, along with the parameter d, determines the solution search area and its choice, in accordance with condition (18), leads to finding the global minimum of the time for passing the schedule T^* and the corresponding variables of this schedule x^*, that is, the solution of the problem.

Thus, the pipeline scheduling problem reduces to one of the traditional problems of global optimization, namely, minimizing the square of the norm of the vector x with quadratic constraints (17).

To demonstrate the efficiency of the above procedure, a pipeline processing model consisting of five tasks ($n=5$) and three sequential processing devices ($m=3$) is chosen. The execution time of each of the tasks on the devices t_{ij} is chosen randomly and given in Table 1.

As a result of the optimization carried out by the method of exact quadratic regularization (EQR), the time values x_{ij} for the beginning of job processing on devices, which values are given in Table 2, are obtained.

Table 1

Time for each task on individual devices

Processing device	Tasks				
	1	2	3	4	5
1	5	4	7	2	3
2	8	4	1	8	5
3	1	9	4	4	1

Table 2

Time to start each task on individual devices

Processing device	Tasks				
	1	2	3	4	5
1	6	2	11	0	18
2	14	10	22	2	23
3	23	14	24	10	28

The completion time for all tasks is 29 and is minimal for this task, and the resulting schedule is optimal. In this case, all restrictions of the sequence of operations (14), presented in Table 1, are satisfied and restrictions (12) for processing only one job on any device at a given time (Table 4). For clarity, constraints (14) are rewritten as:

$$x_{ik+1}-x_{ik}-t_{ik} \ge 0,\ i=1,\dots n,\ k=1,\dots m-1. \tag{20}$$

Table 3

The difference in the times of the beginning of the execution of the subsequent operation and the end of the previous one for each of the tasks

Pairs of operations	Tasks				
	1	2	3	4	5
1–2	3	4	4	0	2
2–3	1	0	1	0	0

Table 4

The numerical values of the constraints (12) for each of the machines and possible pairs of tasks

Number of device	Pairs of tasks									
	1–2	1–3	1–4	1–5	2–3	2–4	2–5	3–4	3–5	4–5
1	00	00	44	105	80	00	228	162	00	336
2	00	00	80	14	104	00	162	252	00	338
3	00	00	126	24	14	00	75	180	00	266

The above solutions are obtained by numerical calculations using Excel Solver (USA).

7. SWOT analysis of research results

Strengths. Using traditional methods of scheduling theory, an increase in the number of tasks or devices leads to an exponential increase in the solution time of the problem. The proposed method for converting the pipeline problem of scheduling reduces it to the problem of convex optimization. The number of constraints in this problem is polynomial with respect to the number of jobs and the number of processing devices. Therefore, an increase in the number of jobs or the number of processing devices leads only to a polynomial increase in the time of solving the problem. Therefore, the time required to calculate the optimal schedule will be orders of magnitude smaller than in the case of traditional methods.

Weaknesses. The convex optimization problem obtained by converting the pipeline problem turns out to be multi-extremal. Therefore, not all methods of convex optimization allow to find a global minimum, that is, an optimal solution.

The found solution may be close to optimal, but not coincide with it. Therefore, it may be necessary to repeat the solution of the problem with other initial conditions.

Opportunities. The method of exact quadratic regularization (EQR) allows to find the global minimum of the multi-extremal problem. Algorithmization of the definition of additional parameters of the method (EQR) and their changes will automate the process of finding the optimal schedule. In this case, the elapsed time for calculating the optimal schedule will be reduced. This will make it possible to optimize the use of resources depending on the time-varying flow of tasks.

Threats. Additional costs will be associated with development and implementation of new software to determine the optimal schedules.

8. Conclusions

1. The formulation of constraints of the flow shop problem is changed and is reduced to a set of analytic functions of the variables of this problem. Thus, we pass from the problems of discrete mathematics to the problems of classical mathematical analysis.

2. The analytic functions of the constraints are convex. Thus, the original problem is reformulated into the problem of convex optimization. This allows to solve the problem by convex optimization methods.

3. A method of exact quadratic regularization (EQR) is chosen for solving the multiply connected convex optimization problem. Its use makes it possible, in polynomial time, to find the optimal schedule for the flow shop problem.

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РЕШЕНИЕ НЕКОТОРЫХ ЗАДАЧ ТЕОРИИ РАСПИСАНИЙ МЕТОДАМИ ВЫПУКЛОЙ ОПТИМИЗАЦИИ

Исследована конвейерная задача теории расписаний (flow shop) со стандартными ограничениями. Путем некоторых преобразований эти ограничения сведены к квадратичным формам переменных задачи. Показано, что любая конвейерная задача теории расписаний сводится к задаче минимума линейной функции с набором линейных и квадратичных ограничений, то есть к задаче выпуклой оптимизации. Рассмотрен модельный пример и методом точной квадратичной регуляризации получено оптимальное расписание.

Ключевые слова: теория расписаний, выпуклая оптимизация, метод точной квадратичной регуляризации.

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