RESEARCH OF THE FUČÍK SPECTRUM FOR THE \((p,q)\)-LAPLACIAN OPERATOR BY MIN-MAX THEORY

The object of research is the Fučík spectrum for the \((p,q)\)-Laplacian operator. In the present paper, we are going to introduce the notion of the Fučík spectrum for a non-linear, non-homogeneous operator, which is the \((p,q)\)-Laplacian operator through the study of the following eigenvalue boundary problem:

\[
\begin{align*}
-\Delta_p u - \Delta_q u &= \lambda (u^+)^{p-1} - \mu (u^-)^{q-1} & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

where \(\Omega \subset \mathbb{R}^N\), \(N \geq 1\) is a bounded open set with smooth boundary and \(\lambda\) and \(\mu\) are two real parameters. In order to establish and show the existence of non-trivial solutions for the problem described above, we will search the weak solution of the energy functional associated to our problem by combining two essential theorems of the Min-Max theory which are the Ljusternick-Schnirelmann (L-S) approach and the Col theorem. In addition to that we are going to use the Lusternik-Schnirelman theorem to show that our problem possesses a critical value \(c_k\) in a suitable manifold that we will define later in the present manuscript. Following to that we will verify the Col geometry by using the critical point associated to the critical value \(c_k\) and by applying the Col theorem, we will find a new critical value \(c_\infty\). After that, by employing the critical value \(c_\infty\) we will demonstrate the existence of the family of curves which generate the set of Fučík spectrum of the \((p,q)\)-Laplacian operator. To complete our research about the structure of the set of the Fučík spectrum of the \((p,q)\)-Laplacian operator we will give the most important properties of the family of curves which are the continuity and the decrease. We have chosen to put our interest on the study of the Fučík spectrum because it’s determination is as important in mathematics as it is in many other fields (physics, plasma-physics, reaction-diffusion equation etc.). We can take as an example it’s use in the field of waves and vibrations where the starting point of the wave or the vibration is influenced by the structure and characteristics of the family of curves which constitute the Fučík spectrum of the \((p,q)\)-Laplacian operator.

Keywords: \((p,q)\)-Laplacian operator, Fučík spectrum, Critical value, Lusternik-Schnirelman Theorem, Col Theorem.

Received date: 17.02.2023
Accepted date: 20.04.2023
Published date: 28.04.2023

1. Introduction

Let \(\lambda\) and \(\mu\) be fixed real. We have the following non-linear problem:

\[
\begin{align*}
-\Delta_p u - \Delta_q u &= \lambda (u^+)^{p-1} - \mu (u^-)^{q-1} & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

where \(\Delta_p\) represents the \(r\)-Laplace operator defined by \(\Delta_p u := \text{div}(\nabla u |^{p-2} \nabla u)\) with \(r \in \{p,q\}, 1 < q < p < \infty, \) and \(u = u^+ - u^-\), \(u^\pm = \max\{0, u^\pm\}\) is the solution of the problem (1).

Hereinafter, the sign \(W^{1,\infty}_p(\Omega)\) denotes the standard Sobolev space equipped with the norm \(\|\cdot\|_p\), and \(\cdot\) will denote the norm in \(L^p(\Omega)\).

We define the Fučík spectrum of the \((p,q)\)-Laplacian operator with the Dirichlet boundary condition as the set \(\Sigma_{p,q}\) of those \((\lambda, \mu)\) \(\in \mathbb{R}^2\) such that the problem (1) has a non-trivial solution in the Sobolev space \(W^{1,\infty}_p(\Omega)\).

The notion of Fučík spectrum was introduced for \(p = 2\) in the 1970s by Fučík [1] and Dancer [2] in connection with the study of the jumping non-linearity. The set \(\Sigma\) itself has attracted an enormous interest among mathematicians for the linearity case we refer to [2] where it is proved that the two line \(\lambda_1 \times \mathbb{R}\) and \(\mathbb{R} \times \lambda_2\) are isolated in \(\Sigma\) and [3] where the author constructed and characterized variationally the first curve in \(\Sigma\) through \((\lambda_1, \lambda_2)\).

In the quasi-linear case \(p \neq 2\), only the ODE situation \(N = 1\) seems to have been investigated in [4].

For the Fučík spectrum of the Laplacian on a two-dimensional torus \(T^2\) we have [5] where the authors show that there exist an explicit global curve in the Fučík spectrum and that their asymptotic limits are positives.

The Fučík spectrum as a notion can be extended to non-linear differential operator. For the \(p\)-Laplacian operator, we refer first to [6] where the author has constructed the curve in \(\Sigma_p\) and he has shown that this is the first non-linear curve in \(\Sigma\).
In [7] the author has studied the following jumping nonlinear problem:
\[-\Delta u = a(u^+) - \beta(u^+) + f, \quad x \in \Omega,\]
\[u = 0, \quad x \in \partial \Omega,\]
where the existence of a non-trivial curve in the Fučík spectrum of the p-Laplacian has been proved by using the sequence of minimax eigenvalues constructed by cohomotological index.

In [8] the authors took interest in to the Fučík spectrum of the p-Laplacian operator with no-flux boundary condition by studying the following problem:
\[-\Delta u = a(u^+) - b(u^+) + f, \quad x \in \Omega,\]
\[u = \text{constant} \quad \text{on} \partial \Omega,\]
\[0 = \int_{\Omega} |u|^{p^*-2} \nabla u \cdot \nabla \sigma.\]

It has been demonstrated that the Fučík spectrum of the p-Laplacian operator with no-flux boundary condition has a first non-trivial curve being Lipchitz, decreasing and with a certain asymptotic behavior.

For the importance and application of the Fučík spectrum of the p-Laplacian, we refer to [9] where the authors studied the existence of sign-changing solutions for the p-Laplacian where the Fučík spectrum possess an important role in the proof of the results.

It’s an evidence that as for the p-Laplacian, it is possible to extend the study of the Fučík spectrum on the (p,q)-Laplacian operator in purpose to exploit at its best any problem that involve this operator (in mathematics, physics).

In the other hand, we need the Fučík spectrum for the study of the existence of nodal solutions of the (p,q)-Laplacian.

Starting with the previous works on the Fučík spectrum about the elliptic operators (Laplacian and p-Laplacian) we were able for the first time to define the Fučík spectrum of the (p,q)-Laplacian and consequently made its study and give its structure.

For the perspectives, we intend to study the existence of nodal solutions of the (p,q)-Laplacian problem based on its Fučík spectrum.

The difficulty of the study of the Fučík spectrum of (p,q)-Laplacian operator is due to its non-homogeneous character which complicates the application of standard theorems of the Min-Max theory. In order to go through this obstacle we had to combine the Ljusternick-Schnirelmann (L-S) theorem and the Col theorem in the manifold \(M_{\alpha,p}\), which we will define later in this paper.

This paper is devoted to study the equation (1) as a constrained problem to which an appropriate min-max approach is applied to establish the existence of non-trivial solutions which determine the Fučík spectrum for the (p,q)-Laplacian operator.

On the one hand the resolution of the problem (1) requires the use of a new method which consists in combining two distinct methods (Col theorem and L-S theory), on the other hand by solving the problem (1) we will be able to define the Fučík spectrum for the (p,q)-Laplacian operator.

In practice, the result of our research will be used in the modeling of several phenomena arising in physics, plasma physics and elementary particles [10–13].

2. Materials and Methods

In this section, we introduce some definitions and theorems, which we will apply to obtain our results. We start with the definition of the Palais-Smale condition. Let \(X\) a Banach space, we consider the manifold:
\[S = \{v \in X : F(v) = \alpha, \alpha \neq 0, \quad \text{with} \quad F \in C^1(\Omega, \mathbb{R}) \quad \text{and} \quad \forall v \in S, F(v) \neq 0,\]
\[\text{Let} \quad f \in C^1(\Omega, \mathbb{R}) \quad \text{and} \quad c \in \mathbb{R}. \quad \text{We can affirm that} \quad f|_c \quad \text{satisfies the Palais-Smale condition (in the level} \quad c\text{) if any sequence} \quad (u_n, b_n) \in \mathbb{S} \times \mathbb{R} \quad \text{such that} \]
\[f(u_n) \to c \quad \text{in} \quad \mathbb{R} \quad \text{and} \quad f'(u_n) - b_n F'(u_n) \to 0 \quad \text{in} \quad \mathbb{R}'.\]

Contains a sub-sequence \((u_{n_k}, b_{n_k})\) that converges to \((u, b)\) in \(S \times \mathbb{R}.

The Ljusternick-Schnirelmann Theorem (L-S) [14] suppose that \(F \text{ and } J \text{ are even, that } J \text{ is not constant, satisfy the Palais-Smale condition on } S \text{ and that } 0 \text{ does not belong to } S.

For any integer, \(k \geq 1\) we put:
\[c_k = \inf_{u \in \mathbb{K}} \sup_{\mathbb{R}} f(v),\]
where \(\mathbb{K} = \{A \in S(X) ; A \subseteq S, \gamma(A) \geq k\} \text{ and } S(X) \text{ designs the set of all closed symmetric subsets of } A \text{ such that } 0 \notin A.

We have for \(k \geq 1\) such that \(k \notin \mathbb{K} \quad \text{and} \quad c_k \in \mathbb{R}. \quad c_k \text{ is the critical value of } f \text{ on } S. \text{ Moreover } c_{k+1} \leq c_k, \text{ and for the integer } j \geq 1 \text{ we have } \mathbb{K}_j \neq \mathbb{K} \text{ and } c_j \leq c_{j+1}, \text{ then:}

\[\gamma(k(c_k)) \geq j + 1,\]
where
\[k(c_j) = \{u \in S, F(u) = c_k, \exists \lambda \in \mathbb{R} \text{ such that } E'(u) = \lambda F'(u)\}.

If for any \(k \geq 1\) we have \(\mathbb{K} \neq \mathbb{R} \text{ and } c_k \text{ in } \mathbb{R} \text{ then:}

\[\lim_{k \to +\infty} c_k = +\infty.\]

Let \(E \text{ a Banach space, } g, f \in C^1(\mathbb{E}, \mathbb{R}), M = \{u \in E : g(u) = 1, \quad u_0, u_1 \in M\}. \text{ Assume that } 1 \text{ is a regular value of } g, \quad e > 0 \text{ such that } f(u - u_0) > e \quad \text{and}

\[\inf_{\mathbb{E}_1} \{f(u) : u \in M, \text{and}\|u - u_0\| = e\} > \max \{f(u_0), f(u_1)\}\]

We also assume that \(f \text{ satisfy the Palais-Smale condition on } M \text{ such that non-empty is:}

\[\Gamma = \{\gamma \in C([-1, 1], M) : \gamma(-1) = u_0, \gamma(1) = u_1\}.

\text{Then by the Col theorem, we have a critical value of } f|_\mathbb{E} \text{ is:}
\[c = \inf_{\gamma \in \Gamma} \max \{f(u)\}.
\]

In order to solve the problem (1) we must apply first the L-S theorem to find the critical value \(c_k\) that we will use to demonstrate the Col geometry.

To show that the Fučík spectrum of the (p,q)-Laplacian is mainly constitute of the family of curves \(c_k\) we use the Col theorem.
3. Results and Discussion

3.1. Existence of solution. Let $\alpha > 0, \beta > 0$, we define the following manifold:

$$M_{p,q} = \left\{ u \in W^{p,q}_0(\Omega) : \frac{\alpha}{p} \int_{\Omega} |\nabla u|^p \, dx + \frac{\beta}{q} \int_{\Omega} |u|^q \, dx = 1 \right\}.$$  

The variational approach of problem (1) is relying on the following functional:

$$I_{\alpha,\beta,\gamma} : W^{p,q}_0(\Omega) \rightarrow \mathbb{R},$$

such that:

$$I_{\alpha,\beta,\gamma}(u) = \int_{\Omega} \left( \frac{\alpha}{p} |\nabla u|^p - \frac{\beta}{q} |u|^q \right) \, dx$$

and

$$G_{\alpha,\beta}(u) = \int_{\Omega} \left( \frac{\alpha}{p} |\nabla u|^p + \frac{\beta}{q} |u|^q \right) \, dx.$$  

Thus, $I_{\alpha,\beta,\gamma}, G_{\alpha,\beta} \in C^1(W^{p,q}_0(\Omega), \mathbb{R})$. Let's define:

$$T = I_{\alpha,\beta,\gamma} |_{M_{p,q}}.$$  

The set $M_{p,q}$ is a smooth sub-manifold of $W^{p,q}_0(\Omega)$ and thus $T$ is $C^1$. By Lagrange multipliers rule, $u \in M_{p,q}$ is a critical point of $T$ if and only if there exists $\lambda \in \mathbb{R}$ such that:

$$T'(u)v = \lambda G_{\alpha,\beta}(u)v, \forall v \in W^{p,q}_0(\Omega).$$  

(2)

Let's describe the relationship between the critical points of $T$ and the Fučík spectrum of problem (1). Given $s > 0$ and $t > 0$, one has that $(\alpha s + \beta t) \in \Sigma_{p,q}$ and if only if there exists a critical point $u \in M_{p,q}$ of $T$ such that $c = T(u)$.

In order to construct a critical point of $T$, let's first check the Palais-Smale condition.

**Lemma 1.** $T$ satisfies the Palais-Smale condition on the sub-manifold $M_{p,q}$.

**Proof.** Let $(u_n)_{n \in \mathbb{N}} \subset M_{p,q}$ and $(v_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ be sequences such that for some constant $K > 0$ we have:

$$|I_{\alpha,\beta,\gamma}(u_n)| \leq K, \quad (3)$$

and

$$\int_{\Omega} |\nabla u_n|^p \, dx + \int_{\Omega} |u_n|^q \, dx - \int_{\Omega} |\nabla u_n|^p \, dx - \int_{\Omega} |u_n|^q \, dx = 0,$$

$$\int_{\Omega} \left( \frac{\alpha}{p} |\nabla u_n|^p + \frac{\beta}{q} |u_n|^q \right) \, dx \leq K$$

(4)

for all $v \in W^{p,q}_0(\Omega)$, where $\varepsilon_n \rightarrow 0$.

From (3) it follows that the sequence $u_n$ remains bounded in $W^{p,q}_0(\Omega)$. Consequently, for a subsequence, $u_n$ converges strongly in $L^p(\Omega)$ and weakly in $W^{p,q}_0(\Omega)$. Note this limit by $u$.

In order to show that $u_n \rightarrow u$ in $W^{p,q}_0(\Omega)$ we remind that:

$$-\Delta : W^{p,q}_0(\Omega) \rightarrow W^{p,q}_0(\Omega)$$

with $r=p$ or $q$, owns the $(S)_r$ property. It is to say that if $u_n \rightarrow u$ in $W^{p,q}_0(\Omega)$ and, $\limsup_{n \rightarrow +} \|\nabla u_n\|_p \leq \|\nabla u\|_p$ then $u_n \rightarrow u$ in $W^{p,q}_0(\Omega)$.

Putting $v = u_n - u$ in (4), we get:

$$\int_{\Omega} |\nabla u_n|^p \, dx \leq \int_{\Omega} |\nabla u|^p \, dx$$

and to the $(S)_r$ property we obtain that $u_n \rightarrow u$ in $W^{p,q}_0(\Omega)$.

In the next step, we will look for local minimizers of the functional:

$$f : W^{p,q}_0(\Omega) \rightarrow \mathbb{R},$$

defined by:

$$f(u) = \int_{\Omega} \left( \frac{\alpha}{p} |\nabla u|^p + \frac{\beta}{q} |u|^q \right) \, dx$$

To fulfill the Mountain-Pass geometry of the functional $\bar{T}$.

**Lemma 2.** For any integer $k \in \mathbb{N}$, the set not empty is:

$$B_k = \{ A \in S \left( W^{p,q}_0(\Omega) \right) ; A \in C, \gamma(A) \geq k \}.$$  

In particular if $X \subset W^{p,q}_0(\Omega)$ is a sub-space of dimension, then:

$$\gamma(M_{p,q} \cap X) = k.$$  

**Proof.** Let $X_k$ a sub-space of $W^{p,q}_0(\Omega)$ such that $\dim X_k = k$

We can show easily that $(X_k \cap M_{p,q})$ is a symmetrical and closed set that does not contain the origin, so $\gamma(M_{p,q} \cap X_k)$ is well defined.

Let now $S$ be the unit sphere in $W^{p,q}_0(\Omega)$. Denote by:

$$P : u \mapsto \frac{1}{\|u\|_r} u, u \neq 0$$

the radial projection in $W^{p,q}_0(\Omega)$.  

---

**ISSN 2664-9969**

**TECHNOLOGY AUDIT AND PRODUCTION RESERVES — № 3/2(71), 2023**
Then $P$ is a bijection between $M_{ab}$ and $S$. We have:

$$P(X_1 \cap M_{ab}) = X_1 \cap P(M_{ab}) = X_1 \cap S.$$ 

So $P$ is a homomorphism between $X_1 \cap M_{ab}$ and $X_1 \cap S$. Since $P$ is odd we get:

$$\gamma(X_1 \cap M_{ab}) = \gamma(X_1 \cap S).$$

According to the genus properties we have:

$$\gamma(X_1 \cap M_{ab}) = k.$$ 

Similar arguments as those used in Lemma 1 show that $f$ satisfies the Palais-Smale condition on $M_{ab}$. Combing this fact and Lemma 2, one can get by the Ljusternick-Schnirelmann theorem that for any $k \in \mathbb{N}$ the quantity:

$$c_k := \inf_{\bar{\varnothing} \in \pi \mathbb{N}} \sup_{u \in M_{ab}} f(u)$$

is a critical value of the functional $f$ with respect to the manifold $M_{ab}$. Hence, a sequence of critical points that we note by $\{u^k\}_{k \in \mathbb{N}} \subset M_{ab}$ also exists.

Next we give the main result of the paper.

**Lemma 3:**

1. For $s > 0$, $t > 0$.
2. $c_k(s, t) = \inf \max I_{s, t, k}(u)$ is a sequence of critical value of $I_{s, t, k}$, where

$$\Gamma = \{\gamma \in C([-1, 1], M_{ab}) : \gamma(-1) = -u_k, \gamma(t) = u_k\}.$$

3. The curve $(s + c_k(s, t), t + c_k(s, t)) \in \Sigma_{pq}$.

**Proof:**

1. First we have: $u_k^p = (-u_k^p) \in M_{ab}$, then for any $\varepsilon > 0$ we have:

$$\|u_k^p - (-u_k^p)\|_p = 2 \|u_k^p\|_p > \varepsilon.$$ 

Now we show that:

$$\inf\left\{I_{s, t, k}(u) : u \in M_{ab}, \|u - (-u_k^p)\|_p = \varepsilon\right\} > \max\{I_{s, t, k}(-u_k^p), I_{s, t, k}(u_k^p)\}.$$ 

Since $c_k$ is a critical value of $f$, there exists a Lagrange multiplier $\bar{\varnothing} \in \mathbb{R}$ and $u \in M_{ab}$ such that:

$$f(u) = \bar{\varnothing} G_{ab}(u).$$

In other words, we have:

$$\int u^{q - 2} u^p \nabla u^p \nabla u^{2} dx + \int u^{q - 2} u^p \nabla u^{2} \nabla u^{2} dx = \alpha \bar{\varnothing} \int u^{q - 2} u^p \nabla u^{2} \nabla u^{2} dx + \bar{\beta} \bar{\varnothing} \int u^{q - 2} u^p \nabla u^{2} \nabla u^{2} dx.$$ 

Taking $u = v$ in the last equation, we get:

$$\frac{1}{p} \int \nabla u^p \nabla u^{2} dx + \frac{1}{q} \int \nabla u^q \nabla u^{2} dx = \bar{\varnothing} \left( \frac{\alpha}{p} \int |u'|^2 dx + \bar{\beta} \frac{1}{q} \int |u'|^2 dx \right).$$

Since $u \in M_{ab}$, we obtain:

$$f(u) = \bar{\varnothing}.$$ 

So $c_k = \bar{\varnothing}$ and $\max\{I_{s, t, k}(-u_k^p), I_{s, t, k}(u_k^p)\} = c_k$.

In the other hand, we have:

$$1 - \frac{1}{q} \int \nabla u^q \nabla u^{2} dx + \frac{1}{p} \int \nabla u^p \nabla u^{2} dx + \frac{1}{q} \int \nabla u^q \nabla u^{2} dx + \frac{1}{p} \int \nabla u^p \nabla u^{2} dx + \frac{1}{q} \int \nabla u^q \nabla u^{2} dx.$$ 

for all $u \in M_{ab}$. Then, it results:

$$\inf \max \{I_{s, t, k}(u) : u \in M_{ab}, \|u - (-u_k^p)\|_p = \varepsilon\} > c_k.$$ 

and this provides the following estimate:

$$\inf \{I(u) : u \in M_{ab}, \|u - (-u_k^p)\|_p = \varepsilon\} > c_k.$$ 

Since $T$ verifies the Palais-Smale condition and 1 is a regular value of $G_{ab}$, then a critical value of $I_{s, t, k}$ is:

$$c_k(s, t) = \inf \max I_{s, t, k}(u).$$

2. $(s + c_k(s, t), t + c_k(s, t)) \in \Sigma_{pq}$ if and only if there exist a critical point $u \in M_{ab}$ such that $c_k = I_{s, t, k}(u)$, and since 1 is satisfies then the curve:

$$(s + c_k(s, t), t + c_k(s, t)) \in \Sigma_{pq}.$$ 

**Lemma 4:** If $c_k(s, t) = \inf \max_{u \in M_{ab}} I_{s, t, k}(u)$ is a critical value of $I_{s, t, k}$, then $c_k = I_{s, t, k}(u)$. 

**Proof:** We have $c_k = I_{s, t, k}(u)$ is a critical value of $I_{s, t, k}$ then:

$$I_{s, t, k}(u) = c_k G_{ab}(u).$$

where $u$ is the critical point associated to $c_k$. 

For any $v \in W_{op}^q(\Omega)$, we have:

$$I_{s, t, k}(u) v = G_{ab}(u) v,$$

that is:

$$\int \nabla u^{q - 2} u^p \nabla u^{2} v dx + \int \nabla u^{q - 2} u^p \nabla u^{2} v dx = -\int u^{q - 2} u^p \nabla u^{2} v dx + \int u^{q - 2} u^p \nabla u^{2} v dx - \int u^{q - 2} u^p \nabla u^{2} v dx + \int u^{q - 2} u^p \nabla u^{2} v dx = c_k \alpha \int |u'|^2 v dx + \bar{\beta} \int |u'|^2 v dx.$$
Consequently,
\[
\int_{\Omega} |\nabla u|^2 \nabla u \nabla x + \int_{\Omega} |\nabla u|^2 \nabla u \nabla x = (\alpha_c + s) \int_{\Omega} |\nabla u|^2 \nabla u \nabla x + (\beta_c + t) \int_{\Omega} |\nabla u|^2 \nabla u \nabla x
\]
\[+ (\alpha_c - s) \int_{\Omega} |\nabla u|^2 \nabla u \nabla x + (\beta_c - t) \int_{\Omega} |\nabla u|^2 \nabla u \nabla x = 0.
\]
Taking,
\[\alpha_c = s_0 \text{ and } \beta_c = t_0,
\]
we get as required:
\[\alpha_c = s_0 \text{ and } \beta_c = t_0.
\]

3.2. Properties of the family of curve. We define the following family of curves:
\[C_\alpha := \{(s+c_\alpha(s,t),t+c_\alpha(s,t)),(t+c_\alpha(s,t),s+c_\alpha(s,t))\}.
\]

Lemma 5: 1. The family of curves \((s,t)\rightarrow (s+c_\alpha(s,t),t+c_\alpha(s,t))\) is Lipschitz and continuous in a way that:
\[s+c_\alpha(s,t)<s'+c_\alpha(s',t),t+c_\alpha(s,t)<t'+c_\alpha(s,t')
\]
and
\[c_\alpha(s,t)>c_\alpha(s',t').
\]
2. The family of curves \((s,t)\rightarrow (s+c_\alpha(s,t),t+c_\alpha(s,t))\) is decreasing in a way that:
\[s+c_\alpha(s,t)<s'+c_\alpha(s',t),t+c_\alpha(s,t)<t'+c_\alpha(s,t')
\]
and
\[c_\alpha(s,t)>c_\alpha(s',t').
\]

Proof. We start by part 1. 1. Let \(s'<s\), respectively \(t'<t\), then:
\[T_{s+c_\alpha(s,t)}(u) \geq T_{s+c_\alpha(s',t)}(u).
\]
for all \(u \in M_{\alpha,\beta}\), then we have:
\[c_\alpha(s',t') \geq c_\alpha(s,t).
\]
then \(\forall \varepsilon > 0\), there exist \(\gamma \in \Gamma\) such that:
\[\max_{\gamma \in [-1,1]} T_{s+c_\alpha(s,t)}(u) \leq c_\alpha(s,t) + \varepsilon,
\]
then
\[0 \leq c_\alpha(s',t') - c_\alpha(s,t) \leq \max_{\gamma \in [-1,1]} T_{s+c_\alpha(s',t')}(u) - \max_{\gamma \in [-1,1]} T_{s+c_\alpha(s,t)}(u) + \varepsilon,
\]
putting now \(u_0 \in \gamma (-1,1)\) such that:
\[\max_{\gamma \in [-1,1]} T_{s+c_\alpha(s',t')}(u) = T_{s+c_\alpha(s',t')} (u_0).
\]
We get:
\[0 \leq c_\alpha(s',t') - c_\alpha(s,t) \leq T_{s+c_\alpha(s',t')} (u_0) - T_{s+c_\alpha(s,t)} (u_0) + \varepsilon,
\]
since \(\varepsilon > 0\), then it's easy to see that:
\[(s,t) \rightarrow (s+c_\alpha(s,t),t+c_\alpha(s,t))
\]
is continuous and Lipchitz.
2. Let \(0<s'<s\), and \(0<t'<t\), then:
\[s'+c_\alpha(s',t'),t'+c_\alpha(s',t') \in P,Q
\]
\[s+c_\alpha(s',t'),t'+c_\alpha(s',t') \in s+c_\alpha(s,t),t+c_\alpha(s,t),
\]
since we have:
\[c_\alpha(s',t') \geq c_\alpha(s,t),
\]
which means that \(c_\alpha\) is decreasing.
These Methods can only be used on elliptic nonlinear operator in bounded spaces of \(\mathbb{R}^N\). If we want to apply them on non-bounded spaces \(\mathbb{R}^N\) we must verify the Sobolev injections.
Based on our work we can further generalize our results by imposing measurable weights under Newmann and Robin boundary conditions which will lead a use of the Fukš spectrum of the \((p,q)\)-Laplacian operator in greater spaces.

4. Conclusions
In this paper, we have shown that the Fukš spectrum of the \((p,q)\)-Laplacian operator is essentially made up by a group of curves \(C_\alpha\) given by:
\[C_\alpha = (s+c_\alpha(s,t),t+c_\alpha(s,t))\]
where \(c_\alpha\) is a sequence of a critical value.
For that we have used a new method that combines the Col and L-S theorems.
This results will lead to a generalization of the Fukš spectrum of the \((p,q)\)-Laplacian operator in a non-bounded spaces.

Acknowledgments
The authors express their thanks to the Laboratory of Applied Mathematics, Mathematics Department, Badji Mokhtar University, Annaba, Algeria, for assistance to carry out this work.

Conflict of interest
The authors declare that they have no conflict of interest in relation to this study, including financial, personal, authorship, or any other, that could affect the study and its results presented in this article.

Financing
There was no external support for this study.

Data availability
The manuscript has associated data in a data repository.
References


Selma Hadjer Djeffal, Postgraduate Student, Laboratory of Applied Mathematics, Department of Mathematics, Badji Mokhtar University, Annaba, Algeria, ORCID: https://orcid.org/0000-0002-1114-4267

Aissa Benselhoub, PhD, Associate Researcher, Environment, Modeling and Climate Change Division, Environmental Research Center (C.R.E), Annaba, Algeria, Visitor Researcher, INFN Frascati National Laboratories, Frascati, Italy, e-mail: aissabenselhoub@cre.dz, ORCID: https://orcid.org/0000-0001-5891-2860

Corresponding author