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ANALYTICAL CONNECTION BETWEEN THE FRENET TRIHEDRON OF A DIRECT CURVE AND THE DARBOUX TRIHEDRON OF THE SAME CURVE ON THE SURFACE

Frenet and Darboux trihedrons are the objects of research.

At the current point of the direction curve of the Frenet trihedron, three mutually perpendicular unit orthogonal vectors can be uniquely constructed. The orthogonal vector of the tangent is directed along the tangent to the curve at the current point. The orthogonal vector of the main normal is located in the plane, which is formed by three points of the curve on different sides from the current one when they are maximally close to the current point. It is directed to the center of the curvature of the curve. The orthogonal vector of the binormal is perpendicular to the two previous orthogonal vectors and has a direction according to the rule of the right coordinate system. Thus, the movement of the Frenet trihedron along the base curve, as a solid body, is determined.

The Darboux trihedron is also a right-hand coordinate system that moves along the base curve lying on the surface. Its orthogonal vector of the tangent is directed identically to the Frenet trihedron, and other orthogonal vectors in pairs form a certain angle ε *with the orthogonal vectors of the Frenet trihedron. This is because one of the orthogonal vectors of the Darboux trihedron is normal to the surface and forms a certain angle* ε *with the binormal. Accordingly, the third orthogonal vector of the Darboux trihedron forms an angle* ε *with the orthogonal vector of the normal of the Frenet trihedron. This orthogonal vector and orthogonal vector of the tangent form the tangent plane to the surface at the current point of the curve, and the corresponding orthogonal vectors of the tangent and the normal of the Frenet trihedron form the tangent plane of the curve at the same point. Thus, when the Frenet and Darboux trihedrons move along a curve with combined vertices, there is a rotation around the common orthogonal vector point of the tangent at an angle* ε *between the osculating plane of the Frenet trihedron and the tangent plane to the surface of the Darboux trihedron. These trihedrons coincide in a separate case (for a flat curve)* (ε=0).

The connection between Frenet and Darboux trihedrons – finding the expression for the angle ε*, is considered in the article. The inverse problem – the determination of the movement of the Darboux trihedron at a given regularity of the change of the angle* ε*, is also considered. A partial case is considered and it is shown that for a flat base curve at* ε=const*, the set of positions of the orthogonal vector of normal forms a developable surface of the same angle of inclination of the generators. In addition, the inverse problem of finding the regularity of the change of the angle* ε *between the corresponding orthogonal vectors of the trihedrons allows constructing a ruled surface for the gravitational descent of the load, conventionally assumed to be a particle. At the same time, the balance of forces in the projections on the orthogonal vectors of the trihedron in the common normal plane of the trajectory is considered. This balance depends on the angle* ε.

Keywords: *orthogonal vector, direction cosines, Euler angles, surface osculating plane, tangent plane.*

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1. Introduction

Using Frenet and Darboux trihedrons, it is possible to construct both ruled and non-ruled surfaces according to a given base curve and a given regularity of the change of the angle ε both as a function of an arbitrary variable (in our case, depending on an arbitrary parameter α) and depending on the length of the arc *s* of the base curve. In the last case, the possibilities of surface construction are expanded precisely thanks to the possibility of applying Frenet's formulas. On the other hand, these possibilities are limited, because among the variety of plane and spatial curves, only a tiny part of them can be described as a function of the arc length *s*. As a rule, these curves are found as a result of many years of research by many scientists from different countries on the subject of their practical application [1–3]. Information about flat curves is given in the article [4], in which some curves are eloquently called «wonderful curves». With the position of the Frenet trihedron in the fixed coordinate system and the regularity of change of the angle ε between the trihedrons, the position of the Darboux trihedron in the fixed coordinate system can be found.

In the dynamics of the movement of a material point along a surface in the vicinity of an infinitely small area, it is customary to consider this movement along a plane tangent to the surface, which corresponds to this area. Since one of the faces of the Darboux trihedron is tangent to the surface during its movement along it, it is convenient to consider the equations of motion of a point in projections onto the orthogonal vectors of this trihedron. A partial case of such motion in the plane, when the Frenet and Darboux trihedrons coincide, is considered in [5].

Research [6, 7] is dedicated to the study of the movement of particles on surfaces. If the curve along which the trihedrons move on the surface is described as a function of the length of its arc, that is, given in a natural parameterization, then the Frenet formulas can be used. Articles [8, 9] are devoted to the construction of such curves. The construction of curves based on other initial conditions is considered in [10, 11].

The aim of this research is to find an analytical connection between Frenet and Darboux trihedrons. It makes it possible to expand the possibilities of surface construction.

2. Materials and Methods

The objects of research are Frenet and Darboux trihedrons. The research was carried out using Matlab software and Mathematica.

Let's consider the Frenet trihedron of a flat base curve. In Fig. 1, *a* it is built at point *A* of the curve, which is located in the horizontal plane of the fixed *Oxyz* coordinate system. In this case, the orthogonal vector of the binormal $\bar{b}$ is parallel to the *Oz* axis and coincides with the normal $\bar{N}$ to the plane in which the curve is located. In general, for a curve on the surface, the binormal $\bar{b}$ and the normal $\bar{N}$ to the surface do not coincide. As already mentioned, the area of the surface around point *A* can be replaced by a tangent plane. At an infinitesimally small size of this area, it will coincide with the tangent plane. Therefore, the normal $\bar{N}$ to the tangent plane is the normal to the surface at point *A*. But in this case, the normal $\bar{N}$ to the surface does not coincide with the binormal $\bar{b}$ of the Frenet trihedron (Fig. 1, *b*). An angle ε will be formed between these vectors. The unit normal vector $\bar{N}$ will be one of the orthogonal vectors of the Darboux trihedron. The second orthogonal vector $\bar{T}$ will always coincide with the orthogonal vector $\bar{\tau}$ of the Frenet trihedron since they move along the same curve. The third orthogonal vector $\bar{P}$ of the Darboux trihedron is perpendicular to the first two, so it lies in the plane μ tangent to the surface. There will also be an angle ε between the orthogonal vector $\bar{P}$ and the orthogonal vector of the main normal $\bar{n}$ (Fig. 1, *b*). Thus, when two trihedrons move along a curve on the surface between their orthogonal vectors $\bar{N}$ and $\bar{b}$, and $\bar{P}$, and $\bar{n}$ in pairs, there is an angle ε, which can be either constant or variable. For the case of a curve located in the horizontal plane *Oxy* (Fig. 1, *a*), these two trihedrons coincide. However, if the flat curve does not lie in the horizontal plane, then trihedrons will not coincide (except for individual points).

A plane cross-section of a vertical cylinder with an inclined plane is shown in Fig. 1, *c*. The base curve is an ellipse. The main normal $\bar{n}$ of the ellipse is located in its plane and is directed to the center of curvature. Between it and the orthogonal vector $\bar{P}$, which is located in the tangent plane *μ*, there is an angle ε, which for point *M* is equal to ε=90°–β, where β is the angle of inclination of the ellipse plane to the horizontal plane. During the trihedrons move along the ellipse, the angle ε will change and at point *L*, will be equal to 90°, since the main normal $\bar{n}$ and the normal to the surface $\bar{N}$ will coincide. For a helical line of constant pitch, the angle ε during the movement of the trihedron will be constant and equal to 90°.

Let's find the regularity of the change of the angle ε during the movement of trihedrons along an ellipse – an inclined cross-section of a cylinder (Fig. 1, *c*). The parametrization of a vertical cylinder is:

$$X = R\cos\alpha;\ Y = R\sin\alpha;\ Z = u, \tag{1}$$

where α is the angle of rotation of a point on the cylinder around its axis; *u* is the length of the straight line (as a generator) of the cylinder, the count of which starts from the base – independent variables; *R* is the radius of the base.

If the cylinder (1) is cut by a plane inclined at an angle β (angle β in Fig. 1 is not shown), then in the cross-section an ellipse will be received, the parametrization of which will be written:

$$x = R\cos\alpha;\ y = R\sin\alpha;\ z = R\,\mathrm{tg}\beta\sin\alpha. \tag{2}$$

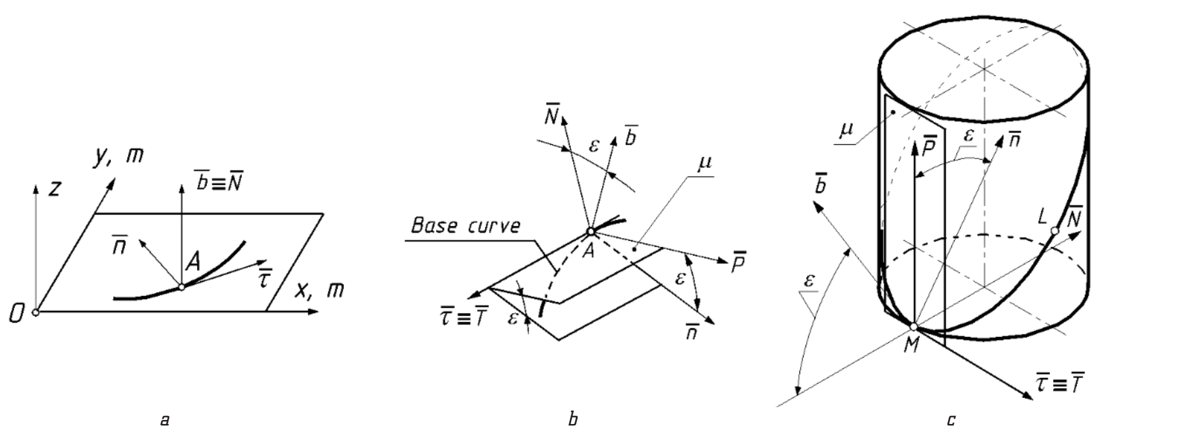

Fig. 1. Graphical illustrations of the connection between Frenet and Darboux trihedrons:
a – Frenet trihedron of a flat curve; *b* – designation of the angle ε between the orthogonal vectors of the trihedrons; *c* – Frenet and Darboux trihedrons of an ellipse – an inclined cross-section of a vertical cylinder

The parametrization of the surface (1) is denoted by uppercase letters «X», «Y», «Z», and the lines (2) on its surface are lowercase. The angle ε between the trihedrons can be found as the angle between the unit orthogonal vectors $\overline{N}$ and $\overline{b}$. The direction of the normal $\overline{N}$ to the surface of the cylinder does not depend on the height u of the point on the cylinder, but only on the angle α. It coincides with the end of the radius vector that describes the circle – the base of the cylinder (the first two expressions in equations (2)). Let's bring it to unit one (shorten it by R) and direct it in the opposite direction, that is, from the circle to the center, as shown in Fig. 1, *c*. Its projections onto the fixed $Oxyz$ coordinate system (in Fig. 1, *b*, *c* it is conditionally absent) will be written:

$$\{-\cos\alpha;\ -\sin\alpha;\ 0\}. \quad (3)$$

Projections of the binormal $\overline{b}$ onto a fixed coordinate system (base cosines of the binormal) can be determined by known formulas through the first and second derivatives of the base curve (2) [12]:

$$\cos\alpha_b = \frac{A}{\sqrt{A^2+B^2+C^2}};\ \cos\beta_b = \frac{B}{\sqrt{A^2+B^2+C^2}};$$
$$\cos\gamma_b = \frac{C}{\sqrt{A^2+B^2+C^2}}, \quad (4)$$

where

$$A = y'z'' - y''z';\ B = z'x'' - z''x';\ C = x'y'' - x''y'. \quad (5)$$

The first and second derivatives of curve (2):

$$x' = -R\sin\alpha;\ y' = R\cos\alpha;\ z' = R\,\mathrm{tg}\beta\cos\alpha; \quad (6)$$

$$x'' = -R\cos\alpha;\ y'' = -R\sin\alpha;\ z = -R\,\mathrm{tg}\beta\sin\alpha. \quad (7)$$

By substituting derivatives (6) and (7) into (5) and after simplifications the expressions A, B, C, which in this case are constant, can be found:

$$A = 0;\ B = -R^2\mathrm{tg}\beta;\ C = R^2. \quad (8)$$

By substituting (8) into (5), the projections of the orthogonal vector of binormal $\overline{b}$ can be received:

$$\{0;\ -\sin\beta;\ \cos\beta\}. \quad (9)$$

The angle ε between the unit vectors (3) and (9) can be found using the well-known formulas:

$$\cos\varepsilon = \sin\beta\sin\alpha;\ \sin\varepsilon = \sqrt{1-\sin^2\beta\sin^2\alpha}. \quad (10)$$

Based on expressions (10), it is possible to find the regularity of the change of the angle ε in the form ε=ε(α). In Fig. 2 graphs of this dependence are shown, and in Fig. 2, *a* the graph is constructed as the arccosine of the first expression (10), and in Fig. 2, *b* – as the arcsine of the second expression.

The different appearance of the graphs of the same dependence ε=ε(α) is explained by the fact that in one case only the absolute value of the angle is taken into account (Fig. 2, *b*), without taking into account the direction of the orthogonal vectors $\overline{N}$ and $\overline{b}$ forming the angle. In the second case (Fig. 2, *a*) this direction is taken into account. Orthogonal vector $\overline{b}$ (9) has a fixed direction, and orthogonal vector $\overline{N}$ (3) has a variable direction. At α=0, both vectors are mutually perpendicular (ε=90°) at the point L (Fig. 1, *c*), so the vector $\overline{b}$ is projected at the point onto the vector $\overline{N}$. At the same point and at the opposite point (after 180°), the direction of the vector's changes to the opposite.

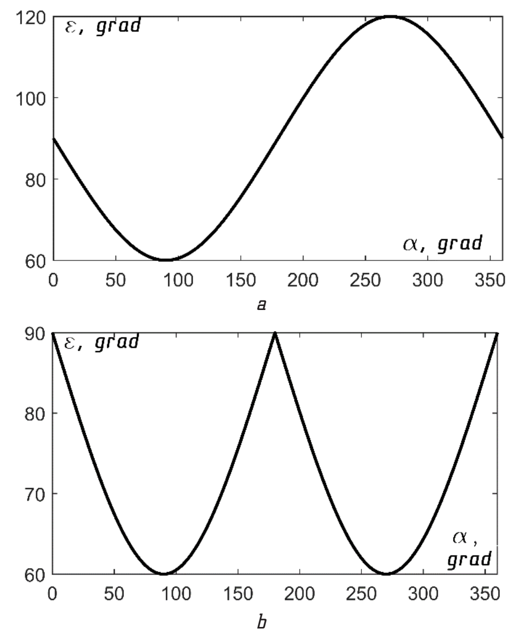

Fig. 2. Graphs of dependence ε=ε(α) for β=30°: *a* – taking into account the rotation of vector $\overline{N}$; *b* – without taking into account the rotation of the vector $\overline{N}$

3. Results and Discussion

If the dependence ε=ε(α) can be found on the given curve on the surface, i. e. the regularity of rotation of trihedrons relative to each other around a common tangent during their movement, then it is quite logical to reproduce the surface itself according to the curve and known regularity ε=ε(α). However, not everything is clear here. A ruled surface can be formed by a set of positions of the orthogonal vector $\overline{P}$ of the Darboux trihedron since it is located in the tangent plane. The order of the creation of the surface is the next: the Frenet trihedron moves along a given curve and the motion of the Darboux trihedron is reproduced according to the given regularity ε=ε(α), the set of positions of the orthogonal vector $\overline{P}$ of which forms the surface. However, other straight lines can be located in the tangent plane of the Darboux trihedron, the set of which forms ruled surfaces during the movement of the trihedron, including the developable ones (such as, for example, the cylinder in Fig. 1, *c*). Therefore, there can be many surfaces that satisfy the movement of the Darboux trihedron along a given curve and a given dependence ε=ε(α), and not only ruled ones but also non-ruled ones. Let's consider an example.

As a given curve an ellipse (2) is taken and the equation of the surface (1) on which it is located. A ruled surface formed by the set of positions of the orthogonal vector $\overline{P}$ of this trihedron should be found. The position of the orthogonal vector $\overline{P}$ of the Darboux trihedron in a fixed coordinate system can be found, based on the remaining two of its orthogonal vectors $\overline{T}$ and $\overline{N}$, the direction of which can be determined by the known equations of the surface and the curve on it. The direction of the orthogonal vector $\overline{P}$ can be determined from the vector product of orthogonal vectors $\overline{T}$ and $\overline{N}$. The projections of orthogonal vector $\overline{N}$ are given in (3). The projections of the orthogonal vector $\overline{T}$ (direction cosines), which coincides with the orthogonal vector $\overline{\tau}$, can be found using the well-known formulas [12]:

$$\cos\alpha_\tau = \frac{x'}{\sqrt{x'^2+y'^2+z'^2}} = -\frac{\cos\beta\sin\alpha}{\sqrt{\cos^2\alpha+\cos^2\beta\sin^2\alpha}};$$
$$\cos\beta_\tau = \frac{y'}{\sqrt{x'^2+y'^2+z'^2}} = \frac{\cos\beta\cos\alpha}{\sqrt{\cos^2\alpha+\cos^2\beta\sin^2\alpha}};$$
$$\cos\gamma_\tau = \frac{z'}{\sqrt{x'^2+y'^2+z'^2}} = \frac{\sin\beta\cos\alpha}{\sqrt{\cos^2\alpha+\cos^2\beta\sin^2\alpha}}. \quad (11)$$

According to the rules of vector multiplication $\overline{N}\times\overline{T}$ and the known coordinates of the orthogonal vectors $\overline{N}$ (3) and $\overline{T}$ (11), the projections of the orthogonal vector $\overline{P}$ on the axis of the fixed coordinate system can be determined:

$$l_P = -\frac{\sin\beta\sin\alpha\cos\alpha}{\sqrt{\cos^2\alpha+\cos^2\beta\sin^2\alpha}};$$
$$m_P = \frac{\sin\beta\cos^2\alpha}{\sqrt{\cos^2\alpha+\cos^2\beta\sin^2\alpha}};$$
$$n_P = -\frac{\cos\beta}{\sqrt{\cos^2\alpha+\cos^2\beta\sin^2\alpha}}. \quad (12)$$

The orthogonal vector $\overline{P}$, which is given by its projections (12) on the axis of the fixed coordinate system, must make an angle ε with the orthogonal vector $\overline{n}$ (Fig. 1, *c*). To check the correctness of these calculations, it is necessary to find the projections of the orthogonal vector $\overline{n}$ on the axis of the fixed coordinate system *Oxyz* according to known formulas [12]:

$$\cos\alpha_n = \frac{Bz'-Cy'}{\sqrt{\sqrt{(x'^2+y'^2+z'^2)}(A^2+B^2+C^2)}} =$$
$$= -\frac{\cos\alpha}{\sqrt{\cos^2\alpha+\cos^2\beta\sin^2\alpha}};$$
$$\cos\beta_n = \frac{Cx'-Az'}{\sqrt{(x'^2+y'^2+z'^2)(A^2+B^2+C^2)}} =$$
$$= -\frac{\cos^2\beta\sin\alpha}{\sqrt{\cos^2\alpha+\cos^2\beta\sin^2\alpha}};$$
$$\cos\gamma_n = \frac{Ay'-Bx'}{\sqrt{(x'^2+y'^2+z'^2)(A^2+B^2+C^2)}} =$$
$$= -\frac{\sin\beta\cos\beta\sin\alpha}{\sqrt{\cos^2\alpha+\cos^2\beta\sin^2\alpha}}, \quad (13)$$

where the expressions for *A, B, C* are given in (5), and the expressions of the derivatives are given in (6).

It is easy to verify that the angle ε between vectors (12) and (13) is described by expressions (10).

Orth $\overline{P}$ (12) determines the direction of the straight lines (as generators) of the ruled surface along which the Darboux trihedron will move along the base curve (2). This movement will be similar to the movement of the Darboux trihedron along the surface of the cylinder (1) along the curve (2) according to the found regularity ε=ε(α) (10). To construct this surface, it is necessary to draw a straight line parallel to the vector (12) through each point of the curve (2). According to this, the parametrization of the ruled surface can be written as:

$$X = x+ul_P = R\cos\alpha - u\frac{\sin\beta\sin\alpha\cos\alpha}{\sqrt{\cos^2\alpha+\cos^2\beta\sin^2\alpha}};$$
$$Y = y+um_P = R\sin\alpha + u\frac{\sin\beta\cos^2\alpha}{\sqrt{\cos^2\alpha+\cos^2\beta\sin^2\alpha}};$$
$$Z = z+un_P = R\text{tg}\beta\sin\alpha - u\frac{\cos\beta}{\sqrt{\cos^2\alpha+\cos^2\beta\sin^2\alpha}}. \quad (14)$$

In Fig. 3 ruled surfaces according to equations (14) with $R=5$, $u=-5...5$, and different values of the angle β are constructed.

For the surfaces, which are shown in Fig. 3, *a*, *b*, the movement of the Darboux trihedron along the ellipse will be the same as for cylindrical surfaces with vertical generators passing through this ellipse. At β=0, the ellipse turns into a circle (Fig. 3, *c*) and both trihedrons coincide.

In Fig. 3, the ruled surfaces, which are formed by the set of positions of the orthogonal vector $\overline{P}$ of the Darboux trihedron, are constructed according to the found regularity of the change of the angle ε=ε(α) (10). In the reverse order, it is possible to construct surfaces of a similar formation with a given dependence ε=ε(α), in particular, with ε=const. In such case, a projection of the orthogonal vector $\overline{P}$ in the system of the Frenet trihedron, are:

$$\tau_P = 0;\ n_P = \cos\varepsilon;\ b_P = \sin\varepsilon. \quad (15)$$

The transition from the coordinates of the vector $\overline{P}$ in the system of the Frenet trihedron to its coordinates in the fixed coordinate system is carried out according to the well-known formulas [12]:

$$x_P = \tau_P\cos\alpha_\tau + n_P\cos\alpha_n + b_P\cos\alpha_b;$$
$$y_P = \tau_P\cos\beta_\tau + n_P\cos\beta_n + b_P\cos\beta_b;$$
$$z_P = \tau_P\cos\gamma_\tau + n_P\cos\gamma_n + b_P\cos\gamma_b, \quad (16)$$

where the expressions of direction cosines are given in (4), (11) and (13).

Vector (13) is the unit one in a fixed coordinate system and a base one for a straight generatrix surface that coincides with the orthogonal vector $\overline{P}$ of the Darboux trihedron. The surface equations according to formulas (14), in which expressions (12) and (16) play the same role, can be written using the expressions (16):

$$X = x+ux_P =$$
$$= R\cos\alpha - u\frac{\cos\varepsilon\cos\alpha}{\sqrt{\cos^2\alpha+\cos^2\beta\sin^2\alpha}};$$
$$Y = y+uy_P =$$
$$= R\sin\alpha - u\left(\frac{\cos\varepsilon\cos^2\beta\sin\alpha}{\sqrt{\cos^2\alpha+\cos^2\beta\sin^2\alpha}} + \sin\varepsilon\sin\beta\right);$$
$$Z = z+uz_P =$$
$$= R\text{tg}\beta\sin\alpha - u\left(\frac{\cos\varepsilon\sin\beta\cos\beta\sin\alpha}{\sqrt{\cos^2\alpha+\cos^2\beta\sin^2\alpha}} - \sin\varepsilon\cos\beta\right). \quad (17)$$

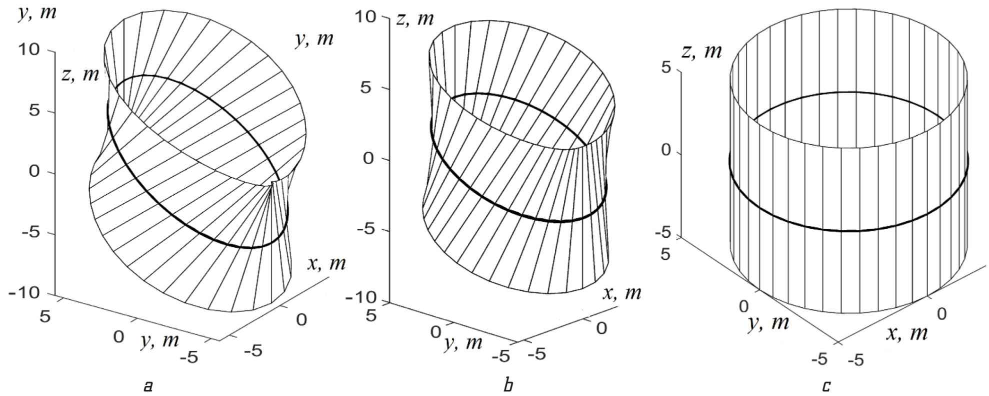

Fig. 3. Ruled surfaces with a base curve in the form of an ellipse, for which the rotation of the Darboux trihedron relative to the Frenet trihedron during their movement along the ellipse is described by expressions (10): *a* – β=45°; *b* – β=30°; *c* – β=0

According to equations (17) in Fig. 4 the surface at R=5, β=30°, and different angle values ε, is constructed.

When ε=const, the formed surfaces for a flat base curve are the surfaces of the same inclination of the generators. In the partial case, at ε=90°, the surface will be a cylinder for which the base curve is an orthogonal cross-section, and at ε=0°, it will be the plane in which the base curve is located.

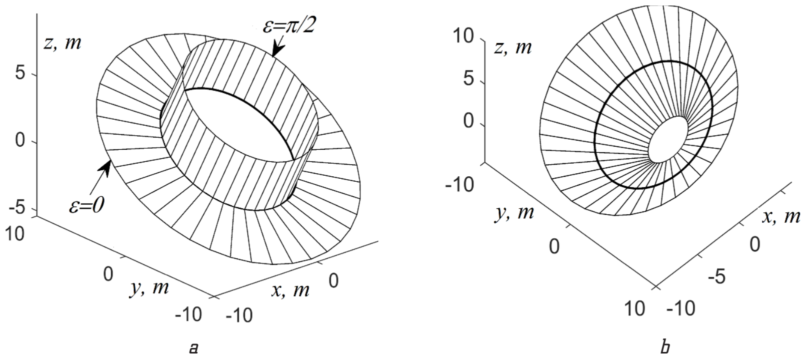

Fig. 4. Ruled surfaces with a base curve in the form of an ellipse and a constant angle ε between Darboux and Frenet trihedrons: *a* – ε=0, ε=90°, u=0…5; *b* – ε=45°, u=–5…5

For the case shown in Fig. 1, *c*, when the base curve is an ellipse on the surface of a vertical cylinder, the dependence ε=ε(α) was found (10). As mentioned, in reverse order for the found dependence (10) and the ellipse (2) it is possible to construct many surfaces, among which there should also be a vertical cylinder (1), which is a developable surface. This surface is an envelope one-parameter set of planes that pass through each point of the base curve (in our case, an ellipse). The position of the plane that passes through a given point in space is given by the normal vector of this plane, that is the vector $\overline{N}$ in our case. The vector $\overline{N}$ is known and has coordinates (3). However, it is possible to found this vector by the known surface of the cylinder (1).

Now this surface is considered to be unknown and needs to be constructed according to the found vector $\overline{N}$. Let's first find the projections of the unit vector $\overline{N}$ onto the orthogonal vectors of the Frenet trihedron. It is located in its normal plane and is projected only on two axes: n_N=sinε, b_N=cosε. Taking into account the expressions (10), which specify the dependence ε=ε(α) during the movement of the trihedron along an ellipse on a vertical cylinder (Fig. 1, *c*), the projections of the vector $\overline{N}$ onto the orthogonal vectors of the trihedron are written:

$$\tau_N = 0;\ n_N = \sqrt{1-\sin^2\beta\sin^2\alpha};\ b_N = \sin\beta\sin\alpha. \quad (18)$$

Applying the formulas (16) for the transition from the projections of the vector (18) on the orthogonal vectors of the Frenet trihedron to the projections on the axis of the fixed coordinate system *Oxyz*, a result that exactly coincides with (3) was obtained. According to this vector, it is possible to construct only one developable surface, as an envelope one-parameter set of planes, which will be a vertical cylinder.

The use of Frenet and Darboux trihedrons *makes it possible* to construct ruled surfaces that pass-through a given curve under given conditions. Such a condition can be, for example, the formation of such a surface that passes through a given curve, so that a material point, which in practice can be a particle, slides along the surface exactly along this curve. If the curve is a helical line, then the surface will be a helicoid in the role of a helical descent.

After stabilization of the movement, taking into account friction, the particle will move at a constant speed and the angle ε between the orthogonal vectors of trihedrons will also be constant. If the helical line is specified, then the angle ε can be found by compiling the differential equations of movement of the particle in the projections onto the orthogonal vectors of the Frenet trihedron according to known formulas. The fact is that when a particle moves along a given trajectory, the balance of forces must be ensured in the projections on the orthogonal vectors of the trihedrons in the common normal plane of the trajectory. This equilibrium depends on the angle ε. However, *there is a limitation:* the trajectory of movement must be set as a function of the traveled path, that is, the length of its arc. However, from among various curves, it is possible to specify a curve by parametrization as a function of the length of its arc only for a limited number of them.

The influence of wartime was reflected in the scientific relations with the scientists of the aggressor country. Before the full-scale invasion, they were quite tight. After the scientific community of the aggressor country supported an unprovoked invasion, this cooperation became impossible.

Prospects of research consist of the application of the proposed approach to finding a surface that would ensure the trajectory of a particle sliding along a given curve.

4. Conclusions

When Frenet and Darboux trihedrons move along a base flat or spatial curve on the surface, there is an analytical connection that describes the relative position of the trihedrons relative to each other. According to this analytical description, in the reverse order, along the same base curve, it is possible to construct many surfaces, including the original one, which specifies the same movement of trihedrons when they are moved along the curve. Among this set of surfaces, there is one developable, which is formed as an envelope one-parameter set of planes, which are perpendicular to the orthogonal vector $\bar{N}$ the Darboux trihedron.

Conflict of interests

The authors declare that they have no conflict of interest about this research, including financial, personal, authorship, or any other, that could affect the research and its results, which are presented in this article.

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Data availability

The manuscript has no associated data.

Use of artificial intelligence

The authors confirm that they did not use artificial intelligence technologies when creating the presented work.

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