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FINDING THE CONDITION OF FINITENESS OF THE DISCRETE SPECTRUM OF A NONSELF-ADJOINT FRIEDRICHS MODEL IN THE CASE OF ONE-DIMENSIONAL PERTURBATION

The object of this study is the Friedrichs model in the case of one-dimensional perturbation of the multiplication operator by an independent variable. One of the most problematic places in this theory is when the number of eigenvalues is infinite. Therefore, the important is to find the conditions under which there is a finite number of eigenvalues.

In this paper used standard methods of functional analysis, namely: calculation of operator norms, finding of conjugate operator, calculation of functional norms, calculation of operator resolvent with substantiation of resolvent existence conditions. Traditionally, the perturbation of the operator is presented in a factorized form (i. e. in the form of the product of two operators, one of which acts from the main space to a certain auxiliary space, and the other, conversely, from the auxiliary space to the main one). In addition to the methods of functional analysis, it is possible to work with improper integrals over an infinite interval. Let's emphasize that in this paper let's also use the concepts of smallness by norm and the concept of smallness by dimension. In this case, the dimension of the perturbation operator is one-dimensional.

The following statement is obtained: if it is established that the integral has a finite number of eigenvalues and if it is established that the resolvent tends to zero for σ→∞, then there will be a finite number of eigenvalues on the entire axis. By superimposing a condition on the difference between the perturbation and the conjugate perturbation, let's find the finiteness of the operator spectrum. Due to the fact that there is a finite range of spectrum, let's obtain the opportunity to work with expressions on various topics. This fact greatly simplifies all calculations, regardless of the nature of the studied expressions: mechanical, physical or otherwise.

Due to the finite number of eigenvalues of the perturbed operator, let's obtain the advantage that there is no need to sum up an infinite number of terms in expressions because it would actually be impossible.

Keywords: *discrete spectrum, Friedrichs model, operator boundedness condition, integral operator, Hilbert space, operator compactness.*

1. Introduction

Information about the finite number of all eigenvalues of this operator is relevant and valuable because it happens that an important object is considered, but it has an infinite number of eigenvalues, which significantly inhibits further research. Thus, the Friedrichs model in an arbitrary interval can have a finite set of eigenvalues under certain restrictions.

Spectral theory is one of the most important areas of the theory of linear operators. Current trends in the theory of operators dictate the need for appropriate knowledge about the number of eigenvalues. It is known that the solution of differential equations after the application of the corresponding Fourier transform in many cases is reduced to the analysis of the non-self-adjoint Friedrichs model, i. e. the sum of the multiplication operator by an independent variable and the operator perturbed by a bounded factor [1].

The Friedrichs model plays a supporting role in the study of the family of some operator matrices [2]. This can be applied to various physical problems using positively defined operators of the Friedrichs model without their spectral decomposition and Parseval equality. In [3] the direct and inverse problem for the Sturm-Liouville operator is investigated, its spectral features are studied and the orthogonality of eigenfunctions and eigenvalues is established. The asymptotic formula of eigenvalues and eigenfunctions of the Sturm-Liouville operator is considered and the spectral decomposition is obtained. It is shown that the eigenfunctions form a complete system and the Weyl function is found. The unity theorem for solving the inverse problem is proved. In this paper, these results are obtained for a finite interval $(0,\pi)$, i. e it is possible to talk about space $L^2(0,\pi)$. But the problem of localization of spectral singularities of dissipative operators from the point of view of asymptotics of the corresponding exponential function is considered in [4] and the solution of

this problem for spectral singularities of higher orders is presented. The work [5] presents the conditions for the Friedrich model, which allow to write a formula for the jump of the resolvent on a continuous spectrum, but they are voluminous and inconvenient to use, and there is no direct connection with the resolvent. The spectral properties of the model operator with emphasis on asymptotics for a number of infinitely many eigenvalues (the case of the Efimov effect) were also studied [6]. And in [7] the finiteness of a number of connected states of the corresponding Schrödinger operator is proved in the case when the potentials satisfy certain conditions and zero is a regular point for a two-part subhamiltonian. Also, such a set of values for the values of the mass of particles is found that the Schrödinger operator can have only a finite number of eigenvalues lying to the left of the essential spectrum. The synthesis of the latest achievements of the spectral theory of the magnetic Schrödinger operator, which can be considered a catalog of specific examples of magnetic spectral asymptotics, is contained in the study [8]. All these works confirm the relevance of the chosen topic. Thus, the object of this study is the Friedrichs model in the case of one-dimensional perturbation of the multiplication operator by an independent variable. And the purpose of the study is to prove the finiteness of the discrete spectrum of the transport operator.

2. Methods of research

In this work the methods of calculation of norms of operators, finding of the conjugate operator, calculation of norms of functionals, calculation of the resolvent of the operator with the substantiation of conditions of existence of the resolvent are used.

It is known that the operation of differentiation after the Fourier transform turns into an operation of multiplication by an independent variable [9]. Let's recall that differential operations often appear in a variety of applications: physical, mechanical and even chemical.

A comfortable space is popular $L^2(a,b)$ is a space of functions integrated squared on an interval (a,b).

The Friedrichs model is an expression in Hilbert space of functions that has the form:

$$Tf = Sf + Vf,$$

where $Sf(x) = xf(x)$ is a multiplication operator, a $Vf(x)$ is integrated operator. If the operator V is «Small», the operator S is «close» to the operator T. The case of smallness on norm is possible, and the case of smallness of dimension, as in this case, one-dimensional perturbation.

Let's consider the Friedrichs model in the space of functions integrated with a square on the half-axis and one-dimensional perturbation. This paper uses methods with a transport operator with matrix potential, methods that are similar to those used in [10].

This work can be considered as a supplement to the work [10].

In space $H = L^2(0,\infty)$ let's consider the operator:

$$T = S + V,$$

where $V = A^*B$, where $A:H \to G$, $B:H \to G$, where G is some Hilbert space. Operators have the form:

$$A\phi = \int_0^\infty \phi(s)\alpha(s)ds, \ B\phi =$$
$$= \int_0^\infty \phi(s)\beta(s)ds, \ \alpha(s) \in G, \beta(s) \in G. \tag{1}$$

The resolvent $T_\zeta = (T-\zeta)^{-1}$ has the form:

$$T_\zeta = S_\zeta - S_\zeta A^* K(\zeta)^{-1} BS_\zeta, \ S_\zeta = (S-\zeta)^{-1}, \tag{2}$$

where

$$K(\zeta) = 1 + BS_\zeta A^*. \tag{3}$$

To calculate the resolvent, let's use the formula obtained in [5]. Indeed, denote $T_\zeta f = g$, then let's obtain:

$$(T-\zeta)g = (S-\zeta)g + A^*Bg = f.$$

Let's act with an operator:

$$S_\zeta : g + S_\zeta A^* Bg = S_\zeta f. \tag{4}$$

Let's act with an operator:

$$S_\zeta : g + S_\zeta A^* Bg = S_\zeta f \ B : Bg + BS_\zeta A^* Bg = BS_\zeta f,$$

or (3):

$$Bg = K(\zeta)^{-1} BS_\zeta.$$

Then:

$$g + S_\zeta A^* K(\zeta)^{-1} BS_\zeta f = S_\zeta f.$$

Substitute in representation (4):

$$g = T_\zeta f, \quad T_\zeta f = S_\zeta f - S_\zeta A^* K(\zeta)^{-1} K(\zeta)^{-1} BS_\zeta f,$$

which proves the form (2).

It is possible to look for operators $A^*, B : G \to H$, what means $(A\phi, c)_G = (\phi, A^*c)_H$. According to equation (1):

$$\int_0^\infty \phi(x)(\alpha(s),c)ds = (\phi, A^*c)_H.$$

So,

$$A^*c(s) = \overline{(\alpha(s),c)_G} = (c, \alpha(s))_G. \tag{5}$$

Let's denote:

$$\phi(s) = A^*c(s) = \overline{(\alpha(s),c)_G} = (c, \alpha(x)). \tag{6}$$

According to the notation (3), let's have (1):

$$K(\zeta)c = c + BS_\zeta A^*c = c + BS_\zeta \phi_1 = c + \int_0^\infty \frac{\phi_1(s)}{s-\zeta}ds.$$

Let's denote:

$$R_\zeta \phi(s) = \frac{\phi(s) - \phi(\zeta)}{s - \zeta}. \tag{7}$$

Then the operator:

$$N(\zeta) = 1 + BR_\zeta A^* \tag{8}$$

is reversible around a point $\zeta = 0$ except perhaps a discrete set.

Let's denote:

$$\phi_\sigma(s) = R_\sigma \alpha(s) = \frac{\alpha(s) - \alpha(\sigma)}{s - \sigma}, \ R_\sigma \alpha(\sigma) = \alpha'(\sigma), \tag{9}$$

$$\phi'_\sigma(s) = \frac{\alpha'(s)(s - \zeta) - 1 \cdot (\alpha(\zeta) - \alpha(\sigma))}{(s - \sigma)^2}. \tag{10}$$

Then,

$$R_\sigma^2(s) = R_\sigma(R_\sigma(s)) =$$

$$= \frac{R_\sigma(s) - R_\sigma(\sigma)}{s - \sigma} = \frac{\frac{\alpha(s) - \alpha(\sigma)}{s - \sigma} - \alpha'(\sigma)}{s - \sigma} =$$

$$= \frac{\alpha(s) - \alpha(\sigma) - \alpha'(\sigma)(s - \sigma)}{(s - \sigma)^2} = -\phi'_\sigma(s).$$

Therefore,

$$\phi'_\sigma(s) = -R_\sigma^2(s). \tag{11}$$

Theorem. Let $V = (\bullet, \beta)_H \alpha$ and $[\sigma_1, \sigma_2]$ is arbitrary finite half-axis interval $[0, \infty]$. Then, if:

$$\|V - V^*\| \le \min_{[\sigma_1, \sigma_2]} \frac{\|R_\sigma \alpha\|_H}{\|R_\sigma^2 \alpha\|_H}. \tag{12}$$

Then the operator $T = S + V$ in the interval $[\sigma_1, \sigma_2]$ may have only a finite set of eigenvalues.

Proof. To prove this theorem, let's use methods which are similar to the methods in [7]. If $(T - \sigma)\phi = 0$, then its own element has the form $\phi = \phi_0$ (9), where $\alpha(\sigma) = 0$. Really, if $T = S + (\bullet, \beta)_H \alpha = 0$, then $(S - \sigma)\phi + (\phi, \beta)_H \alpha = 0$ or $(S - \sigma)\phi(s) + (\phi, \beta)_H \alpha(s) = 0$, from where:

$$\phi(s) = -\frac{(\phi, \beta)_H}{s - \sigma} \alpha(s).$$

As $\phi(s)$ is integrated, then $\alpha(\sigma) = 0$ and then:

$$\phi(s) = -(\phi, \beta)_H \frac{\alpha(s) - \alpha(\sigma)}{s - \sigma} = -(\phi, \beta) R_\sigma \alpha(s).$$

Let's suppose that the operator T under condition (10) has infinity of eigenvalues in $[\sigma_1, \sigma_2]$, let's find a convergent subsequence $\sigma_k \to \sigma_0 \in [\sigma_1, \sigma_2]$ of eigenvalues. Due to the closed nature of the operator T, it follows that σ_0 is also its eigenvalue. Since $D(T^*) = D(T)$, then:

$$(T_{\sigma_k}, \phi_{\sigma_0})_H = (\phi_{\sigma_k}, T\phi_{\sigma_0})_H + (\phi_{\sigma_k}, (T^* - T)\phi_\sigma)_H,$$

or

$$\sigma_k(\phi_{\sigma_k}, \phi_{\sigma_0}) - \sigma_0(\phi_{\sigma_k}, \phi_{\sigma_0}) = (\phi_{\sigma_k}, (V^* - V)\phi_{\sigma_0}),$$

$$(\sigma_k - \sigma_0)(\phi_{\sigma_k}, \phi_{\sigma_0}) = (\phi_{\sigma_k}, (V^* - V)\phi_{\sigma_0})_H.$$

When $k \to \infty$:

$$0 = (\phi_{\sigma_0}, (V^* - V)\phi_{\sigma_0})_H,$$

therefore

$$(\sigma_k - \sigma_0)(\phi_{\sigma_k}, \phi_{\sigma_0})_H = (\phi_{\sigma_k} - \sigma_{\sigma_0}, (V^* - V)\phi_{\sigma_0})_H.$$

Let's divide by $\sigma_k - \sigma_0 \to \infty$ and $k \to \infty$, then:

$$\|\phi_\sigma\|_H^2 = (\phi'_\sigma, (V^* - V)\phi_{\sigma_0})_H,$$

$$\|\phi_{\sigma_0}\|^2 = (\phi'_{\sigma_0}, (V^* - V)\phi_{\sigma_0}),$$

$$\|\sigma_{\sigma_0}\| \le \|\phi'_\sigma\| \cdot \|V^* - V'\|,$$

contrary to equality (10).

3. Results of research and discussion

The finiteness of the discrete spectrum, which was obtained earlier in [10], is now obtained for the Friedrichs model.

The Friedrichs model in an arbitrary interval can have a finite set of eigenvalues under certain restrictions. In the future, it is possible to enhance the results: instead of the finiteness of the spectrum in a finite interval, it is possible to obtain finiteness in the whole half-axis. If to make sure that the resolvent goes to zero when the parameter goes to infinity and if the finite interval contains all eigenvalues.

Sometimes it happens that the eigenvalues are on a continuous spectrum. The finiteness of the discrete spectrum, which was obtained in [10], in this paper was obtained for the Friedrichs model.

4. Conclusions

An important result of a theoretical nature is the assertion that, under certain constraints, the Friedrichs model in an arbitrary interval can have a finite set of eigenvalues.

Thus, the Friedrichs model $T = S + V$ is in an arbitrary interval $[\sigma_1, \sigma_2] \subset [0, \infty)$ can have a finite set of eigenvalues under certain restrictions on perturbations, namely:

$$\|V - V^*\| < \min_{\sigma \in [\sigma_1, \sigma_2]} \frac{\|R_\sigma \alpha\|}{\|R_\sigma^2 \alpha\|_H}, \ V = (\cdot, \beta)\alpha.$$

In the future, it is possible to enhance the results: instead of the finiteness of the spectrum in $[\sigma_1, \sigma_2] \subset [0, \infty)$ it is possible to get finiteness in the whole half-axis $[0, \infty)$, if to make sure the resolvent $T_\sigma f$ goes to zero at $\sigma \to \infty$, if $[\sigma_1, \sigma_2]$ contains all eigenvalues of the operator T.

The result of this research has an important theoretical character.

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