UDC 539.3
DOI: 10.15587/2706-5448.2022.263238
Article type «Reports on Research Projects»

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CONSTRUCTION OF HOMOGENEOUS SOLUTIONS IN THE TORSION PROBLEM FOR A TRANSVERSALLY ISOTROPIC SPHERE WITH VARIABLE ELASTIC MODULI

The object of research is the problem of torsion for a radially inhomogeneous transversally isotropic sphere and the study based on this three-dimensional stress-strain state.

To establish the scope of applicability of existing applied theories and to create more refined applied theories of inhomogeneous shells, it is important to study the stress-strain state of inhomogeneous bodies based on three-dimensional equations of elasticity theory.

The problem of torsion of a radially inhomogeneous transversally isotropic non-closed sphere containing none of the poles 0 and ϖ is considered. It is believed that the elastic moduli are linear functions of the radius of the sphere. It is assumed that the lateral surface of the sphere is free from stresses, and arbitrary stresses are given on the conic sections, leaving the sphere in equilibrium.

The formulated boundary value problem is reduced to a spectral problem. After fulfilling the homogeneous boundary conditions specified on the side surfaces of the sphere, a characteristic equation is obtained with respect to the spectral parameter. The corresponding solutions are constructed depending on the roots of the characteristic equation. It is shown that the solution corresponding to the first group of roots is penetrating, and the stress state determined by this solution is equivalent to the torques of the stresses acting in an arbitrary section θ=const. The solutions corresponding to the countable set of the second group of roots have the character of a boundary layer localized in conic slices. In the case of significant anisotropy, some boundary layer solutions decay weakly and can cover the entire region occupied by the sphere.

On the basis of the performed three-dimensional analysis, new classes of solutions (solutions having the character of a boundary layer) are obtained, which are absent in applied theories. In contrast to an isotropic radially inhomogeneous sphere, for a transversely isotropic radially inhomogeneous sphere, a weakly damped boundary layer solution appears, which can penetrate deep far from the conical sections and change the picture of the stress-strain state.

Keywords: *torsion problem, elastic moduli, Legendre equations, penetrating solutions, boundary layer solutions, torque.*

Received date: 10.07.2022
Accepted date: 17.08.2022
Published date: 26.08.2022

How to cite
Yusubova, S. (2022). Construction of homogeneous solutions in the torsion problem for a transversally isotropic sphere with variable elastic moduli. Technology Audit and Production Reserves, 4 (1 (66)), 15–20. doi: http://doi.org/10.15587/2706-5448.2022.263238

1. Introduction

One of the properties of materials that affect the stress-strain state of elastic bodies is their inhomogeneity. Various materials are being developed and created, the characteristics of which, in particular, the elastic moduli, can change continuously along certain directions [1]. These materials offer unique advantages over traditional materials.

Despite the existence of a number of applied theories of shells based on various hypotheses, the areas of their applicability have been little studied. To establish the scope of applicability of existing applied theories and to create new, more refined, applied theories of inhomogeneous shells, it is important to analyze the stress-strain state of inhomogeneous bodies from the standpoint of three-dimensional equations of elasticity theory.

The study of the stress-strain state of inhomogeneous bodies on the basis of three-dimensional equations of the theory of elasticity is associated with significant mathematical difficulties. Along with this, from the physical point of view, new qualitative and quantitative effects arise.

A number of studies [2] are devoted to the study of three-dimensional problems of elasticity theory for a sphere. The problem of elasticity theory for a sphere was studied by Saint-Venant [3]. In [4], on the basis of the equations of the theory of elasticity for a sphere, a general solution was obtained that satisfies the boundary conditions on the contour in the sense of Saint-Venant, and an analysis of the stress-strain state of the sphere was carried out. In [5], based on the equations of elasticity theory for a thick isotropic sphere, homogeneous solutions were constructed that depend on the roots of the transcendental equation.

In [6], on the basis of solving three-dimensional problems of the theory of elasticity for a sphere of small thickness, the accuracy of existing applied theories was studied and a method for constructing refined applied theories was given. In [7], a three-dimensional asymptotic theory of a spherical shell of small thickness is presented. In [8], an analysis of the three-dimensional stress-strain state of a three-layer sphere with soft filler is given. In [9], the problem of torsion was studied for a radially layered sphere with an arbitrary number of alternating hard and soft layers. The existence of weakly damped boundary layer solutions and the possible violation of the Saint-Venant principle in its classical formulation are shown. The work [10] is devoted to the study of thermal stresses in electromagnetically elastic hollow balls made of a functionally graded material. In [11, 12], thermal and mechanical stresses in a hollow thick radially inhomogeneous sphere are studied, when the properties of the material change along the radius according to power laws. In [13], using the finite element method and spline collocation, the problem of elasticity theory for a radially inhomogeneous hollow ball was studied. The results obtained by finite element methods and spline collocation are compared. In [14], an axisymmetric problem of elasticity theory for a radially inhomogeneous transversely isotropic sphere of small thickness was studied by the method of asymptotic integration of the equations of elasticity theory. The nature of the stress-strain state is established. In [15], an axisymmetric problem of elasticity theory for a sphere of small thickness with variable moduli of elasticity was considered by the method of homogeneous solutions. Asymptotic formulas for displacements and stresses are obtained, which make it possible to calculate the three-dimensional stress-strain state of a radially inhomogeneous sphere. In [16], an axisymmetric problem of elasticity theory for a radially inhomogeneous transversally isotropic sphere of small thickness was considered by the method of homogeneous solutions. Based on the asymptotic analysis carried out, three groups of solutions are obtained: a penetrating solution, a solution having the nature of an edge effect, and a solution having the nature of a boundary layer. The branching of the third group of roots of the characteristic equation generates a countable set of new solutions. A weakly damped boundary layer solution appears.

In order to construct an applied theory of torsion for a radially inhomogeneous transversally isotropic sphere, which adequately takes into account the occurrence of weakly damped boundary layer solutions, it is important to analyze its stress-strain state based on the equations of elasticity theory.

Thus, *the object of research* is the problem of torsion for a radially inhomogeneous transversely isotropic sphere and the study on the basis of this three-dimensional stress-strain state.

The aim of this research is to construct a solution and reveal the features of the stress-strain state for the problem of torsion of a radially inhomogeneous sphere. This will allow to evaluate the areas of applicability of existing applied theories for a radially inhomogeneous sphere.

2. Research methodology

The problem of torsion for a radially inhomogeneous sphere is studied on the basis of the equation of elasticity theory. Equilibrium equations are given that describe the torsion of a radially inhomogeneous transversally isotropic sphere in a spherical coordinate system, and a boundary value problem is formulated.

3. Research results and discussion

Let's consider the problem of torsion for a radially inhomogeneous transversely isotropic non-closed hollow sphere of small thickness. Let's assume that the sphere does not contain any of the poles 0 and π (Fig. 1). In the spherical coordinate system, the area occupied by the sphere will be denoted by $\Gamma=\left\{r\in\left[r_1;r_2\right],\ \theta\in\left[\theta_1;\theta_2\right],\ \varphi\in\left[0;2\pi\right]\right\}$.

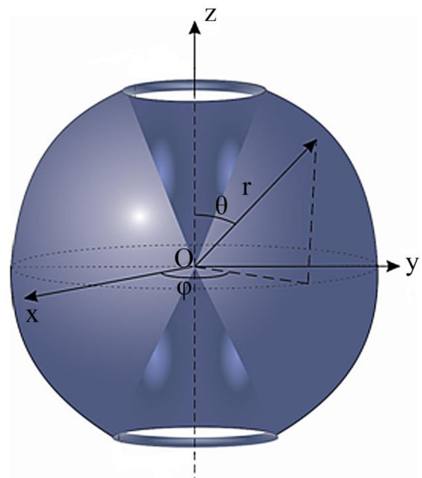

Fig. 1. An open hollow sphere that does not contain any of the poles 0 and π

Let's assume that the change in the elastic modulus along the radius occurs linearly:

$$A_{22}=a_{22}^{(0)}r,\ A_{23}=a_{23}^{(0)}r,\ A_{44}=a_{44}^{(0)}r, \tag{1}$$

where $a_{22}^{(0)}$, $a_{23}^{(0)}$, $a_{44}^{(0)}$ are some constant values.

The equilibrium equation in the absence of body forces in a spherical coordinate system r, θ, φ has the form [17]:

$$\frac{\partial\sigma_{r\varphi}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\varphi\theta}}{\partial\theta}+\frac{3\sigma_{r\varphi}+2\sigma_{\varphi\theta}\operatorname{ctg}\theta}{r}=0, \tag{2}$$

where $\sigma_{r\varphi}$, $\sigma_{\varphi\theta}$ – the components of the stress tensor, which are expressed in terms of displacement vectors $\upsilon_\varphi=\upsilon_\varphi(r,\theta)$ as follows [17]:

$$\sigma_{\varphi\theta}=\frac{\left(A_{22}-A_{23}\right)}{2}\frac{1}{r}\left(\frac{\partial\upsilon_\varphi}{\partial\theta}-\upsilon_\varphi\operatorname{ctg}\theta\right), \tag{3}$$

$$\sigma_{r\varphi}=A_{44}\left(\frac{\partial\upsilon_\varphi}{\partial r}-\frac{\upsilon_\varphi}{r}\right). \tag{4}$$

Substituting (3), (4) into (2), taking into account (1), let's obtain the equations of equilibrium in displacements:

$$a_{44}^{(0)}\left(r\frac{\partial^2\upsilon_\varphi}{\partial r^2}+3\frac{\partial\upsilon_\varphi}{\partial r}-\frac{3\upsilon_\varphi}{r}\right)+\frac{\left(a_{22}^{(0)}-a_{23}^{(0)}\right)}{2r^2}\times$$

$$\times\left(\frac{\partial^2\upsilon_\varphi}{\partial\theta^2}+\frac{\partial\upsilon_\varphi}{\partial\theta}\operatorname{ctg}\theta-\frac{\cos 2\theta}{\sin^2\theta}\upsilon_\varphi\right)=0. \tag{5}$$

Let's introduce a new dimensionless radial variable ρ related to r by the relation:

$$\rho=\frac{1}{\varepsilon}\ln\left(\frac{r}{r_0}\right), \quad (6)$$

where $\varepsilon=1/2\ln(r_2/r_1)$ – small parameter characterizing the thickness of the sphere; $r_0=\sqrt{r_1 r_2}$; $\rho\in[-1;1]$.

Taking into account (6), the equilibrium equations take the form:

$$b_{44}^{(0)}\left(\frac{\partial^2 u_\varphi}{\partial\rho^2}+2\varepsilon\frac{\partial u_\varphi}{\partial\rho}-3\varepsilon^2 u_\varphi\right)+\frac{\left(b_{22}^{(0)}-b_{23}^{(0)}\right)\varepsilon^2}{2}\times$$
$$\times\left(\frac{\partial^2 u_\varphi}{\partial\theta^2}+\frac{\partial u_\varphi}{\partial\theta}\operatorname{ctg}\theta-\frac{\cos 2\theta}{\sin^2\theta}u_\varphi\right)=0, \quad (7)$$

where $u_\varphi=\upsilon_\varphi/r_0$, $b_{ij}^{(0)}=a_{ij}^{(0)}r_0/G_0$ – dimensionless quantities; G_0 – some parameter having the dimension of the modulus of elasticity.

Let's assume that the lateral part of the sphere boundary is stress-free:

$$\sigma_{\rho\varphi}=\frac{b_{44}^{(0)}}{\varepsilon}\left(\frac{\partial u_\varphi}{\partial\rho}-\varepsilon u_\varphi\right)\bigg|_{\rho=\pm1}=0, \quad (8)$$

where $\sigma_{\rho\varphi}=\sigma_{r\varphi}/G_0$ – dimensionless quantity.

Let's consider that stresses are given at the ends of the sphere (on conical sections):

$$\sigma_{\varphi\theta}\big|_{\theta=\theta_s}=f_s(\rho), \quad (9)$$

where $f_s(\rho)(s=1;2)$ – sufficiently smooth functions that have an order $O(1)$ with respect to ε and satisfy the equilibrium conditions.

A homogeneous solution is any solution of the equilibrium equation (7) that satisfies the condition of the absence of stresses on the side surfaces.

Let's construct homogeneous solutions. Solution (7) is sought in the form:

$$u_\varphi(\rho;\theta)=c(\rho)m'(\theta). \quad (10)$$

Here, the function $m(\theta)$ satisfies the Legendre equation [7]:

$$m''(\theta)+\operatorname{ctg}\theta\cdot m'(\theta)+\left(z^2-\frac{1}{4}\right)m(\theta)=0, \quad (11)$$

moreover, the parameter z is determined after the fulfillment of the boundary conditions on the lateral surface.

Substituting (10) into (7), (8) let's obtain:

$$c''(\rho)+2\varepsilon c'(\rho)+\left[\frac{\left(b_{22}^{(0)}-b_{23}^{(0)}\right)}{2b_{44}^{(0)}}\left(\frac{9}{4}-z^2\right)-3\right]\varepsilon^2 c(\rho)=0, \quad (12)$$

$$b_{44}^{(0)}\left(c'(\rho)-\varepsilon c(\rho)\right)\big|_{\rho=\pm1}=0. \quad (13)$$

Solution (12) has the form:

$$c(\rho)=D_1 e^{\varepsilon(t-1)\rho}+D_2 e^{-\varepsilon(t+1)\rho}, \quad (14)$$

where D_1, D_2 are arbitrary constants;

$$t=\sqrt{4+\frac{\left(b_{22}^{(0)}-b_{23}^{(0)}\right)}{2b_{44}^{(0)}}\left(z^2-\frac{9}{4}\right)}.$$

With the help of (14), satisfying the boundary conditions (13), with respect to D_1, D_2, let's obtain a homogeneous linear system of algebraic equations:

$$\begin{cases}(t-2)e^{-\varepsilon(t-1)}D_1-(t+2)e^{\varepsilon(t+1)}D_2=0,\\(t-2)e^{\varepsilon(t-1)}D_1-(t+2)e^{-\varepsilon(t+1)}D_2=0.\end{cases} \quad (15)$$

From the existence condition for nontrivial solutions (15), there is the characteristic equation:

$$\Delta(z;\varepsilon)=\left(z^2-\frac{9}{4}\right)sh\left(2\varepsilon\sqrt{4+\frac{\left(b_{22}^{(0)}-b_{23}^{(0)}\right)}{2b_{44}^{(0)}}\left(z^2-\frac{9}{4}\right)}\right)=0. \quad (16)$$

The function $\Delta(z;\varepsilon)$ has two groups of zeros with the following properties:

1) the first group consists of zeros $z_0^{\pm}=\pm3/2$;
2) the second group consists of a countable set of zeros:

$$z_k=\pm\sqrt{\frac{9}{4}-\frac{2b_{44}^{(0)}}{b_{22}^{(0)}-b_{23}^{(0)}}\left(\frac{\pi^2k^2}{4\varepsilon^2}+4\right)}, \quad (17)$$

which tend to infinity as $\varepsilon\to0$.

Displacements and stresses corresponding to zeros $z_0^{\pm}=\pm3/2$ have the form:

$$u_\varphi^{(1)}(\rho;\theta)=D_0 e^{\varepsilon\rho}\left(\frac{1}{2}\sin\theta\ln\left(\operatorname{ctg}^2\left(\frac{\theta}{2}\right)\right)+\operatorname{ctg}\theta\right), \quad (18)$$

$$\sigma_{\rho\varphi}^{(1)}=0, \sigma_{\theta\varphi}^{(1)}=-\frac{\left(b_{22}^{(0)}-b_{23}^{(0)}\right)e^{\varepsilon\rho}}{\sin^2\theta}D_0. \quad (19)$$

Displacements and stresses corresponding to the second group of zeros have the form:

$$u_\varphi^{(2)}(\rho;\theta)=\sum_{k=1}^{\infty}e^{-\varepsilon(\rho+1)}\begin{bmatrix}\pi k\cos\left(\frac{\pi k}{2}(1-\rho)\right)-\\-4\varepsilon\sin\left(\frac{\pi k}{2}(1-\rho)\right)\end{bmatrix}m_k'(\theta), \quad (20)$$

$$\sigma_{\rho\varphi}^{(2)}=\sum_{k=1}^{\infty}2b_{44}^{(0)}e^{-\varepsilon(\rho+1)}\left(\frac{\pi^2k^2}{4\varepsilon}+4\varepsilon\right)\sin\left(\frac{\pi k}{2}(1-\rho)\right)m_k'(\theta), \quad (21)$$

$$\sigma_{\theta\varphi}^{(2)}=\sum_{k=1}^{\infty}\left(b_{22}^{(0)}-b_{23}^{(0)}\right)e^{-\varepsilon(\rho+1)}\begin{bmatrix}-\frac{\pi k}{2}\cos\left(\frac{\pi k}{2}(1-\rho)\right)+\\+2\varepsilon\sin\left(\frac{\pi k}{2}(1-\rho)\right)\end{bmatrix}\times$$
$$\times\left[2\operatorname{ctg}\theta m_k'(\theta)+\left(z_k^2-\frac{1}{4}\right)m_k(\theta)\right], \quad (22)$$

where

$$m_k(\theta)=N_{1k}P_{z_k-\frac{1}{2}}(\cos\theta)+N_{2k}Q_{z_k-\frac{1}{2}}(\cos\theta);$$

$$P_{z_k-\frac{1}{2}}(\cos\theta),\ Q_{z_k-\frac{1}{2}}(\cos\theta)$$

are the Legendre functions of the first and second kind, respectively; N_{1k}, N_{2k} are unknown constants.

Equations (12), (13) will be represented in the following form:

$$Ac=\lambda c, \tag{23}$$

where

$$Ac= =\left\{q\left(-c''(\rho)-2\varepsilon c'(\rho)+3\varepsilon^2 c(\rho)\right);b_{44}^{(0)}\left(c'(\rho)-\varepsilon c(\rho)\right)\Big|_{\rho=\pm1}=0\right\},$$

$$\lambda=\frac{9}{4}-z^2,\ q=\frac{2b_{44}^{(0)}}{\varepsilon^2\left(b_{22}^{(0)}-b_{23}^{(0)}\right)}.$$

Let's introduce a Hilbert space H with inner product:

$$(c,v)=\int_{-1}^{1}c(\rho)v(\rho)e^{2\varepsilon\rho}d\rho.$$

Let's prove that A is a symmetric operator in a Hilbert space $H(-1;1)$ having a weight $e^{2\varepsilon\rho}$. For any function, $c(\rho)\in D_A$, $v(\rho)\in D_A$ there is:

$$(Ac,v)-(c,Av)=\int_{-1}^{1}(vAc-cAv)e^{2\varepsilon\rho}d\rho=$$

$$\int_{-1}^{1}\begin{bmatrix}q(-c''(\rho)-2\varepsilon c'(\rho)+3\varepsilon^2 c(\rho))v(\rho)-\\-q\left(-v''(\rho)-2\varepsilon v'(\rho)+3\varepsilon^2 v(\rho)\right)c(\rho)\end{bmatrix}e^{2\varepsilon\rho}d\rho=$$

$$=q\int_{-1}^{1}\begin{bmatrix}2\varepsilon\left(c(\rho)v'(\rho)-c'(\rho)v(\rho)\right)+\\+c(\rho)v''(\rho)-v(\rho)c''(\rho)\end{bmatrix}e^{2\varepsilon\rho}d\rho. \tag{24}$$

Using integration by parts and taking into account the boundary conditions (13) from (24), let's find that:

$$(Ac,v)-(c,Av)=0,\ \text{i. e.}\ (Ac,v)=(c,Av). \tag{25}$$

All eigenvalues $\lambda_k(A)$ are real and their corresponding eigenfunctions are orthogonal:

$$(c_k,c_n)=0;\ (k\neq n), \tag{26}$$

where

$$c_n=\left(\pi n\cos\left(\frac{\pi n}{2}(1-\rho)\right)-4\varepsilon\sin\left(\frac{\pi n}{2}(1-\rho)\right)\right)e^{-\varepsilon(\rho+1)}.$$

The general solution of problem (7), (8) will be a superposition of solutions corresponding to the different groups of roots of the characteristic equation (16) found above:

$$u_\varphi(\rho,\theta)=D_0e^{\varepsilon\rho}\left(\frac{1}{2}\sin\theta\ln\left(\operatorname{ctg}^2\frac{\theta}{2}\right)+\operatorname{ctg}\theta\right)+$$

$$+\sum_{k=1}^{\infty}e^{-\varepsilon(\rho+1}\begin{bmatrix}\pi k\cos\left(\frac{\pi k}{2}(1-\rho)\right)-\\-4\varepsilon\sin\left(\frac{\pi k}{2}(1-\rho)\right)\end{bmatrix}m_k'(\theta), \tag{27}$$

$$\sigma_{\rho\varphi}=\sum_{k=1}^{\infty}2b_{44}^{(0)}e^{-\varepsilon(\rho+1)}\left(\frac{\pi^2k^2}{4\varepsilon}+4\varepsilon\right)\sin\left(\frac{\pi k}{2}(1-\rho)\right)m_k'(\theta), \tag{28}$$

$$\sigma_{\theta\varphi}=-\frac{\left(b_{22}^{(0)}-b_{23}^{(0)}\right)e^{\varepsilon\rho}}{\sin^2\theta}D_0+\sum_{k=1}^{\infty}\left(b_{22}^{(0)}-b_{23}^{(0)}\right)e^{-\varepsilon(\rho+1)}\times$$

$$\times\begin{bmatrix}2\varepsilon\sin\left(\frac{\pi k}{2}(1-\rho)\right)-\\-\frac{\pi k}{2}\cos\left(\frac{\pi k}{2}(1-\rho)\right)\end{bmatrix}\begin{bmatrix}2\operatorname{ctg}\theta m_k'(\theta)+\\+\left(z_k^2-\frac{1}{4}\right)m_k(\theta)\end{bmatrix}. \tag{29}$$

Let's prove that the constant D_0, in the absence of external forces on the side surfaces, is proportional to the torque M of the stresses acting in the section $\theta=\text{const}$.

For torques M stresses acting in the section $\theta=\text{const}$ there is [7]:

$$M=2\pi\sin^2\theta\int_{r_1}^{r_2}\sigma_{\theta\varphi}r^2dr, \tag{30}$$

so

$$M=2\pi\varepsilon\sin^2\theta\int_{-1}^{1}\sigma_{\theta\varphi}e^{3\varepsilon\rho}d\rho. \tag{31}$$

Substituting (29) into (31) using integration by parts, let's obtain:

$$M=2\pi\varepsilon\sin^2\theta\times$$

$$\times\int_{-1}^{1}\left\{-\frac{\left(b_{22}^{(0)}-b_{23}^{(0)}\right)}{\sin^2\theta}e^{\varepsilon\rho}D_0+\sum_{k=1}^{\infty}\left(b_{22}^{(0)}-b_{23}^{(0)}\right)e^{-\varepsilon(\rho+1)}\times\right.$$

$$\left.\times\begin{bmatrix}2\varepsilon\sin\left(\frac{\pi k}{2}(1-\rho)\right)-\\-\frac{\pi k}{2}\cos\left(\frac{\pi k}{2}(1-\rho)\right)\end{bmatrix}\begin{pmatrix}2\operatorname{ctg}\theta m_k'(\theta)+\\+\left(z_k^2-\frac{1}{4}\right)m_k(\theta)\end{pmatrix}\right\}e^{3\varepsilon\rho}d\rho=$$

$$=2\pi\varepsilon\sin^2\theta\left\{-\frac{\left(b_{22}^{(0)}-b_{23}^{(0)}\right)}{\sin^2\theta}D_0\int_{-1}^{1}e^{4\varepsilon\rho}d\rho+\right.$$

$$+\sum_{k=1}^{\infty}\left(b_{22}^{(0)}-b_{23}^{(0)}\right)\left(2\operatorname{ctg}\theta m_k'(\theta)+\left(z_k^2-\frac{1}{4}\right)m_k(\theta)\right)\times$$

$$\times\begin{bmatrix}2\varepsilon\int_{-1}^{1}\sin\left(\frac{\pi k}{2}(1-\rho)\right)e^{\varepsilon(2\rho-1)}d\rho-\\-\frac{\pi k}{2}\int_{-1}^{1}\cos\left(\frac{\pi k}{2}(1-\rho)\right)e^{\varepsilon(2\rho-1)}d\rho\end{bmatrix}=$$

$$=\pi\left(b_{23}^{(0)}-b_{22}^{(0)}\right)sh(4\varepsilon)D_0. \tag{32}$$

So,

$$D_0=\frac{M}{\pi\left(b_{23}^{(0)}-b_{22}^{(0)}\right)sh(4\varepsilon)}. \tag{33}$$

Equation (18) is a penetrating solution and determines the internal stress-strain state of the sphere. The stress state corresponding to solution (20) is self-balanced in each section $\theta=\text{const}$ and has the nature of a boundary layer localized at the ends.

The main term of the asymptotic solution of equation (24) for the second group of roots of equation (16) has the following form [14]:

$$m_k(\theta)=$$

$$=\begin{cases}(\mu\pi^2k^2)^{-\frac{1}{4}}\dfrac{1}{\sqrt{\sin\theta}}\exp\left(-\dfrac{\pi k}{\varepsilon}\sqrt{\mu}(\theta-\theta_1)\right)(1+O(\varepsilon))\\ \text{in the vicinity of } \theta=\theta_1,\\ (\mu\pi^2k^2)^{-\frac{1}{4}}\dfrac{1}{\sqrt{\sin\theta}}\exp\left(\dfrac{\pi k}{\varepsilon}\sqrt{\mu}(\theta-\theta_2)\right)(1+O(\varepsilon))\\ \text{in the vicinity of } \theta=\theta_2.\end{cases} \quad (34)$$

where

$$\mu=\frac{b_{44}^{(0)}}{2\left(b_{22}^{(0)}-b_{23}^{(0)}\right)}.$$

Solutions (20)–(22) decrease exponentially as to move away from the conic sections $\theta=\theta_j\ (j=1,2)$. It can be seen from (34) that for fixed values of «k» and for smaller values of μ, some boundary layer solutions do not decay. They can penetrate deeply and significantly change the picture of the stress-strain state far from the ends. In this case, the stress-strain state of a transversely isotropic inhomogeneous and isotropic inhomogeneous sphere is qualitatively different [14].

Let's substitute equation (29) into (9) and take into account (33):

$$\sum_{k=1}^{\infty}e^{-\varepsilon(\rho+1)}\left[2\varepsilon\sin\left(\frac{\pi k}{2}(1-\rho)\right)-\frac{\pi k}{2}\cos\left(\frac{\pi k}{2}(1-\rho)\right)\right]\times$$

$$\times\left[2\operatorname{ctg}\theta_s m_k'(\theta_s)+\left(z_k^2-\frac{1}{4}\right)m_k(\theta_s)\right]=f_s^*(\rho), \quad (35)$$

where

$$f_s^*(\rho)=f_s(\rho)-\frac{Me^{\varepsilon\rho}}{\pi\left(b_{22}^{(0)}-b_{23}^{(0)}\right)sh(4\varepsilon)\sin^2\theta_s}.$$

Multiplying (35) by $c_n(\rho)e^{2\varepsilon\rho}$ and integrating within limits $[-1;1]$, taking into account (26):

$$\left[2\operatorname{ctg}\theta m_k'(\theta)+\left(z_k^2-\frac{1}{4}\right)m_k(\theta)\right]\Bigg|_{\theta=\theta_s}=h_{sk},\ (s=1;2), \quad (36)$$

where

$$h_{sk}=-\frac{1}{16\varepsilon^2+\pi^2k^2}\int_{-1}^{1}\left[4\varepsilon\sin\left(\frac{\pi k}{2}(1-\rho)\right)-\pi k\cos\left(\frac{\pi k}{2}(1-\rho)\right)\right]\times e^{\varepsilon(\rho-1)}f_{3s}^*(\rho)\mathrm{d}\rho.$$

After solving system (36), let's determine the unknown constants in the following form:

$$N_{1k}=\frac{W_{1k}}{W_k};$$

$$N_{2k}=\frac{W_{2k}}{W_k},$$

where

$$W_k=4\operatorname{ctg}\theta_1\operatorname{ctg}\theta_2L^{(1;1)}_{z_k-\frac{1}{2}}(\theta_1;\theta_2)+$$

$$+2\left(z_k^2-\frac{1}{4}\right)\begin{bmatrix}\operatorname{ctg}\theta_1L^{(1;0)}_{z_k-\frac{1}{2}}(\theta_1;\theta_2)+\\+\operatorname{ctg}\theta_2L^{(0;1)}_{z_k-\frac{1}{2}}(\theta_1;\theta_2)\end{bmatrix}+\left(z_k^2-\frac{1}{4}\right)^2L^{(0;0)}_{z_k-\frac{1}{2}}(\theta_1;\theta_2);$$

$$W_{1k}=h_{1k}\left[2\operatorname{ctg}\theta_2Q^{(1)}_{z_k-\frac{1}{2}}(\cos\theta_2)+\left(z_k^2-\frac{1}{4}\right)Q_{z_k-\frac{1}{2}}(\cos\theta_2)\right]-$$

$$-h_{2k}\left[2\operatorname{ctg}\theta_1Q^{(1)}_{z_k-\frac{1}{2}}(\cos\theta_1)+\left(z_k^2-\frac{1}{4}\right)Q_{z_k-\frac{1}{2}}(\cos\theta_1)\right];$$

$$W_{2k}=h_{2k}\left[2\operatorname{ctg}\theta_1P^{(1)}_{z_k-\frac{1}{2}}(\cos\theta_1)+\left(z_k^2-\frac{1}{4}\right)P_{z_k-\frac{1}{2}}(\cos\theta_1)\right]-$$

$$-h_{1k}\left[2\operatorname{ctg}\theta_2P^{(1)}_{z_k-\frac{1}{2}}(\cos\theta_2)+\left(z_k^2-\frac{1}{4}\right)P_{z_k-\frac{1}{2}}(\cos\theta_2)\right];$$

$$L^{(n_1,n_2)}_{z_k-\frac{1}{2}}(\theta_1;\theta_2)=P^{(n_1)}_{z_k-\frac{1}{2}}(\cos\theta_1)Q^{(n_2)}_{z_k-\frac{1}{2}}(\cos\theta_2)-$$

$$-P^{(n_2)}_{z_k-\frac{1}{2}}(\cos\theta_2)Q^{(n_1)}_{z_k-\frac{1}{2}}(\cos\theta_1);\ (n_1,n_2=0.1).$$

It is found that homogeneous solutions are composed of two types: penetrating solution and boundary layer solutions.

The penetrating solution (18), (19) determines the internal stress-strain state of a radially inhomogeneous spherical shell. Solutions (20)–(22), which have the nature of a boundary layer, are localized at the ends, and as they move away from the ends, they decrease exponentially. Such solutions are absent in applied shell theories. The division of the stress-strain state into internal and boundary layer solutions are valid only for a thin shell.

When, for a radially inhomogeneous spherical shell, the values of the elastic modulus do not change within the same order, but differ greatly from each other, then a weak boundary layer appears and the processes of determining the penetrating solution, the weak boundary layer, are not separated. Then, for solving the torsion problem, the use of the above method is not effective.

4. Conclusions

The nature of the stress-strain state of a radially inhomogeneous transversely isotropic sphere of small thickness is determined. Based on the analysis carried out, a penetrating solution and a solution that has the character of a boundary layer are obtained. The stress state determined by the penetrating solution is equivalent to the torque of the stresses acting in an arbitrary section θ=const of the sphere.

The second group of solutions is localized in conic sections and decreases exponentially as to move away from conic sections. For a radially inhomogeneous transversely isotropic sphere, some boundary layer solutions can penetrate deep far from the conic sections and change the pattern of the stress-strain state.

Formulas for displacements and stresses are obtained, which make it possible to accurately calculate the three-

ISSN 2664-9969

dimensional stress-strain state of a radially inhomogeneous transversely isotropic sphere.

It is shown that, in contrast to an isotropic radially inhomogeneous sphere, new solutions appear that are characteristic only of a transversely isotropic radially inhomogeneous sphere.

Conflict of interest

The author declares that they have no conflict of interest in relation to this research, whether financial, personal, authorship or otherwise, that could affect the research and its results presented in this paper.

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