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Selma Hadjer Djeffal,
Aissa Benselhoub

RESEARCH OF THE FUČIK SPECTRUM FOR THE (p,q)-LAPLACIAN OPERATOR BY MIN-MAX THEORY

The object of research is the Fučik spectrum for the (p,q)-Laplacian operator. In the present paper, we are going to introduce the notion of the Fučik spectrum for a non-linear, non-homogeneous operator, which is the (p,q)-Laplacian operator through the study of the following eigenvalue boundary problem:

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda(u^+)^{p-1} - \mu(u^-)^{q-1} & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$ is a bounded open subset with smooth boundary and λ and μ are two real parameters. In order to establish and show the existence of non-trivial solutions for the problem described above, we will search the weak solution of the energy functional associated to our problem by combining two essentials theorems of the Min-Max theory which are the Ljusternick-Schnirelmann (L-S) approach and the Col theorem. In addition to that we are going to use the Ljusternick-Schnirelman theorem to show that our problem possesses a critical value c_k in a suitable manifold that we will define later in the present manuscript. Following to that we will verify the Col geometry by using the critical point associated to the critical value c_k and by applying the Col theorem, we will find a new critical value c_n. After that, by employing the critical value c_n we will demonstrate the existence of the family of curves which generate the set of Fučik spectrum of the (p,q)-Laplacian operator. To complete our research about the structure of the set of the Fučik spectrum of the (p,q)-Laplacian operator we will give the most important properties of the family of curves which are the continuity and the decrease. We have chosen to put our interest on the study of the Fučik spectrum because it's determination is as important in mathematics as it is in many other fields (physics, plasma-physics, reaction-diffusion equation etc.). We can take as an example it's use in the field of waves and vibrations where the starting point of the wave or the vibration is influenced by the structure and characteristics of the family of curves which constitute the Fučik spectrum of the (p,q)-Laplacian operator.

Keywords: *(p,q)-Laplacian operator, Fučik spectrum, Critical value, Ljusternick-Schnirelmann Theorem, Col Theorem.*

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1. Introduction

Let λ and μ be fixed real. We have the following non-linear problem:

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda(u^+)^{p-1} - \mu(u^-)^{q-1} & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where Δ_r represents the r-Laplace operator defined by $\Delta_r u := div(|\nabla u|^{r-2}\nabla u)$ with $r \in \{p,q\}, 1<q\leq p<\infty$, and $u=u^+-u^-$, $u^\pm = \max\{0,u^\pm\}$ is the solution of the problem (1).

Hereinafter, the sign $W_0^{1,p}(\Omega)$ denotes the standard Sobolev space equipped with the norm $\|\cdot\|_{1,r}$, and $\|\cdot\|_r$ will denote the norm in $L^r(\Omega)$.

We define the Fučik spectrum of the (p,q)-Laplacian operator with the Dirichlet boundary condition as the set $\Sigma_{p,q}$ of those $(\lambda,\mu)\in\mathbb{R}^2$ such that the problem (1) has a non-trivial solution in the Sobolev space $W_0^{1,p}(\Omega)$.

The notion of Fučik spectrum was introduced for $p=2$ in the 1970s by Fučik [1] and Dancer [2] in connection with the study of the jumping non linearity. The set $\sum\limits_2$ itself has attracted an enormous interest among mathematicians for the linear case we refer to [2] where it is proved that the two line $\lambda_1\times\mathbb{R}$ and $\mathbb{R}\times\lambda_1$ are isolated in $\sum\limits_2$ and [3] where the author constructed and characterized variationally the first curve in $\sum\limits_2$ through (λ_2,λ_2).

In the quasi-linear case $p\neq 0$, only the ODE situation $N=1$ seems to have been investigated in [4].

For the Fučik spectrum of the Laplacian on a two-dimensional torus T^2 we have [5] where the authors show that there exist an explicit global curve in the Fučik spectrum and that their asymptotic limits are positives.

The Fučik spectrum as a notion can be extended to non-linear differential operator. For the p-Laplacian operator, we refer first to [6] where the author has constructed the curve in $\sum\limits_p$, and he has shown that this is the first non-linear curve in $\sum\limits_2$.

In [7] the author has studied the following jumping nonlinear problem:

$$\begin{cases}-\Delta_p u=\alpha(u^+)^{p-1}-\beta(u^-)^{p-1}+f, & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases}$$

where the existence of a non-trivial curve in the Fučik spectrum of the p-Laplacian has been proved by using the sequence of minimax eigenvalues constructed by cohotnological index.

In [8] the authors took interest in to the Fučik spectrum of the p-Laplacian operator with no-flux boundary condition by studying the following problem:

$$\begin{aligned}&-\Delta_p u=a(u^+)^{p-1}-b(u^-)^{p-1} && \text{in } \Omega,\\ &u=\text{constant} && \text{on } \partial\Omega,\\ &0=\int_{\partial\Omega}|\nabla u|^{p-2}\,\nabla u\cdot v\,d\sigma.\end{aligned}$$

It has been demonstrated that the Fučik spectrum of the p-Laplacian operator with no-flux boundary condition has a first non-trivial curve being Lipchitz, decreasing and with a certain asymptotic behavior.

For the importance and application of the Fučik spectrum of the p-Laplacian, we refer to [9] where the authors studied the existence of sign-changing solution for the p-Laplacian where the Fučik spectrum possess an important role in the proof of the results.

It's an evidence that as for the p-Laplacian, it is possible to extend the study of the Fučik spectrum on the (p,q)-Laplacian operator in purpose to exploit at its best any problem that involve this operator (in mathematics, physics). In the other hand, we need the Fučik spectrum for the study of the existence of nodals solutions (sign-changing solutions) of the (p,q)-Laplacian.

Starting with the previous works on the Fučik spectrum about the elliptic operators (Laplacian and p-Laplacian) we were able for the first time to define the Fučik spectrum of the (p,q)-Laplacian and consequently made its study and give its structure.

For the perspectives, we intend to study the existence of nodal solutions of the (p,q)-Laplacian problem based on its Fučik spectrum.

The difficulty of the study of the Fučik spectrum of (p,q)-Laplacian operator is due to its non-homogenous character which complicates the application of standard theorems of the Min-Max theory. In order to go through this obstacle we had to combine the Ljusternick-Schnirelmann (L-S) theorem and the Col theorem in the manifold $M_{\alpha,\beta}$, which we will define later in this paper.

This paper is devoted to study the equation (1) as a constrained problem to which an appropriate min-max approach is applied to establish the existence of non-trivial solution which determinate the Fučik spectrum for the (p,q)-Laplacian operator.

On the one hand the resolution of the problem (1) requires the use of a new method which consists in combining two distinct methods (Col theorem and L-S theory), on the other hand by solving the problem (1) we will able to define the Fučik spectrum for the (p,q)-Laplacian operator.

In practice, the result of our research will be used in the modeling of several phenomena arising in physics, plasma physics and elementary particles [10–13].

2. Materials and Methods

In this section, we introduce some definitions and theorems, which we will apply to obtain our results. We start with the definition of the Palais-Smale condition. Let X a Banach space, we consider the manifold:

$$S=\{v\in X: F(v)=\alpha\},\ \alpha\neq 0,$$

with $F\in C^1(X,\mathbb{R})$ and $\forall v\in S, F(v)\neq 0$.

Let $J\in C^1(X,\mathbb{R})$ and $c\in\mathbb{R}$. We can affirm that $J|_S$ satisfies the Palais-Smale condition (in the level c) if any sequence $(u_n,b_n)\in S\times\mathbb{R}$ such that:

$$J(u_n)\to c \text{ in } \mathbb{R} \text{ and} J'(u_n)-b_nF'(u_n)\to 0 \text{ in } X'.$$

Contains a sub-sequence $(u_{n_k},b_{n_k})_k$ that converges to (u,b) in $S\times\mathbb{R}$.

The Ljusternick-Schnirelmann Theorem (L-S) [14] suppose that F and J are even, that J is not constant, satisfy the Palais-Smale condition on S and that 0 does not belong to S. For any integer, $k\geq 1$ we put:

$$c_k=\inf_{A\in B_k}\sup_{u\in A} J(v),$$

where $B_k=\{A\in S(X); A\subset S, \gamma(A)\geq k\}$ and $S(X)$ designs the set of all closed symmetric subsets A of X such that $0\notin A$.

We have for $k\geq 1$ such that $B_k\neq\varnothing$ and $c_k\in\mathbb{R}$, c_k is the critical value of j on S. Moreover $c_k\leq c_{k+1}$, and for the integer $j\geq 1$ we have $B_{k+j}\neq\varnothing$ and $c_k\leq c_{k+j}\in\mathbb{R}$, then:

$$\gamma(k(c_k))\geq j+1,$$

where

$$k(c_k)=\{u\in S; J(u)=c_k, \exists\lambda\in\mathbb{R} \text{ such that } E'(u)=\lambda F'(u)\}.$$

If for any $k\geq 1$ we have $B_k\neq\varnothing$ and $c_k\in\mathbb{R}$ then:

$$\lim_{k\to+\infty} c_k=+\infty.$$

Let E a Banach space, $g,f\in C^1(E,\mathbb{R})$, $M=\{u\in E: g(u)=1\}$, $u_0,u_1\in M$. Assume that 1 is a regular value of g, $\in>0$ such that $\|u_1-u_0\|_E>\in$ and

$$\inf\{f(u): u\in M, \text{and}, \|u-u_0\|=\varepsilon\}>\max\{f(u_0),f(u_1)\}.$$

We also assume that f satisfy the Palais-Smale condition on M such that non-empty is:

$$\Gamma=\{\gamma\in C([-1,1],M):\gamma(-1)=u_0,\gamma(1)=u_1\}.$$

Then by the Col theorem, we have a critical value of $f|_M$ is:

$$c=\inf_{\gamma\in\Gamma}\max_{u\in\gamma[-1,1]} f(u).$$

In order to solve the problem (1) we must apply first the L-S theorem to find the critical value c_k that we will use to demonstrate the Col geometry.

To show that the Fucik spectrum of the (p,q)-Laplacian is mainly constitute of the family of curves c_n we use the Col theorem.

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3. Results and Discussion

3.1. Existence of solution. Let $\alpha>0, \beta>0$, we define the following manifold:

$$M_{\alpha\beta}=\left\{u\in W_0^{1,p}(\Omega):\frac{\alpha}{p}\int_\Omega |u|^p\,dx+\frac{\beta}{q}\int_\Omega |u|^q\,dx=1\right\}.$$

The variational approach of problem (1) is relying on the following functional:

$$I_{s,s_0,t,t_0}, G_{\alpha\beta}: W_0^{1,p}(\Omega)\mapsto\mathbb{R},$$

such that:

$$I_{s,s_0,t,t_0}=\frac{1}{p}\int_\Omega |\nabla u|^p\,dx+\frac{1}{q}\int_\Omega |\nabla u|^q\,dx-$$
$$-\frac{s}{p}\int_\Omega |u^+|^p\,dx+\frac{s_0}{p}\int_\Omega |u^-|^p\,dx,$$
$$-\frac{t}{q}\int_\Omega |u^-|^q\,dx+\frac{t_0}{q}\int_\Omega |u^+|^q\,dx,$$

and

$$G_{\alpha\beta}(u)=\frac{\alpha}{p}\int_\Omega |u|^p\,dx+\frac{\beta}{q}\int_\Omega |u|^q\,dx.$$

Thus, $I_{s,s_0,t,t_0}, G_{\alpha\beta}\in C^1\left(W_0^{1,p}(\Omega),\mathbb{R}\right)$. Let's define:

$$\bar{I}=I_{s,s_0,t,t_0}|_{M_{\alpha\beta}}.$$

The set $M_{\alpha\beta}$ is a smooth sub-manifold of $W_0^{1,p}(\Omega)$ and thus $\bar{I}$ is C^1. By Lagrange multipliers rule, $u\in M_{\alpha\beta}$ is a critical point of $\bar{I}$ if and only if there exists $\lambda\in\mathbb{R}$ such that:

$$\bar{I}'(u)v=\lambda G'_{\alpha\beta}(u)v, \forall v\in W_0^{1,p}(\Omega). \tag{2}$$

Let's describe the relationship between the critical points of $\bar{I}$ and the Fučik spectrum of problem (1). Given $s>0$ and $t>0$, one has that $(\alpha c+s, \beta c+t)$ belongs to the spectrum $\Sigma_{p,q}$ if and only if there exists a critical point $u\in M_{\alpha\beta}$ of $\bar{I}$ such that $c=\bar{I}(u)$.

In order to construct a critical point of $\bar{I}$, let's first check the Palais-Smale condition.

Lemma 1. $\bar{I}$ satisfies the Palais-Smale condition on the sub-manifold $M_{\alpha\beta}$.

Proof. Let $\{u_n\}_{n\in\mathbb{N}}\subset M_{\alpha\beta}$ and $\{c_n\}_{n\in\mathbb{N}}\subset\mathbb{R}$ be sequences such that for some constant $K>0$ we have:

$$\left|I_{s,s_0,t,t_0}(u_n)\right|\le K, \tag{3}$$

and

$$\left|\int_\Omega \left|\nabla u_n\right|^{p-2}\nabla u_n\nabla v\,dx+\int_\Omega \left|\nabla u_n\right|^{q-2}\nabla u_n\nabla v\,dx-\right.$$
$$-(\alpha c_n+s)\int_\Omega (u_n^+)^{p-1}v\,dx-(\beta c_n+t)\int_\Omega (u_n^-)^{q-1}v\,dx-$$
$$-(\alpha c_n-s_0)\int_\Omega (u_n^-)^{p-1}v\,dx-$$
$$\left.-(\beta c_n-t_0)\int_\Omega (u_n^+)^{q-1}v\,dx\right|\le\xi_n\|v\|_{W_0^{1,p}}, \tag{4}$$

for all $v\in W_0^{1,p}(\Omega)$, where $\in_n\to 0$.

From (3) it follows that the sequence u_n remains bounded in $W_0^{1,p}(\Omega)$. Consequently, for a subsequence, u_n converges strongly in $L^p(\Omega)$ and weakly in $W_0^{1,p}(\Omega)$. Note this limit by u.

In order to show that $u_n\to u$ in $W_0^{1,p}(\Omega)$ we remind that:

$$-\Delta_r: W_0^{1,r}(\Omega)\to W_0^{1,r}((\Omega))^*,$$

with $r=p$ or q, owns the (S_+) property. It is to say that if $u_n\rightharpoonup u$ in $W_0^{1,r}(\Omega)$ and, $\limsup_{n\to\infty}\int_\Omega |\nabla u_n|^{r-2}\nabla u_n\nabla(u_n-u)\le 0$ then $u_n\to u$ in $W_0^{1,r}(\Omega)$.

Putting $v=u_n-u$ in (4), we get:

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla(u_n-u)\,dx+$$
$$+\int_\Omega |\nabla u_n|^{q-2}\nabla u_n\nabla(u_n-u)\,dx=$$
$$=(\alpha c_n+s)\int_\Omega (u_n^+)^{p-1}(u_n-u)\,dx+$$
$$+(\beta c_n+t)\int_\Omega (u_n^-)^{q-1}(u_n-u)\,dx+$$
$$+(\alpha c_n-s_0)\int_\Omega (u_n^-)^{p-1}(u_n-u)\,dx+$$
$$+(\beta c_n-t_0)\int_\Omega (u_n^+)^{q-1}(u_n-u)\,dx.$$

Since,

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla(u_n-u)\underset{n\to+\infty}{\to}0,$$

and

$$\int_\Omega |\nabla u_n|^{q-2}\nabla u_n\nabla(u_n-u)\underset{n\to+\infty}{\to}0,$$

and according to the (S_+) property we obtain that $u_n\to u$ in $W_0^{1,p}(\Omega)$.

In the next step, we will look for local minimizers of the functional:

$$J: W_0^{1,p}(\Omega)\mapsto\mathbb{R},$$

defined by:

$$J(u)=\frac{1}{p}\int_\Omega |\nabla u|^p\,dx+\frac{1}{q}\int_\Omega |\nabla u|^q\,dx.$$

To fulfill the Mountain-Pass geometry of the functional $\bar{I}$.

Lemma 2. For any integer $k\in\mathbb{N}$, the set not empty is:

$$B_k=\left\{A\in S\left(W_0^{1,p}(\Omega)\right); A\subset S, \gamma(A)\ge k\right\}.$$

In particular if $X_k\subset W_0^{1,p}(\Omega)$ is a sub-space of dimension, then:

$$\gamma(M_{\alpha\beta}\cap X_k)=k.$$

Proof. Let X_k a sub-space of $W_0^{1,p}(\Omega)$ such that $\dim X_k=k$. We can show easily that $(X_k\cap M_{\alpha\beta})$ is a symmetrical and closed set that does not contain the origin, so $\gamma(M_{\alpha\beta}\cap X_k)$ is well defined.

Let now S be the unit sphere in $W_0^{1,p}(\Omega)$. Denote by:

$$P: u\mapsto\frac{1}{\|u\|_{1,p}}u, u\ne 0$$

the radial projection in $W_0^{1,p}(\Omega)$.

Then P is a bijection between $M_{\alpha\beta}$ and S. We have:

$$P(X_k \cap M_{\alpha\beta}) = X_k \cap P(M_{\alpha\beta}) = X_k \cap S.$$

So P is a homomorphism between $X_k \cap M_{\alpha\beta}$ and $X_k \cap S$. Since P is odd we get:

$$\gamma(X_k \cap M_{\alpha\beta}) = \gamma(X_k \cap S).$$

According to the genus properties we have:

$$\gamma(X_k \cap M_{\alpha\beta}) = k.$$

Similar arguments as those used in Lemma 1 show that J satisfies the Palais-Smale condition on $M_{\alpha\beta}$. Combing this fact and Lemma 2, one can get by the Ljusternick-Schnirelmann theorem that for any $k \in \mathbb{N}$ the quantity:

$$c_k := \inf_{A \in B_k} \sup_{u \in A} J(u)$$

is a critical value of the functional J with respect to the manifold $M_{\alpha\beta}$. Hence, a sequence of critical points that we note by $\{u_k^1\}_{k\in\mathbb{N}} \subset M_{\alpha\beta}$ also exists.

Next we give the main result of the paper.

Lemma 3:

1. For $s>0$, $t>0$.
2. $c_n(s,t) = \inf_{\gamma\in\Gamma} \max_{u\in\gamma[-1,1]} I_{s,s_0,t,t_0}(u)$ is a sequence of critical value of I_{s,s_0,t,t_0}, where

$$\Gamma = \left\{\gamma \in C([-1,1], M_{\alpha\beta}) : \gamma(-1) = -u_k^1, \gamma(1) = u_k^1\right\}.$$

3. The curve $(s + c_n(s,t), t + c_n(s,t)) \in \Sigma_{p,q}$.

Proof:

1. First we have: $u_k^1, (-u_k^1) \in M_{\alpha\beta}$, then for any $\in > 0$ we have:

$$\left\|u_k^1 - (-u_k^1)\right\|_{1,p} = 2\left\|u_k^1\right\|_{1,p} > \in.$$

Now we show that:

$$\inf\left\{I_{s,s_0,t,t_0}(u) : u \in M_{\alpha\beta}, \left\|u - (-u_k^1)\right\|_{1,p} = \in\right\} >$$
$$> \max\left\{I_{s,s_0,t,t_0}(-u_k^1), I_{s,s_0,t,t_0}(u_k^1)\right\}.$$

Since c_k is a critical value of J, there exists a Lagrange multiplier $\vartheta_k \in \mathbb{R}$ and $u \in M_{\alpha\beta}$ such that:

$$J'(u) = \vartheta_k G'_{\alpha\beta}(u).$$

In other words, we have:

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \mathrm{d}x + \int_\Omega |\nabla u|^{q-2} \nabla u \nabla v \mathrm{d}x =$$
$$= \alpha\vartheta_k \int_\Omega |u|^{p-2} uv + \beta\vartheta_k \int_\Omega |u|^{q-2} uv\mathrm{d}x.$$

Taking $u = v$ in the last equation, we get:

$$\frac{1}{p}\int_\Omega |\nabla u|^p \mathrm{d}x + \frac{1}{q}\int_\Omega |\nabla u|^q \mathrm{d}x =$$
$$= \vartheta_k\left(\frac{\alpha}{p}\int_\Omega |u|^p \mathrm{d}x + \frac{\beta}{q}\int_\Omega |u|^q\right)\mathrm{d}x.$$

Since $u \in M_{\alpha\beta}$, we obtain:

$$J(u) = \vartheta_k.$$

So $c_k = \vartheta_k$ and $\max\left\{\bar{I}_{s,s_0,t,t_0}(-u_k^1), \bar{I}_{s,s_0,t,t_0}(u_k^1)\right\} = c_k$. In the other hand, we have:

$$\frac{1}{p}\int_\Omega |\nabla u|^p \mathrm{d}x + \frac{1}{q}\int_\Omega |\nabla u|^q \mathrm{d}x < \frac{1}{p}\int_\Omega |\nabla u|^p \mathrm{d}x +$$
$$+ \frac{1}{q}\int_\Omega |\nabla u|^q \mathrm{d}x - \frac{s}{p}\int_\Omega |u^+|^p \mathrm{d}x + \frac{s_0}{p}\int_\Omega |u^-|^p \mathrm{d}x -$$
$$- \frac{t}{q}\int_\Omega |u^-|^q \mathrm{d}x + \frac{t_0}{q}\int_\Omega |u^+|^q \mathrm{d}x,$$

for all $u \in M_{\alpha\beta}$. Then, it results:

$$\inf_{A\in B_k} \sup_{w\in A} \bar{I}(w) < \bar{I}(u),$$

witch apply that:

$$\inf_{A\in B_k} \sup_{w\in A} \bar{I}(w) < \inf \bar{I}(u),$$

for any $u \in M_{\alpha\beta}$. Consequently,

$$\inf_{A\in B_k} \sup_{w\in A} \bar{I}(w) < \inf\left\{\bar{I}(u); u \in M_{\alpha\beta}, \left\|u - (-u_k^1)\right\|_{1,p} = \in\right\},$$

and this provides the following estimate:

$$\inf\left\{I(u); u \in M_{\alpha\beta}, \left\|u - (-u_k^1)\right\|_{1,p} = \in\right\} > c_k.$$

Since $\bar{I}$ verifies the Palais-Smale condition and 1 is a regular value of $G_{\alpha\beta}$, then a critical value of I_{s,s_0,t,t_0} is:

$$c_n(s,t) = \inf_{\gamma\in\Gamma} \max_{u\in\gamma[-1,1]} I_{s,s_0,t,t_0}(u).$$

2. $(s + c_n(s,t), t + c_n(s,t)) \in \Sigma_{p,q}$ if and only if there exist a critical point $u \in M_{\alpha\beta}$ such that $c_n = I_{s,s_0,t,t_0}(u)$, and since 1 is satisfies then the curve:

$$(s + c_n(s,t), t + c_n(s,t)) \in \Sigma_{p,q}.$$

Lemma 4. If $c_n(s,t) = \inf_{\gamma\in\Gamma}\max_{u\in\gamma[-1,1]} I_{s,s_0,t,t_0}(u)$ is a critical value of I_{s,s_0,t,t_0} then $s_0 = \alpha c_n$ and $t_0 = \beta c_n$.

Proof. We have $c_n = c_n(s,t)$ is a critical value of I_{s,s_0,t,t_0} then:

$$I'_{s,s_0,t,t_0}(u_n) = c_n G'_{\alpha\beta}(u_n),$$

where u_n is the critical point associated to c_n.

For any $v \in W_0^{1,p}(\Omega)$, we have:

$$I'_{s,s_0,t,t_0}(u_n)v = G'_{\alpha\beta}(u_n)v,$$

that is,

$$\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla v \mathrm{d}x + \int_\Omega |\nabla u_n|^{q-2} \nabla u_n \nabla v \mathrm{d}x -$$
$$- s\int_\Omega |u_n^+|^{p-1} v\mathrm{d}x + s_0 \int_\Omega |u_n^-|^{p-1} v\mathrm{d}x -$$
$$- t\int_\Omega |u_n^-|^{q-1} v\mathrm{d}x + t_0 \int_\Omega |u_n^+|^{q-1} v\mathrm{d}x =$$
$$= c_n\left(\alpha\int_\Omega |u_n|^{p-2} u_n v\mathrm{d}x + \beta\int_\Omega |u_n|^{q-2} u_n v\mathrm{d}x\right).$$

Consequently,

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla v \mathrm{d}x + \int_\Omega |\nabla u_n|^{q-2}\nabla u_n \nabla v \mathrm{d}x =$$
$$= (\alpha c_n + s)\int_\Omega |u_n^+|^{p-1} v\mathrm{d}x + (\beta c_n + t)\int_\Omega |u_n^-|^{q-1} v\mathrm{d}x +$$
$$+ (\alpha c_n - s_0)\int_\Omega |u_n^-|^{p-1} v\mathrm{d}x + (\beta c_n - t_0)\int_\Omega |u_n^+|^{q-1} v\mathrm{d}x.$$

Taking,

$$(\alpha c_n - s_0)\int_\Omega |u_n^-|^{p-1} v\mathrm{d}x + (\beta c_n - t_0)\int_\Omega |u_n^+|^{q-1} v\mathrm{d}x = 0,$$

we get as required:

$$\alpha c_n = s_0 \text{ and } \beta c_n = t_0.$$

3.2. Properties of the family of curve. We define the following family of curves:

$$C_n := \{(s + c_n(s,t), t + c_n(s,t)), (t + c_n(s,t), s + c_n(s,t))\}.$$

Lemma 5:

1. The family of curves $(s,t) \mapsto (s + c_n(s,t), t + c_n(s,t))$ is Lipschitz and continuous in a way that:

$$s + c_n(s,t) < s' + c_n(s',t), t + c_n(s,t) < t' + c_n(s,t')$$

and

$$c_n(s,t) > c_n(s',t').$$

2. The family of curves $(s,t) \mapsto (s + c_n(s,t), t + c_n(s,t))$ is decreasing in a way that:

$$s + c_n(s,t) < s' + c_n(s',t), t + c_n(s,t) < t' + c_n(s,t')$$

and

$$c_n(s,t) > c_n(s',t').$$

Proof. We start by part 1.

1. Let $s' < s$, respectively $t' < t$, then:

$$\overline{I}_{s',s_0,t',t_0}(u) \geq \overline{I}_{s,s_0,t,t_0}(u),$$

for all $u \in M_{\alpha,\beta}$, then we have:

$$c_n(s',t') \geq c_n(s,t),$$

then $\forall \in > 0$, there exist $\gamma \in \Gamma$ such that:

$$\max_{u\in\gamma[-1,1]} \overline{I}_{s,s_0,t,t_0}(u) \leq c_n(s,t) + \in,$$

then

$$0 \leq c_n(s',t') - c_n(s,t) \leq$$
$$\leq \max_{u\in\gamma[-1,1]} \overline{I}_{s',s_0,t',t_0}(u) - \max_{u\in\gamma[-1,1]} \overline{I}_{s,s_0,t,t_0}(u) + \in,$$

putting now $u_0 \in \gamma[-1,1]$ such that:

$$\max_{u\in\gamma[-1,1]} \overline{I}_{s',s_0,t',t_0}(u) = \overline{I}_{s',s_0,t',t_0}(u_0).$$

We get:

$$0 \leq c_n(s',t') - c_n(s,t) \leq \overline{I}_{s',s_0,t',t_0}(u_0) - \overline{I}_{s,s_0,t,t_0}(u_0) + \in,$$

since $\in > 0$, then it's easy to sea that:

$$(s,t) \mapsto (s + c_n(s,t), t + c_n(s,t))$$

is continuous and Lipchitz.

2. Let $0 < s' < s$, and $0 < t' < t$, then:

$$(s' + c_n(s',t'), t' + c_n(s',t')), (s + c_n(s,t), t + c_n(s,t)) \in \sum_{p,q} P,Q$$
$$\Downarrow$$
$$s' + c_n(s',t'), t' + c_n(s',t') < s + c_n(s,t), t + c_n(s,t),$$

since we have:

$$c_n(s',t') \geq c_n(s,t),$$

which means that c_n is decreasing.

These Methods can only be used on elliptic nonlinear operator in bounded spaces of $\mathbb{R}^N$. If we want to apply them on non-bounded spaces $\mathbb{R}^N$ we must verify the Sobolev injections.

Based on our work we can further generalize our results by imposing measurable weights under Newmann and Robin boundary conditions which will lead a use of the Fučik sepectrum of the (p,q)-Laplacian operator in greater spaces.

4. Conclusions

In this paper, we have shown that the Fučik spectrum of the (p,q)-Laplacian operator is essentially made up by a group of curves C_n given by:

$$C_n = (s + c_n(s,t), t + c_n(s,t)),$$

where c_n is a sequence of a critical value.

For that we have used a new method that combines the Col and L-S theorems.

This results will lead to a generalization of the Fučik spectrum of the (p,q)-Laplacian operator in a non-bounded spaces.

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Conflict of interest

The authors declare that they have no conflict of interest in relation to this study, including financial, personal, authorship, or any other, that could affect the study and its results presented in this article.

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Data availability

The manuscript has associated data in a data repository.

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***Selma Hadjer Djeffal**, Posgradute Student, Laboratory of Applied Mathematics, Departement of Mathematics, Badji Mokhtar University, Annaba, Algeria, ORCID: https://orcid.org/0000-0002-1114-4267*

✉***Aissa Benselhoub,** PhD, Associate Researcher, Environment, Modeling and Climate Change Division, Environmental Research Center (C.R.E), Annaba, Algeria; Visitor Researcher, INFN Frascati National Laboratories, Frascati, Italy, e-mail: aissabenselhoub@cre.dz, ORCID: https://orcid.org/0000-0001-5891-2860*

✉*Corresponding author*